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Exact controllability of scalar conservation laws with strict convex flux

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Abstract

We consider one space dimension scalar conservation law with strict convex flux. The problem is to study exact controllability of entropy solutions. Some partial results have been obtained in [17] and we investigate the precise conditions under which exact controllability problem admits a solution. The basic ingredients in the proof of these results are [14], Lax-Oleinik [12] explicit formula and finer properties of generalized characteristics introduced by Dafermos [11].

Key words: Hamilton-Jacobi equation, scalar conservation laws, characteristic lines, controllability.

1 Introduction:

In this paper we consider the following scalar conservation law in one space dimension. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a strictly convex C^1 function satisfying the super linear growth,

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{|u|} = \infty. \quad (1.1)$$

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Let $T > 0$, $0 \leq \delta < T$, $A < B$, $I = (A, B)$, $\Omega = I \times (\delta, T)$, $u_0 \in L^\infty(I)$, $b_0, b_1 \in L^\infty((0, T))$ and consider the problem

$$u_t + f(u)_x = 0 \quad (x, t) \in \Omega, \quad (1.2)$$

$$u(x, \delta) = u_0(x) \quad x \in I, \quad (1.3)$$

$$u(A, t) = b_0(t) \quad t \in (\delta, T), \quad (1.4)$$

$$u(B, t) = b_1(t) \quad t \in (\delta, T). \quad (1.5)$$

This problem was well studied from last several decades starting from the pioneering works of Lax-Oleinik [12], Kruzkov [19], Bardaux-Leraux-Nedeleck [8]. They have studied the existence and uniqueness of weak solutions to (1.2)-(1.5) satisfying the entropy condition. In spite of being well studied, still there are problems which are open. Notably among them are

1. Profile of a solution, for example how many shocks can a solution exhibit and the nature of the shocks.
2. Optimal controllability for initial and initial-boundary value problem.
3. Exact controllability of initial and initial-boundary value problem.

Problem (1) and (2) has been dealt in [3] and [2] respectively. In this paper we investigate problem (3) for the entropy solution of (1.2). Through out the paper solution of (1.2) always means a weak solution satisfying the entropy condition. The basic ingredient in studying all these problems comes from the analysis of characteristic curves $R_\pm$. Originally this was introduced by Hopf [14] and later by Dafermos [11], who studied them quite extensively to obtain information on the nature of solutions. Independently this was used in [5] to obtain the explicit formula for solution of discontinuous flux.

The plan of the paper is as follows:

In section (1) we state the main results. In section (2) we prove these results assuming four Lemmas without proof. First two Lemma deal with backward construction which will be proved in section (3). The remaining two Lemma deals with free regions. In order to prove these Lemmas, one has to study the finer properties of the generalized characteristics namely

- (i). Comparison properties with respect to the initial data.
- (ii). Failure of the continuity with respect to the initial data.
- (iii). Behavior of the characteristics when one side of the initial data is large.

This has been carried out in section (4). Main tool to study all these properties are the Hopf [14], Lax-Oleinik [12] explicit formulas and we recall them without proof.

Main results, Exact Controllability: Normally for the non linear evolution equations, technique of linearization is adopted to study controllability problems. Unfortunately this method does not work (see Horsin [17]) and very few results are available on this subject. Here we consider the following three problems of controllability. Let $u_0 \in L^\infty(\mathbb{R})$ and

- (I) **Controllability for pure initial value problem:** Assume that $I = \mathbb{R}, \Omega = \mathbb{R} \times (0, T)$. Let $J_1 = (C_1, C_2), J_2 = (B_1, B_2), g \in L^\infty(J_1)$, a target be given. The question is, does there exists a $\bar{u}_0 \in L^\infty(J_2)$ and u in $L^\infty(\Omega)$ such that u is a solution of (1.2) satisfying

$$u(x, T) = g(x) \quad x \in J_1, \quad (1.6)$$

$$u(x, 0) = \begin{cases} u_0(x) & \text{if } x \notin J_2, \\ \bar{u}_0(x) & \text{if } x \in J_2. \end{cases} \quad (1.7)$$

- (II) **Controllability for one sided initial boundary value problem:** Assume that $I = (0, \infty), \Omega = \mathbb{R} \times (0, T), J = (0, C)$ and a target function $g \in L^\infty(J)$ be given. The question is, does there exists a $u \in L^\infty(\Omega)$ and $b \in L^\infty((0, T))$ such that u is a solution of (1.2) satisfying

$$u(x, T) = g(x) \quad \text{if } x \in J, \quad (1.8)$$

$$u(x, 0) = u_0(x) \quad \text{if } x \in (0, \infty), \quad (1.9)$$

$$u(0, t) = b(t) \quad \text{if } t \in (0, T). \quad (1.10)$$

- (III) **Controllability from two sided initial boundary value problem:**

(a). Let $\Omega = \mathbb{R} \times (0, T), I_1 = (B_1, B_2), B_1 \leq C \leq B_2$. Given the target functions $g_1 \in L^\infty(B_1, C), g_2 \in L^\infty(C, B_2)$, does there exists a $\bar{u}_0 \in L^\infty(\mathbb{R} \setminus I_1)$ and $u \in L^\infty(\Omega)$ such that u is a solution of (1.2) satisfying

$$u(x, T) = \begin{cases} g_1(x) & \text{if } B_1 < x < C, \\ g_2(x) & \text{if } C < x < B_2. \end{cases} \quad (1.11)$$

and

$$u(x, 0) = \begin{cases} u_0(x) & \text{if } B_1 < x < B_2, \\ \bar{u}_0(x) & \text{if } x < B_1 \text{ or } x > B_2. \end{cases} \quad (1.12)$$

(b). Here we consider controllability in a strip. Let $I = (B_1, B_2), \Omega = I \times (0, T), B_1 < C < B_2$. Let $g_1 \in L^\infty((B_1, C)), g_2 \in L^\infty((C, B_2))$ be given. Then

the question is, does there exist $b_0, b_1 \in L^\infty((0, T))$ and a $u \in L^\infty(\Omega)$ such that u is a solution of (1.2) and satisfying

$$u(x, 0) = u_0(x), \quad (1.13)$$

$$u(x, T) = \begin{cases} g_1(x) & \text{if } B_1 < x < C, \\ g_2(x) & \text{if } C < x < B_2. \end{cases} \quad (1.14)$$

$$u(B_1, t) = b_0(t), \quad (1.15)$$

$$u(B_2, t) = b_1(t). \quad (1.16)$$

In view of the Lax-Oleinik (Chapter (3) of [12]) explicit formula for solutions of pure initial value problem and by Joseph-Gowda [18] for initial boundary value problem, the targets g or g_1, g_2 cannot be arbitrary. They must satisfy the compatibility condition, for example in the case of problem (I), there exists a non-decreasing function ρ in (C_1, C_2) such that for a.e $x \in (C_1, C_2)$

$$f'(g(x)) = \frac{x - \rho(x)}{T}. \quad (1.17)$$

In the case of problem (II), there exists a non-decreasing function ρ in $(0, C)$ such that

$$f'(g(x)) = \frac{x}{T - \rho(x)}. \quad (1.18)$$

Assuming that the target functions satisfies the compatibility conditions, then the question is

whether the problems (I),(II) and (III) admit a solution? In fact, **it is true** and we have the following results. First we describe the class of functions satisfying compatibility conditions.

Definition (Admissible functions): Let $J = (M, N)$ and $T > 0$,

$$S(J) = \{\rho : J \rightarrow \mathbb{R} : \rho \text{ is monotone and left or right continuous function}\}.$$

Then define admissible class of target functions by

(i) Target space for initial value problem (IA):

$$IA(J) = \{g \in L^\infty(J) : f'(g(x)) = \frac{x - \rho(x)}{T}, \rho \in S(J), \\ \rho \text{ is a non-decreasing function}\}. \quad (1.19)$$

(ii) Target space for left boundary problem (LA):

$$LA(J) = \{g \in L^\infty(J) : f'(g(x)) = \frac{x - M}{T - \rho(x)}, \rho \in S(J), \\ \rho \text{ is a non-increasing right continuous function}\}. \quad (1.20)$$

(iii) Target space for right boundary problem (RA):

$$RA(J) = \{g \in L^\infty(J) : f'(g(x)) = \frac{x - N}{T - \rho(x)}, \rho \in S(J), \\ \rho \text{ is a non-decreasing left continuous function}\}. \quad (1.21)$$

Then we have the following

Main Theorems :

THEOREM 1.1 Let $J_1 = (C_1, C_2), J_2 = (B_1, B_2)$. Let $g(x) = (f')^{-1} \left(\frac{x - \rho(x)}{T} \right)$ be in $IA(J_1)$ and $B_1 < A_1 < A_2 < B_2$, satisfying

$$A_1 \leq \rho(x) \leq A_2 \quad \text{if } x \in J_1, \quad (1.22)$$

then there exists a $\bar{u}_0 \in L^\infty(J_2)$, $u \in L^\infty(\Omega)$ such that $(u, \bar{u}_0)$ is a solution to problem (I) (see Figure 1).

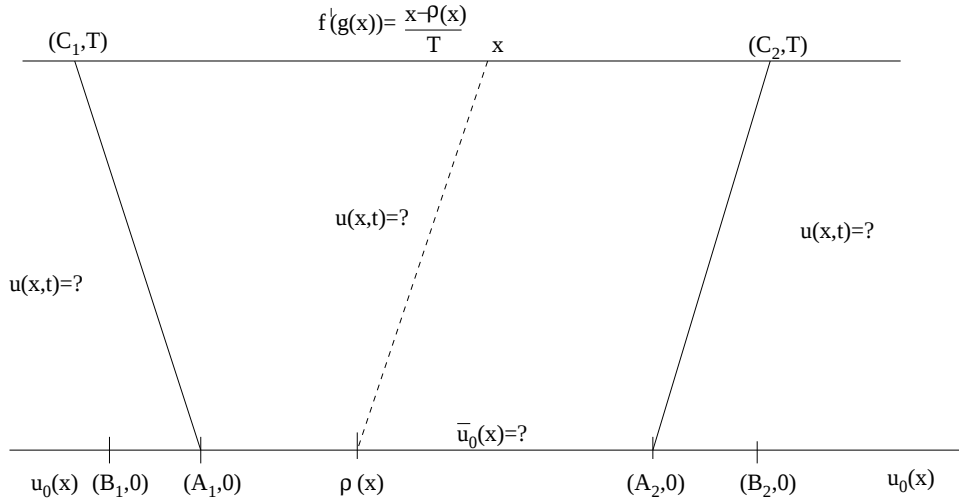


Figure 1:

THEOREM 1.2 Let $\wedge > 0, C > 0, \delta > 0, J = (0, C)$. Let $g \in LA(J)$ given by $f'(g(x)) = \frac{x}{T - \rho(x)}$ for $x \in J$ and satisfying

$$\delta \leq \rho(x) \leq T, \quad (1.23)$$

$$\left| \frac{x}{T - \rho(x)} \right| \leq \wedge. \quad (1.24)$$

Then there exist a $b \in L^\infty(0, T), u \in L^\infty(\Omega)$ such that (u, b) is a solution to Problem II (see Figure 2).

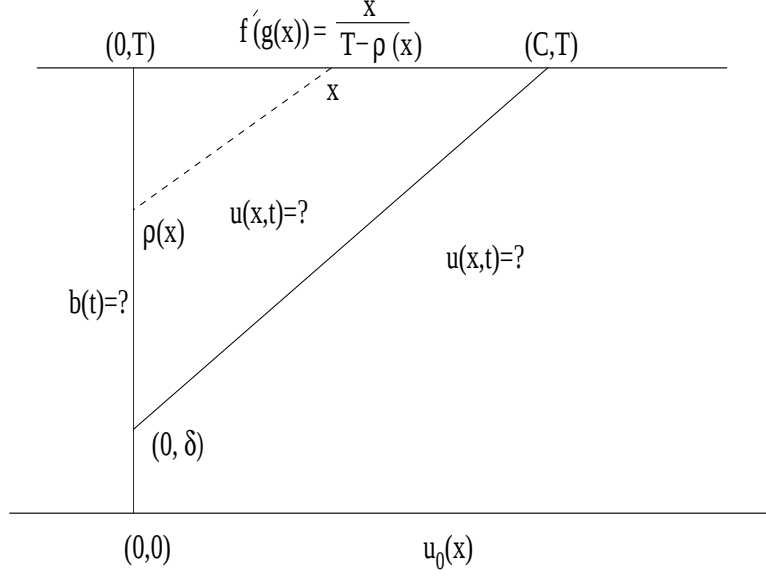


Figure 2:

THEOREM 1.3 *Let $I_1 = (B_1, B_2)$, $B_1 < C < B_2$, $J_1 = (B_1, C)$, $J_2 = (C, B_2)$, then*

(a). Let $A_1 < B_1 < B_2 < A_2$ and $g_1 \in IA(J_1)$, $g_2 \in IA(J_2)$ given by $f'(g_1(x)) = \frac{x - \rho_1(x)}{T}$, $f'(g_2(x)) = \frac{x - \rho_2(x)}{T}$, satisfying

$$\rho_1(x) \leq A_1 \quad \text{if } x \in J_1, \quad (1.25)$$

$$\rho_2(x) \geq A_2 \quad \text{if } x \in J_2. \quad (1.26)$$

Then there exists $\bar{u}_0 \in L^\infty(\mathbb{R} \setminus I_1)$, $u \in L^\infty(\Omega)$ such that $(u, \bar{u}_0)$ is a solution to problem (a) of III (see Figure 3).

(b). Let $\wedge > 0$, $0 < \delta < T$, $g_1 \in LA(J_1)$, $g_2 \in RA(J_2)$, given by $f'(g_1(x)) = \frac{x - B_1}{T - \rho_1(x)}$, $f'(g_2(x)) = \frac{x - B_2}{T - \rho_2(x)}$ satisfying for $i = 1, 2$, $x \in J_i$

$$\delta \leq \rho_i(x) \leq T, \quad (1.27)$$

$$\left| \frac{x - B_i}{T - \rho_i(x)} \right| \leq \wedge. \quad (1.28)$$

Then there exists $b_0, b_1 \in L^\infty((0, T))$ and $u \in L^\infty(\Omega)$ such that (u, b_0, b_1) is a solution to problem (b) of III (see Figure 4).

Before going for further results, let us recall some of the earlier works in this direction and compare them with these results.

Problem (a) in III was considered by Horsin [17] for the Burger's equation under similar assumptions on g_1 and g_2 as in (a) of Theorem 1.3. He proves that given any

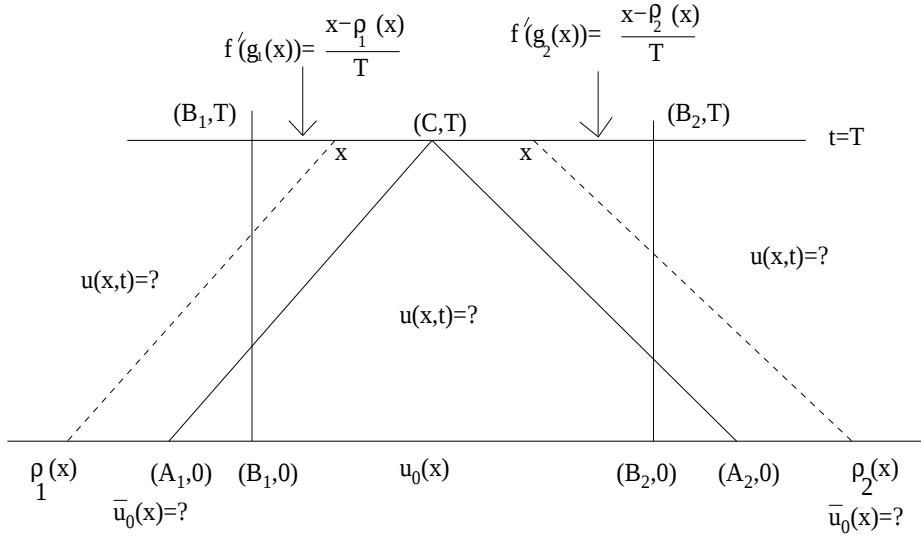


Figure 3:

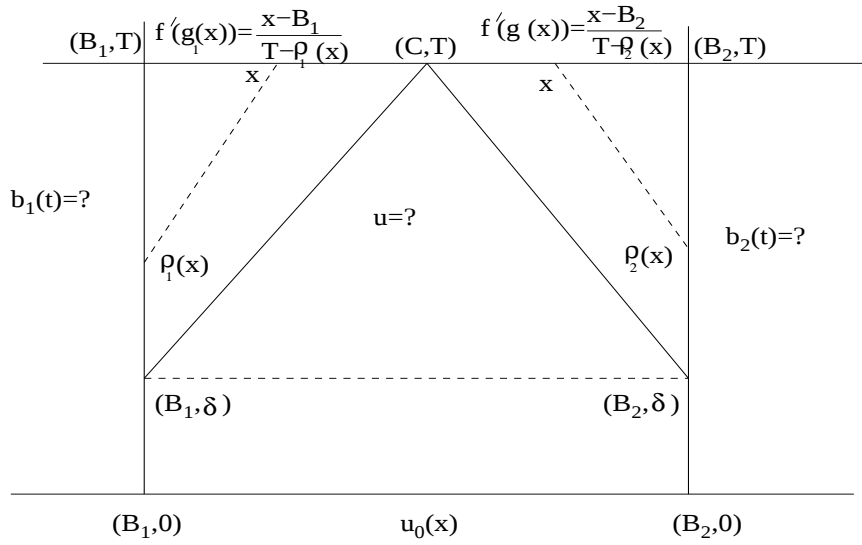


Figure 4:

$T > 2$ there exists a $T_c \geq T$, such that (a) of problem III has an approximate controllability solution. That is given $\epsilon > 0$, there exist $(u, \bar{u}_0)$ such that

$$\int_{B_1}^{B_2} |u(x, T_c) - g(x)| dx = O(\epsilon),$$

and $u(x, T_c) = g(x) = \chi_{(B_1, C)}(x)g_1(x) + \chi_{(C, B_2)}(x)g_2(x)$, outside an interval of length ϵ .

In the viscous case the same problem was considered by Glass-Guerrero [13] for the control $u(x, T) = M$ is constant. Using the Cole-Hopf transformation, they show that there exist $T_0 > 0$ such that for all time $T > T_0$ and small viscosity, they prove the exact controllability. Also Guerrero-Imanuvilov [15] proves a negative result by showing that $M = 0$ cannot be controllable.

Theorem 1.3 is stronger and much more precise result in the non viscous case because

- (i). It removes the condition on time T_c and obtains exact controllability.
- (ii). It deals with general convex flux instead of Burger's equation.
- (iii). In section (5) we give a criterion when the constants are controllable.

In the case of problem (II), Fabio-Ancona and Andrea-Marson [6],[7] studied the problem from the point of view of Hamilton-Jacobi equations and studies the compactness properties of $\{u(\cdot, T)\}$ when $u(x, 0) = 0$ and $u(\cdot, 0) \in \mathcal{U}$, here $\mathcal{U}$ is a set of controls satisfying some properties.

In our results on controllability, superlinearity of f plays an important role in removing the condition on T_c and by creating free regions (see Lemmas 2.3 and 2.4). Next using convexity and backward construction, we explicitly construct solutions in these free regions for particular data which allow to obtain solutions for control problems (see Lemmas 2.1 and 2.2).

REMARK 1.1 *The conditions in Theorems 1.1-1.3 are optimal. That is, in general, we cannot take $A_1 = B_1$, $A_2 = B_2$ in Theorem 1.1 and $\delta = 0$ in Theorems 1.2 and 1.3. This can be illustrated by a simple counter example (see counter example 4.2).*

2 Exact controllability and main results

In this section we give the proof of Theorems 1.1 to 1.3. Basically following two main ideas are used to prove these results

- (a) Free regions : By suitable variations of parameters in the initial data, one can obtain sub region in Ω where the solution are prescribed as constants. These sub regions are called free regions. This is achieved in Lemmas 4.7 and 4.8. For example from (4.95), the region

$$\{(x, t) : x < L_1(t)\} \quad (2.1)$$

is a free region since u^λ is constant.

- (b) Backword construction : This is a method where for a given target function satisfying the compatibility conditions, one can construct a solution which achieves the given target at $t = T$.

Then the Theorems will follow from backword construction in free regions and gluing the different solutions using Rankine-Hugoniot conditions across the boundaries of free regions.

We state the following Lemmas which deal with this construction.

Let $u_0 \in L^\infty(\mathbb{R})$, $0 \leq \delta < T$, $A, C \in \mathbb{R}$. Let $l(\cdot, \delta, A, C)$ be the line joining between (C, T) and (A, δ) with slope $1/f'(a(\delta, A, C))$, intersecting $t = 0$ axis at $D(\delta, A, C)$ and is given by

$$f'(a(\delta, A, C)) = \frac{C - A}{T - \delta}, \quad (2.2)$$

$$l(t, \delta, A, C) = A + f'(a(\delta, A, C))(t - \delta), \quad (2.3)$$

$$D(\delta, A, C) = A - \delta f'(a(\delta, A, C)), \quad (2.4)$$

$$= A - \frac{\delta(C - A)}{T - \delta}. \quad (2.5)$$

LEMMA 2.1 1. Let $\wedge > 0$, $A < C$ and $\rho \in LA((A, C))$ satisfying

$$0 \leq \delta \leq \rho(x) \leq T, \quad (2.6)$$

$$\left| \frac{x - A}{T - \rho(x)} \right| \leq \wedge. \quad (2.7)$$

Let $\Omega = (A, \infty) \times (\delta, T)$. Then there exists a $\tilde{b}_1(t, \delta, A, C) \in L^\infty((\delta, T))$ and a solution $\tilde{u}_1(x, t, \delta, A, C)$ of (1.2) satisfying

$$f'(\tilde{u}_1(x, T, \delta, A, c)) = \frac{x - A}{T - \rho(x)}, \quad x \in (A, C), \quad (2.8)$$

with initial and boundary conditions

$$\tilde{u}_1(A, t, \delta, a, C) = \tilde{b}_1(t, \delta, A, C), \quad t \in (\delta, T), \quad (2.9)$$

$$\tilde{u}_1(x, t, \delta, a, C) = a(\delta, A, C), \quad x > l(t, \delta, A, C), \quad (2.10)$$

$$\tilde{u}_1(l(t, \delta, A, C) -, t, \delta, a, C) = a(\delta, A, C). \quad (2.11)$$

2. Let $C < A$ and $\rho \in RA((C, A))$ satisfying (2.6) and (2.7) for $x \in (C, A)$. Let $\Omega = (-\infty, A) \times (\delta, T)$. Then there exist $b_2(t, \delta, A, C) \in L^\infty((\delta, T))$ and a solution $\tilde{u}_2(x, t, \delta, a, C)$ of (1.2) satisfying

$$f'(\tilde{u}_2(x, T, \delta, A, C)) = \frac{x - A}{T - \rho(x)}, \quad x \in (C, A), \quad (2.12)$$

with initial and boundary conditions

$$\tilde{u}_2(A, t, \delta, A, C) = \tilde{b}_2(t, \delta, A, C), \quad t \in (\delta, T), \quad (2.13)$$

$$\tilde{u}_2(x, t, \delta, A, C) = a(\delta, a, C), \quad x < l(t, \delta, A, C), \quad (2.14)$$

$$\tilde{u}_2(l(t, \delta, A, C)+, t, \delta, A, C) = a(\delta, A, C), \quad t \in (\delta, T). \quad (2.15)$$

LEMMA 2.2 Let $A_1 < A_2$, $C_1 < C_2$, $\rho \in IA((C_1, C_2))$ such that for all $x \in (C_1, C_2)$,

$$A_1 \leq \rho(x) \leq A_2. \quad (2.16)$$

Let $\Omega = \mathbb{R} \times \mathbb{R}_+$, for $i = 1, 2$, $l_i(t) = l(t, 0, A_i, C_i)$, $a_i = a_i(0, A_i, C_i)$, then there exist $\tilde{u}_0 \in L^\infty((A_1, A_2))$ and a solution $\tilde{u}$ of (1.2) such that for $0 < t < T$,

$$f'(\tilde{u}(x, T)) = \frac{x - \rho(x)}{T}, \quad \text{for } x \in (C_1, C_2), \quad (2.17)$$

$$\tilde{u}(l_1(t)+, t) = a_1, \quad (2.18)$$

$$\tilde{u}(l_2(t)-, t) = a_2, \quad (2.19)$$

with initial conditions

$$\tilde{u}(x, 0) = \begin{cases} a_1 & \text{if } x < A_1, \\ \tilde{u}_0(x) & \text{if } A_1 < x < A_2, \\ a_2 & \text{if } x > A_2. \end{cases} \quad (2.20)$$

Let $T > 0, \mu, \lambda \in \mathbb{R}, A < B, l_1(t) = l(t, 0, A, C), l_2(t) = l(t, 0, B, C), a_1 = a_1(0, A, C), a_2 = a_2(0, B, C)$. Define u_0^λ and u_0^μ by

$$u_0^\lambda(x) = \begin{cases} a_1 & \text{if } x < A, \\ \lambda & \text{if } A < x < B, \\ u_0(x) & \text{if } x > B \end{cases} \quad (2.21)$$

and

$$u_0^\mu(x) = \begin{cases} a_2 & \text{if } x > B, \\ \mu & \text{if } A < x < B, \\ u_0(x) & \text{if } x < A. \end{cases} \quad (2.22)$$

Let $u_\lambda(x, t)$ and $u_\mu(x, t)$ be the solutions of (1.2) with initial data u_0^λ and u_0^μ respectively. Then we have the following

LEMMA 2.3 *There exist $\mu_0 < \lambda_0$ such that for all $\mu \leq \mu_0$, $\lambda \geq \lambda_0$, $0 < t < T$, $x \in \mathbb{R}$, u_λ and u_μ satisfies*

$$u_\lambda(x, t) = a_1, \quad \text{if } x < l_1(t), \quad u_\lambda(l_1(t)+, t) = a_1 \quad (2.23)$$

$$u_\mu(x, t) = a_2, \quad \text{if } x > l_2(t), \quad u_\mu(l_2(t)-, t) = a_2. \quad (2.24)$$

Let $\delta > 0, T > 0, B_1 \leq B_2$, $l_1(t) = l(t, \delta, B_1, C)$, $l_2(t) = l(t, \delta, B_2, C)$, $A_1 = l_1(0) < B_1$, $A_2 = l_2(0) > B_2$, $a_1 = a(\delta, B_1, C)$, $a_2 = a(\delta, B_2, C)$. For $\lambda, \mu \in \mathbb{R}$, define $u_0^{\lambda, \mu}$ by

$$u_0^{\lambda, \mu}(x) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ a_2 & \text{if } x > A_2 \end{cases} \quad (2.25)$$

and $u_{\lambda, \mu}$ be the solution of (1.2) with initial data $u_0^{\lambda, \mu}$.

Then we have the following

LEMMA 2.4 *Given any λ_0, μ_0 , there exist $\lambda_2 \geq \lambda_0$, $\mu_2 \leq \mu_0$ such that for $0 \leq t \leq T$, u_{λ_2, μ_2} satisfies*

$$u_{\lambda_2, \mu_2}(x, t) = \begin{cases} a_1 & \text{if } x < l_1(t), \\ a_2 & \text{if } x > l_2(t), \end{cases} \quad (2.26)$$

$$u_{\lambda_2, \mu_2}(l_1(t)+, t) = a_1, \quad u_{\lambda_2, \mu_2}(l_2(t)-, t) = a_2. \quad (2.27)$$

Proof of Theorem 1.1 Let $\Omega = \mathbb{R} \times (0, T)$, A_i, B_i, C_i, g and ρ be as in Theorem 1.1
Let

$$f'(a_1) = \frac{C_1 - A_1}{T}, \quad f'(a_2) = \frac{C_2 - A_2}{T}, \quad l_1(t) = A_1 + tf'(a_1), \quad l_2(t) = A_2 + tf'(a_2).$$

Then from Lemma 2.3 choose λ, μ and solutions u_λ and u_μ of (1.2) such that

$$u_\lambda(x, t) = a_2 \quad \text{if } x < l_2(t), \quad u_\lambda(l_2(t)+, t) = a_2 \quad (2.28)$$

$$u_\mu(x, t) = a_1 \quad \text{if } x > l_1(t), \quad u_\mu(l_1(t)-, t) = a_1 \quad (2.29)$$

with

$$u_\lambda(x, 0) = \begin{cases} a_2 & \text{if } x < A_2, \\ \lambda & \text{if } A_2 < x < B_2, \\ u_0(x) & \text{if } x > B_2 \end{cases} \quad (2.30)$$

and

$$u_\mu(x, 0) = \begin{cases} a_1 & \text{if } x > A_1, \\ \mu & \text{if } B_1 < x < A_1, \\ u_0(x) & \text{if } x < B_1 \end{cases} \quad (2.31)$$

From (1.22) and Lemma 2.1 there exist a solution u_1 of (1.2) and $\tilde{u}_0 \in L^\infty(A_1, A_2)$ satisfying

$$u_1(x, T) = g(x), \text{ if } x \in (C_1, C_2) \quad (2.32)$$

$$u_1(x, 0) = \begin{cases} a_1 & \text{if } x < A_1, \\ \tilde{u}_0(x) & \text{if } A_1 < x < A_2, \\ a_2 & \text{if } x > A_2, \end{cases} \quad (2.33)$$

and

$$u_1(x, t) = \begin{cases} a_1 & \text{if } x < l_1(t), \\ a_2 & \text{if } x > l_2(t). \end{cases} \quad (2.34)$$

$$u_1(l_1(t)+, t) = a_1, u_1(l_2(t)-, t) = a_2. \quad (2.35)$$

Let

$$\bar{u}_0(x) = \begin{cases} u_0(x) & \text{if } x \notin (B_1, B_2) \\ \lambda & \text{if } A_2 < x < B_2 \\ \tilde{u}_0(x) & \text{if } A_1 < x < A_2 \\ \mu & \text{if } B_1 < x < A_1. \end{cases}$$

From (2.28), (2.29), (2.34) and RH condition, glue u_λ, u_μ, u_1 to form a single solution u of (1.2) for $0 < t < T$ by

$$u(x, t) = \begin{cases} u_\mu(x, t) & \text{if } x < l_1(t), \\ u_1(x, t) & \text{if } l_1(t) < x < l_2(t), \\ u_\lambda(x, t) & \text{if } l_2(t) < x. \end{cases} \quad (2.36)$$

Then from (2.30), (2.31) and (2.33), $(u, \bar{u}_0)$ is the required solution. This proves the Theorem.

Proof of Theorem 1.2 Let $f'(a) = \frac{C}{T-\delta}$ and $l(t)$ be the line joining (C, T) and $(0, \delta)$ given by $l(t) = (t - \delta)f'(a)$. Let $A = l(0) = -\delta f'(a) < 0$. From Lemma 2.3 by choosing λ large, we can find a solution u_λ of (1.2) in $\Omega = \mathbb{R} \times (0, T)$ satisfying

$$u_\lambda(x, 0) = \begin{cases} a & \text{if } x < A, \\ \lambda & \text{if } A < x < 0, \\ u_0(x) & \text{if } x > 0. \end{cases} \quad (2.37)$$

$$u_\lambda(x, t) = a \text{ if } x < l(t), \quad (2.38)$$

$$u_\lambda(l(t)+, t) = a. \quad (2.39)$$

From (1.23) , (1.24) and (1) of Lemma 2.1, choose a solution u_1 of (1.2) and $b_1 \in L^\infty(\delta, T)$ such that

$$u_1(x, T) = g(x) \quad (2.40)$$

$$u_1(0, t) = b_1(t) \text{ if } \delta < t < T, \quad (2.41)$$

$$u_1(x, t) = a \quad \text{if } x > l(t), t > \delta, \quad (2.42)$$

$$u_1(l(t)-, t) = a \quad \text{if } t > \delta. \quad (2.43)$$

From (2.39), (2.43) and RH conditions we glue the solutions u_λ and u_1 to obtain a solution u of (1.2) by

$$u(x, t) = \begin{cases} u_\lambda(x, t) & \text{if } x > l(t), 0 < t < T, \\ u_1(x, t) & \text{if } 0 < x < l(t), \delta < t < T. \end{cases} \quad (2.44)$$

Define $b \in L^\infty(0, T)$ by

$$b(t) = \begin{cases} u_\lambda(0+, t) & \text{if } 0 < t < \delta, \\ b_1(t) & \text{if } \delta < t < T. \end{cases} \quad (2.45)$$

Then from (2.37) , (2.40), (u, b) is the required solution. This proves the theorem.

Proof of Theorem 1.3 Let $f'(a_1) = \frac{C-A_1}{T}$, $f'(a_2) = \frac{C-A_2}{T}$, $l_1(t) = A_1 + tf'(a_1) = A_2 + tf'(a_2)$ be the respective lines joining (C, T) , $(A_1, 0)$ and (C, T) , $(A_2, 0)$.

From Lemma 2.4 choose (λ, μ) and a solution $u_{\lambda, \mu}$ of (1.2) in $\mathbb{R} \times (0, T)$ satisfying

$$u_{\lambda, \mu}(x, t) = \begin{cases} a_1 & \text{if } x < l_1(t), \\ a_2 & \text{if } x > l_2(t), \end{cases} \quad (2.46)$$

with initial condition

$$u_{\lambda, \mu}(x, 0) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ a_2 & \text{if } x > A_2. \end{cases} \quad (2.47)$$

(a). Since g_i is a non decreasing function for $i = 1, 2$ satisfying (1.23) , (1.26) and hence

$$D_1 = \rho_1(B_1) \leq A_1, A_2 \leq \rho_2(B_2) = D_2.$$

Let η_i be the line joining (B_i, T) and $(D_i, 0)$ with $f'(m_i) = \frac{B_i - D_i}{T}$ for $i = 1, 2$. Then from Lemma 2.2, there exist solutions u_i of (1.2) in $\mathbb{R} \times (0, T)$ with initial

condition $u_0^i \in L^\infty(D_i, A_i)$ for $i = 1, 2$ such that

$$u_1(x, T) = g_1(x, T) \quad \text{if } x \in (B_1, C), \quad (2.48)$$

$$u_2(x, T) = g_2(x, T) \quad \text{if } x \in (C, B_2), \quad (2.49)$$

$$u_1(x, t) = m_1 \quad \text{if } x < \eta_1(t), \quad (2.50)$$

$$u_1(l_1(t)-, t) = u_1(l_1(t)+, t) = a_1, \quad (2.51)$$

$$u_2(x, t) = m_2 \quad \text{if } x > \eta_2(t), \quad (2.52)$$

$$u_2(l_2(t)-, t) = u_2(l_2(t)+, t) = a_2, \quad (2.53)$$

and

$$u_1(x, 0) = \begin{cases} m_1 & \text{if } x < D_1, \\ u_0^1(x) & \text{if } D_1 < x < A_1, \\ a_1 & \text{if } x > A_1. \end{cases} \quad (2.54)$$

$$u_2(x, 0) = \begin{cases} m_2 & \text{if } x > D_2, \\ u_0^2(x) & \text{if } A_2 < x < D_2, \\ a_2 & \text{if } x < A_2. \end{cases} \quad (2.55)$$

From (2.46) , (2.47) ,(2.51), (2.53) and from RH conditions, we can glue $u_1, u_2, u_{\lambda, \mu}$ to a solution u of (1.2) with initial data $u(x, 0)$ given by

$$u(x, t) = \begin{cases} u_1(x, t) & \text{if } x < l_1(t), \\ u_{\lambda, \mu}(x, t) & \text{if } l_1(t) < x < l_2(t), \\ u_2(x, t) & \text{if } x > l_2(t), \end{cases} \quad (2.56)$$

$$u(x, 0) = \begin{cases} m_1 & \text{if } x < D_1, \\ u_0^1(x) & \text{if } D_1 < x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ u_0^2(x) & \text{if } A_2 < x < D_2, \\ m_2 & \text{if } x > D_2. \end{cases}$$

Define $\bar{u}_0$ by

$$\bar{u}_0(x) = \begin{cases} m_1 & \text{if } x < D_1, \\ u_0^1(x) & \text{if } D_1 < x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ \mu & \text{if } B_2 < x < A_2, \\ u_0^2(x) & \text{if } A_2 < x < D_2, \\ m_2 & \text{if } x > D_2. \end{cases} \quad (2.57)$$

From (2.52) , (2.53) u satisfies

$$u(x, T) = \begin{cases} g_1(x) & \text{if } B_1 < x < C, \\ g_2(x) & \text{if } C < x < B_2, \end{cases} \quad (2.58)$$

and $(u, \bar{u}_0)$ is the required solution. This proves (a).

(b). Given $\delta > 0$ choose $A_1 < B_1 < B_2 < A_2$ such that $\max(l_1(B_1), l_2(B_2)) = \delta$ and $u_{\lambda, \mu}$ be the solution of (1.2) as in (2.46). From (1.27), (1.28) and from Lemma 2.1, there exist solutions u_1 of (1.2) in $(B_1, \infty) \times (\delta, T)$ and boundary data $\tilde{b}_1$, u_2 of (1.2) in $(-\infty, B_2) \times (\delta, T)$ and boundary data $\tilde{b}_2$ such that

$$\begin{aligned} u_1(x, T) &= g_1(x) & \text{if } x \in (B_1, C), \\ u_2(x, T) &= g_2(x) & \text{if } x \in (C, B_2), \end{aligned}$$

and for $\delta < t < T$,

$$\begin{aligned} u_1(B_1, t) &= \tilde{b}_1(t), & u_1(l_1(t)-, t) &= a_1, \\ u_2(B_2, t) &= \tilde{b}_2(t), & u_2(l_2(t)+, t) &= a_2. \end{aligned}$$

Then from RH condition glue $u_1, u_2, u_{\lambda, \mu}$ in $\Omega = (B_1, B_2) \times (0, T)$ by

$$u(x, t) = \begin{cases} u_1(x, t) & \text{if } 0 < t < \delta, B_1 < l_1(x) < t, \\ u_2(x, t) & \text{if } 0 < t < \delta, t < l_1(x) < B_2, \\ u_{\lambda, \mu}(x, t) & \text{otherwise.} \end{cases}$$

Then u is a solution of (1.2) satisfying the boundary conditions (b_1, b_2) given by

$$\begin{aligned} b_1(t) &= \begin{cases} \tilde{b}_1(t) & \text{if } \delta < t < T, \\ u_{\lambda, \mu}(B_1+, t) & \text{if } 0 < t < \delta, \end{cases} \\ b_2(t) &= \begin{cases} \tilde{b}_2(t) & \text{if } \delta < t < T, \\ u_{\lambda, \mu}(B_2-, t) & \text{if } 0 < t < \delta. \end{cases} \end{aligned}$$

Then (u, b_1, b_2) is the solution for problem (1.3). This proves the Theorem.

3 Proof of Lemmas 2.1 and 2.2 :

Proof of Lemma 2.2 follows as that of Lemma 3.5 of [2]. Where as proof of Lemma 2.1 is quite involve and we give the proof here.

Boundary value partition: (See Figure 5) Let $0 \leq \delta < T$, $A < C$, $I = (A, C)$, $J = (\delta, T)$. Let $P = \{t_0, t_1 \dots t_n, x_0, x_1 \dots x_n\}$ is called a boundary value partition if

$$T = t_0 > t_1 > t_2 \dots > t_n = \delta, \quad A = x_0 \leq x_1 \leq x_2 < \dots \leq x_n = C.$$

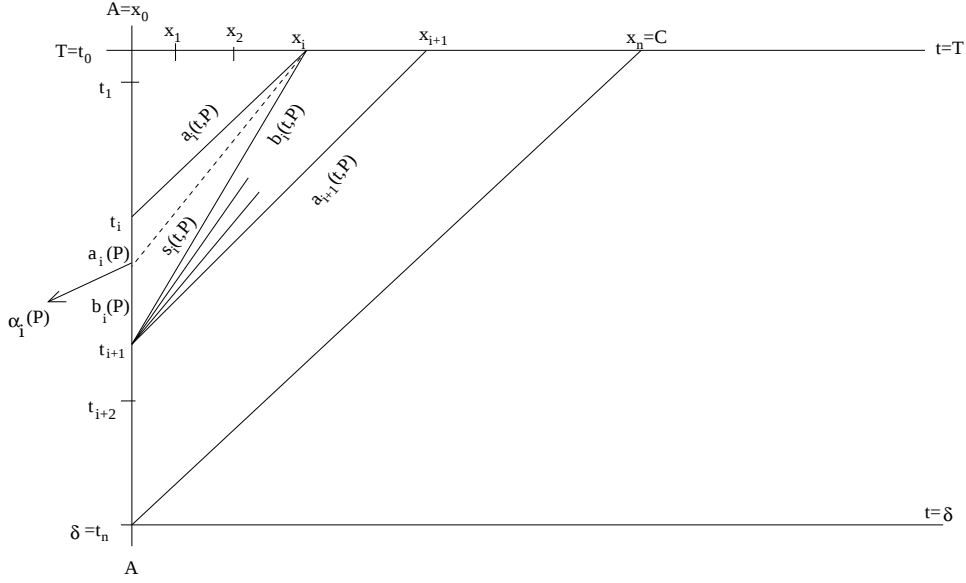


Figure 5:

Let $P(I, J) = \{P : P \text{ is a boundary value partition of } I, J\}$. (3.1)

For a $P \in P(I, J)$ denote $a_i(P), s_i(P), b_i(P), a_i(t, P), s_i(t, P), b_i(t, P)$ by

$$\begin{aligned} f'(a_i(P)) &= \frac{x_i - A}{T - t_i}, \\ f'(b_i(P)) &= \frac{x_i - A}{T - t_{i+1}}, \\ s_i(P) &= \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)}, \\ a_i(t, P) &= x_i + f'(a_i(P))(t - T), \\ b_i(t, P) &= x_i + f'(b_i(P))(t - T), \\ s_i(t, P) &= x_i + s_i(P)(t - T), \end{aligned}$$

Clearly $a_i(t_i, P) = t_i$, $b_i(t_{i+1}, P) = t_{i+1}$.

LEMMA 3.1 Define $\alpha_i(P)$ such that $s_i(\alpha_i(P), P) = A$. Then for $t \leq T$

$$a_i(P) > b_i(P), \quad a_{i+1}(P) \geq b_i(P), \quad (3.2)$$

$$t_i > \alpha_i(P) > t_{i+1}, \quad a_i(t, P) \leq s_i(t, P) \leq b_i(t, P), \quad (3.3)$$

Proof. Since $t_i > t_{i+1}$, $x_i \leq x_{i+1}$, hence

$$\frac{x_i - A}{T - t_i} > \frac{x_i - A}{T - t_{i+1}}, \quad \frac{x_i - A}{T - t_{i+1}} \leq \frac{x_{i+1} - A}{T - t_{i+1}}.$$

This implies (3.2). From strict convexity of f and (3.2), we have

$$f'(a_i(P)) > \frac{f(a_i(P)) - f(b_i(P))}{a_i(P) - b_i(P)} > f'(b_i(P)),$$

hence $t_i > \alpha_i(P) > t_{i+1}$ and for all $t < T$, $a_i(t, P) \leq s_i(t, P) \leq b_i(t, P)$. This proves the Lemma.

Let $\Omega_i(P) = \{(x, t) : a_i(t, P) < x < a_{i+1}(t, P), t_{i+1} < t < T\}$. In view of Lemma 3.1, let $u_i(x, t, P)$ be a solution of (1.2) in $\Omega_i(P)$ defined by

$$u_i(x, t, P) = \begin{cases} a_i(P) & \text{if } a_i(t, P) < x < s_i(t, P), \\ b_i(P) & \text{if } s_i(t, P) < x < b_i(t, P), \\ (f')^{-1}\left(\frac{x-A}{T-t_{i+1}}\right) & \text{if } b_i(t, P) < x < a_{i+1}(t, P). \end{cases} \quad (3.4)$$

Then

$$u_{i+1}(a_{i+1}(t, P)+, t, P) = a_{i+1}(P) = u_i(a_{i+1}(t, P)-, t, P). \quad (3.5)$$

Also $a_n(P)$ and $a_n(t, P)$ are independent of P and denote by $a_n, a_n(t)$. Then from (3.5) it follows that $u_{n-1}(a_n(t)-, t, P) = a_n$. Therefore define the solution $u(x, t, P)$ of (1.2) in $\Omega = (A, \infty) \times (\delta, T)$ by

$$u(x, t, P) = \begin{cases} u_i(x, t, P) & \text{if } (x, t) \in \Omega_i(P), 0 < i \leq n-1, \\ a_n & \text{if } x > a_n(t), \delta < t < T, \end{cases} \quad (3.6)$$

and $u(x, t, P)$ takes the boundary value $b(t, P)$ and initial value a_n given by

$$u(A, t, P) = b(t, P) = \begin{cases} \theta_f & \text{if } t_1 < t < T, \\ a_i(P) & \text{if } \alpha_i(P) < t < t_i, \\ b_i(P) & \text{if } t_{i+1} < t < \alpha_i(P). \end{cases} \quad (3.7)$$

$$u(x, \delta, P) = u_0(x, P) = a_n = (f)^{-1}\left(\frac{C-A}{T-\delta}\right) \quad \text{if } x \in (A, \infty). \quad (3.8)$$

Further more at $t = T$, and $x \in (A, C)$, u satisfies

$$f'(u(x, T, P)) = \sum_{i=1}^n \chi_{[x_i, x_{i+1})}(x) \left(\frac{x-A}{T-t_i}\right). \quad (3.9)$$

Next we calculate the L^∞ and TV bounds of the boundary value $b(\cdot, P)$.

$$\begin{aligned} |f'(b(t, P))| &= \max_{1 \leq i \leq n} (|f'(a_i(P))|, |f'(b_i(P))|) \\ &= \max_{1 \leq i \leq n} \left(\left| \frac{x_i-A}{T-t_i} \right|, \left| \frac{x_i-A}{T-t_{i+1}} \right| \right) \\ &= \max_{1 \leq i \leq n} \left(\left| \frac{x_i-A}{T-t_i} \right| \right). \end{aligned} \quad (3.10)$$

$$\begin{aligned}
TV(f'(b(\cdot, P))) &= \sum_{i=1}^{n-1} |f'(a_i(P)) - f'(b_i(P))| + \sum_{i=0}^{n-1} |f'(b_i(P)) - f'(a_{i+1}(P))| \\
&= \sum_{i=1}^{n-1} \left| \frac{x_i - A}{T - t_i} - \frac{x_i - A}{T - t_{i+1}} \right| + \sum_{i=1}^{n-1} \left| \frac{x_i - A}{T - t_{i+1}} - \frac{x_{i+1} - A}{T - t_{i+1}} \right| + \left| \frac{x_1 - A}{T - t_1} \right| \\
&= \sum_{i=1}^{n-1} \frac{(x_i - A)(t_i - t_{i+1})}{(T - t_i)(T - t_{i+1})} + \sum_{i=1}^{n-1} \frac{(x_{i+1} - x_i)}{(T - t_{i+1})} + \left| \frac{x_1 - A}{T - t_1} \right| \\
&\leq \left(\frac{T - \delta}{T - t_1} \right) \max_{1 \leq i \leq n} \left| \frac{x_i - A}{T - t_i} \right| + \left(\frac{C - A}{T - t_1} \right).
\end{aligned} \tag{3.11}$$

Analysis of Discretization and Convergence: Let $\rho : [A, C] \rightarrow [\delta, T]$ be a non increasing right continuous function. Then it follows that $\{x : \rho(x) \leq t\}$ is a closed interval for any t . Let $0 < \epsilon < C - A$, define

$$\rho_\epsilon(x) = \min\{\rho(x), \rho(A + \epsilon)\}.$$

Then ρ_ϵ is a non-increasing right continuous function. Let m, n be non negative integers and let $T = t_0 > t_1 = \rho(A + \epsilon) > t_2 > \dots > t_n = \delta$ be such that $|t_i - t_{i+1}| \leq \frac{1}{m}$ for all $i \geq 1$. Let $k \leq n - 1$ such that $\{x : \rho_\epsilon(x) \leq t_{k+1}\} = \phi$, $\{x : \rho(x) \leq t_k\} \neq \phi$ and define $\{x_i\}$ by $x_i = C$ if $i \geq k + 1$ and for $1 \leq i \leq k$,

$$\{x : \rho_\epsilon(x) > t_i\} = (x_i, C).$$

Denote $P_{n,m,\epsilon}$ by $P_{m,n,\epsilon} = \{t_0, t_1, \dots, t_n, x_0, x_1, \dots, x_n\}$ the partition depending on n, m and ϵ . Associate to $P_{m,n,\epsilon}$ define

$$\rho(x, P_{m,n,\epsilon}) = \sum_{i=1}^{n-1} t_i \chi_{[x_{i-1}, x_i)}(x) + t_n \chi_{[x_{n-1}, x_n]}(x). \tag{3.12}$$

Then it follows from definition,

$$\sup_n |\rho_\epsilon(x) - \rho(x)| \leq \sup_{A < x < \epsilon} |\rho(x) - \rho(A + \epsilon)| \tag{3.13}$$

$$\sup_n |\rho_\epsilon(x) - \rho(x, P_{m,n,\epsilon})| \leq \frac{1}{m}. \tag{3.14}$$

Definition: Let $\epsilon_2 < \epsilon_1, n_2 \geq n_1$. For $i = 1, 2$, let $P_{m,n_i,\epsilon_i} = \{t_0, t_{1,i}, \dots, t_{n_i,i}, x_0, x_{1,i}, \dots, x_{n_i,i}\}$ be the partitions. Then we say P_{m,n_2,ϵ_2} dominates P_{m,n_1,ϵ_1} and is denoted by $P_{m,n_2,\epsilon_2} \geq P_{m,n_1,\epsilon_1}$ if for $1 \leq j \leq n_1$

$$\begin{aligned}
t_{j,1} &= t_{n_2 - n_1 + j, 2}, \\
x_{j,1} &= x_{n_2 - n_1 + j, 2}.
\end{aligned} \tag{3.15}$$

For a partition $P_{m,n,\epsilon}$, define $\Omega(P_{m,n,\epsilon})$ by

$$\Omega(P_{m,n,\epsilon}) = \{(x, t) : a_1(t, P_{m,n,\epsilon}) < x, \delta < t < T\}. \tag{3.16}$$

Properties of the domination: Let $\epsilon_2 < \epsilon_1, n_2 \geq n_1$ and let for $i = 1, 2, u_i(x, t) = u(x, t, P_{m, n_i, \epsilon_i}), b_i(t) = b(t, P_{m, n_i, \epsilon_i})$ as in (3.6) and (3.7) respectively. Let $P_{m, n_2, \epsilon_2} \geq P_{m, n_1, \epsilon_1}$, then from the construction it follows

$$\rho_{\epsilon_1}(x) = \rho_{\epsilon_2}(x) \quad \text{if} \quad x \geq \epsilon_1 + A, \quad (3.17)$$

$$u_1(x, t) = u_2(x, t) \quad \text{if} \quad (x, t) \in \Omega(P_{m, n, \epsilon_1}), \quad (3.18)$$

$$b_1(t) = b_2(t) \quad \text{if} \quad \delta < t \leq \rho(A + \epsilon_1), \quad (3.19)$$

$$f'(u_i(x, T)) = \frac{x - A}{T - \rho(x, p_{m, n_i, \epsilon_i})}, \quad i = 1, 2. \quad (3.20)$$

$$\begin{aligned} \int_{\delta}^T |f'(b_1(t)) - f'(b_2(t))| dt &= \int_{\rho(\epsilon_1 + A)}^{\rho(\epsilon_2 + A)} |f'(b_2(t)) - f'(b_1(t))| dt \\ &\leq |\rho(A + \epsilon_1) - \rho(A + \epsilon_2)| \max_j \left\{ \left| \frac{x_{j,2} - A}{T - t_{j,2}} \right| \right\} \\ &= |\rho(A + \epsilon_1) - \rho(A + \epsilon_2)| \max_j \left\{ \left| \frac{x_{j,2} - A}{T - \rho_{\epsilon_2}(x_{j,2})} \right| \right\}. \end{aligned} \quad (3.21)$$

Construction of dominations: Let $\epsilon_2 < \epsilon_1$ and $P_{m, n_1, \epsilon_1} = \{t_0, t_{1,1}, \dots, t_{n_1,1}, x_0, x_{1,1}, \dots, x_{n_1,1}\}$. Now choose $\rho(\epsilon_2 + A) = t_{1,2} > t_{2,2} > \dots, t_{r_2,2} = t_{11} = \rho(\epsilon_1 + A)$ such that $|t_{i,2} - t_{i+1,2}| \leq \frac{1}{m}$ for $1 \leq i \leq r_2 - 1$. Let $n_2 = n_1 + n_2$ and define $t_{i,2}$ for $i \geq r_2$ by

$$t_{i,2} = t_{i-r_2+1,1},$$

and $\{x_{i,2}\}$ be associated to $\{t_{i,2}\}$. Let $n_2 = r_2 + n_1 - 1$ and $P_{m, n_2, \epsilon_2} = \{t_0, t_{1,2}, \dots, t_{n_2,2}, x_0, x_{1,2}, \dots, x_{n_2,2}\}$, then $P_{m, n_2, \epsilon_2} \geq P_{m, n_1, \epsilon_1}$.

Let $0 < \epsilon_{i+1} < \epsilon_i < C - A, \lim_{i \rightarrow \infty} \epsilon_i = 0$. Let $m \geq 1$ and $\{P_{m, n_1, \epsilon_1}\}_m$ be a partition corresponding to ρ_{ϵ_1} . From the above construction, extend this partition to $\{P_{m, n_2, \epsilon_2}\}_m$ to ρ_{ϵ_2} such that $P_{m, n_2, \epsilon_2} \geq P_{m, n_1, \epsilon_1}$. By induction there exist partitions $\{P_{m, n_j, \epsilon_j}\}_m$ of ρ_{ϵ_j} such that

$$P_{m, n_j, \epsilon_j} \geq P_{m, n_{j-1}, \epsilon_{j-1}}. \quad (3.22)$$

Denote $P_{m, n_j, \epsilon_j} = \{t_0, t_{1,m,j}, \dots, t_{n_j,m,j}, x_0, x_{1,m,j}, \dots, x_{n_j,m,j}\}$. Since $\rho_{\epsilon_j} \leq \rho$ and hence

$$\left| \frac{x - A}{T - \rho_{\epsilon_j}(x)} \right| \leq \left| \frac{x - A}{T - \rho(x)} \right|,$$

and

$$\left| \frac{x_{k,m,j} - A}{T - t_{k,m,j}} \right| = \left| \frac{x_{k,m,j} - A}{T - \rho_{\epsilon_j}(x_{k,m,j})} \right| \leq \max_x \left| \frac{x - A}{T - \rho(x)} \right|. \quad (3.23)$$

Assume that ρ satisfies (2.7). Then from (3.23)

$$\max_{k \leq n_j} \left\{ \left| \frac{x_{k,m,j} - A}{T - t_{k,m,j}} \right| \right\} \leq \wedge. \quad (3.24)$$

For each m, j , let

$$\begin{aligned} u_{m,j}(x, t) &= u(x, t, P_{m,n_j,\epsilon_j}), \\ b_{m,j}(t) &= b(t, P_{m,n_j,\epsilon_j}), \end{aligned}$$

where u and b are given in (3.6) and (3.7) respectively. From (3.10), (3.11) and (3.24) we have for all m, j

$$\left| f'(b_{m,j}(t)) \right| \leq \wedge. \quad (3.25)$$

$$TV(f'(b_{m,j})) \leq \left(\frac{T - \delta}{T - \rho(\epsilon_j)} \right) \wedge + \frac{C - A}{T - \rho(\epsilon_j + A)}. \quad (3.26)$$

Let $j > k$, then from (3.21)

$$\int_{\delta}^T \left| f'(b_{m,j}(t)) - f'(b_{m,k}(t)) \right| dt \leq \wedge |\rho(\epsilon_j + A) - \rho(\epsilon_k + A)|. \quad (3.27)$$

Under the above notations we have

Proof of Lemma 2.1 Let ρ satisfies (2.7), then for $\rho(\epsilon_j + A) < T$ and from (3.25), (3.26), for each j , $\{f'(b_{m,j})\}_{m \in \mathbb{N}}$ is bounded in total variation norm. Therefore from super linearity of f , $\{b_{m,j}\}_{m \in \mathbb{N}}$ is uniformly bounded in L^∞ for all j, m . Hence from Helly's theorem and Cantors diagonalization, we can extract a subsequence still denoted by $\{b_{m,j}\}$ such that for every j , $f'(b_{m,j}) \rightarrow f'(b_j)$ as $m \rightarrow \infty$ in L^1 and for *a.e.* t . Since $(f')^{-1}$ exist and hence $b_{m,j} \rightarrow b_j$ *a.e.* t and by dominated convergence Theorem, $b_{m,j} \rightarrow b_j$ in L^1 . Let $\rho_{m,j}(x) = \rho(x, P_{m,n_j,\epsilon_j})$, then from (3.14) $\rho_{m,j}(x) \rightarrow \rho_{\epsilon_j}(x)$ uniformly. Since $f'(u_{m,j}(x, \delta)) = \frac{C-A}{T-\delta}$, hence by L^1_{loc} contraction, $u_{m,j}$ converges in L^1_{loc} and for *a.e.* (x, t) to a solution u_j of (1.2) with initial boundary condition

$$u_j(A, t) = b_j(t) \quad (3.28)$$

$$f'(u_j(x, \delta)) = \frac{C - A}{T - \delta}. \quad (3.29)$$

From (3.9), (3.12) and (3.14), for *a.e.* $x \in (A, C)$

$$f'(u_j(x, T)) = \frac{x - A}{T - \rho_{\epsilon_j}(x)}. \quad (3.30)$$

Letting $m \rightarrow \infty$ in (3.27) to obtain

$$\int_{\delta}^T \left| f'(b_j(t)) - f'(b_k(t)) \right| dt \leq \wedge |\rho(A + \epsilon_j) - \rho(A + \epsilon_k)|. \quad (3.31)$$

Since ρ is right continuous and hence $|\rho(A + \epsilon_j) - \rho(A + \epsilon_k)| \rightarrow 0$ as $j, k \rightarrow \infty$. Therefore from L^1_{loc} contractivity, there exist a subsequence still denoted by j such

that $u_j \rightarrow \tilde{u}_1$, a solution of (1.2), $b_j \rightarrow \tilde{b}_1$ in L^1_{loc} and *a.e.* Letting $j \rightarrow \infty$ in (3.28) to (3.30), then $(\tilde{u}_1, \tilde{b}_1)$ satisfies (2.7) to (2.10). From Rankine-Hugoniot condition across $a_n(t, \delta)$, $\tilde{u}_1$ satisfies (2.10). This proves (1). Similarly (2) follows and hence the Lemma.

REMARK 3.1 *Given ρ , we have exhibited a method to construct an initial data $\bar{u}_0$ and the solution u such that at $t = T$*

$$f'(u(x, T)) = \frac{x - \rho(x)}{T}. \quad (3.32)$$

This method is not unique. In fact we can construct infinitely many initial datas and all the solutions to these initial datas satisfy (3.32). Here we illustrate this method with an example.

EXAMPLE 3.1 *Let $T > 0$ and $x_1 < x_2$, $y_1 < y_2$. Define*

$$\rho(x) = \begin{cases} x - x_1 + Ty_1 & \text{if } x < x_1, \\ y_2 & \text{if } x_1 < x < x_2, \\ x - x_2 + Ty_2 & \text{if } x > x_2. \end{cases}$$

Let $f'(a_1) = \frac{x_1 - y_1}{T}$, $f'(b_1) = \frac{x_1 - y_2}{T}$, $f'(a_2) = \frac{x_2 - y_2}{T}$. By strict convexity, it follows that $b_1 < \min\{a_1, a_2\}$. Let $y_1 = \xi_1 < \xi_2 < \dots < \xi_n = y_2$ be a sequence and define $a_1 = c_1 < c_2 < \dots < c_n = b_1$ and $\{d_i\}$ by

$$f'(c_i) = \frac{x_1 - \xi_i}{T}, \quad f'(d_i) = \frac{f(c_{i+1}) - f(c_i)}{c_{i+1} - c_i}.$$

By strict convexity $c_i < d_i < c_{i+1}$. For $0 \leq t \leq T$, let

$$\begin{aligned} \alpha_i(t) &= x_1 + f'(c_i)(t - T) \\ s_i(t) &= x_1 + f'(d_i)(t - T) \\ \beta(t) &= x_2 + f'(a_2)(t - T), \end{aligned}$$

then $\alpha_i(t) < s_i(t) < \alpha_{i+1}(t) < \beta(t)$ for $1 \leq i \leq n - 1$, $t \in (0, T)$. Let $s_i = s_i(0) = x_i - Tf'(d_i)$, then $\xi_i < s_i < \xi_{i+1}$. Now define u and $\bar{u}_0$ by (see figure 6)

$$\bar{u}_0 = \begin{cases} c_1 = a_1 & \text{if } x < s_1, \\ c_i & \text{if } \xi_i < x < s_i, \\ c_{i+1} & \text{if } s_i < x < \xi_{i+1}, \\ c_n = a_2 & \text{if } x > s_{n-1}, \end{cases}$$

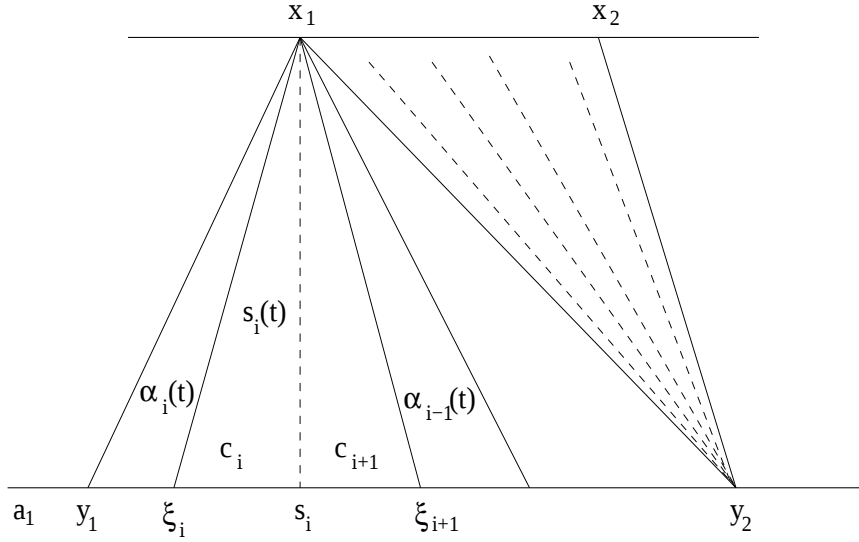


Figure 6:

then the solution u with initial data $\bar{u}_0$ in $(0, T)$ is given by

$$u(x, t) = \begin{cases} c_1 & \text{if } x < s_1, \\ c_i & \text{if } \alpha_i(t) < x < s_i(t) \\ c_{i+1} & \text{if } s_i(t) < x < s_{i+1}(t) \\ c_n & \text{if } s_{n-1}(t) < x < \alpha_n \\ (f')^{-1} \left(\frac{x-y_2}{t} \right) & \text{if } \alpha_n(t) < x < \beta(t) \\ a_2 & \text{if } x > \beta(t). \end{cases}$$

Clearly u satisfies (3.32).

Since $\{\xi_i\}$ are arbitrary and hence there exist infinitely many solutions satisfying (3.32). In the above example, $s_i(t)$ are shock curves. In fact one can also introduce the backward rarefaction in the region $\alpha_i(t) < x < \alpha_{i+1}(t)$ by

$$u(x, t) = (f')^{-1} \left(\frac{x - x_1}{t - T} \right) \quad \text{for } \alpha_i(t) < x < \alpha_{i+1}(t).$$

4 Finer Analysis of Characteristics

In a beautiful paper, Dafermos [11] had extensively studied the properties of characteristic curves. Here we make a finer analysis of these characteristics curves and then use them to obtain our results. In order to do this, first we recollect the results of

Lax-Olenik explicit formula and a good reference for this, is third chapter in [12].

Let $f^*(p) = \sup_q \{pq - f(q)\}$ denote the Legendre transform of f .

Then f^* is in C^1 , strictly convex, super linear growth and satisfies

$$\begin{aligned} f &= f^{**}, \\ f^{*'}(p) &= (f')^{-1}(p), \\ f^*(f'(p)) &= pf'(p) - f(p), \\ f(f^{*'}(p)) &= pf^{*'}(p) - f^*(p). \end{aligned} \tag{4.1}$$

Controlled Curves: Let $x \in \mathbb{R}$, $0 \leq s < t$ and define the controlled curves $\Gamma(x, s, t)$ by

$$\Gamma(x, s, t) = \{r : [s, t] \rightarrow \mathbb{R}; \text{ } r \text{ is linear and } r(t) = x\}, \tag{4.2}$$

and denote $\Gamma(x, t) = \Gamma(x, 0, t)$.

Value function: Let $u_0 \in L^\infty(\mathbb{R})$, $x_0 \in \mathbb{R}$, define

$$v_0(x) = \int_{x_0}^x u_0(\theta) d\theta, \tag{4.3}$$

be its primitive. Define the value function $v(x, t)$ by

$$\begin{aligned} v(x, t) &= \min_{r \in \Gamma(x, t)} \left\{ v_0(r(0)) + tf^* \left(\frac{x-r(0)}{t} \right) \right\} \\ &= \min_{\beta \in \mathbb{R}} \left\{ v_0(\beta) + tf^* \left(\frac{x-\beta}{t} \right) \right\}. \end{aligned} \tag{4.4}$$

Then v satisfies the

Dynamic Programming principle: For $0 \leq s < t$,

$$v(x, t) = \min_{r \in \Gamma(x, s, t)} \left\{ v(r(s), s) + (t-s)f^* \left(\frac{x-r(s)}{t-s} \right) \right\}. \tag{4.5}$$

Define the characteristic set $ch(x, s, t, u_0)$ and extreme characteristics $y_\pm(x, s, t, u_0)$ by

$$ch(x, s, t, u_0) = \{r \in \Gamma(x, s, t); \text{ } r \text{ is a minimizer in (4.5)}\}, \tag{4.6}$$

$$y_-(x, s, t, u_0) = \min\{r(s) : r \in ch(x, s, t, u_0)\}, \tag{4.7}$$

$$y_+(x, s, t, u_0) = \max\{r(s); r \in ch(x, s, t, u_0)\}, \tag{4.8}$$

Denote $ch(x, t, u_0) = ch(x, 0, t, u_0)$, $y_\pm(x, t, u_0) = y_\pm(x, 0, t, u_0)$. Then we have the following result due to Hopf, Lax -Oleinik:

THEOREM 4.1 *Let $0 \leq s < t, u_0, v_0, v$ be as above, then*

1. *v is a uniformly Lipschitz continuous function and is a unique viscosity solution of the Hamilton-Jacobi equation*

$$\begin{aligned} v_t + f(v_x) &= 0 & (x, t) \in \mathbb{R} \times (0, \infty), \\ v(x, 0) &= v_0(x) & x \in \mathbb{R}. \end{aligned} \quad (4.9)$$

2. *There exist $M > 0$, depending only on $\|u_0\|_\infty$ and Lipschitz constant of f restricted to the interval $[-\|u_0\|_\infty, \|u_0\|_\infty]$ such that for $(x, t) \in \mathbb{R} \times \mathbb{R}_+$, $ch(x, s, t, u_0) \neq \emptyset$ and for $r \in ch(x, s, t, u_0)$*

$$\left| \frac{x - r(s)}{t - s} \right| \leq M. \quad (4.10)$$

3. *NIP (Non intersecting property of characteristics): Let $x_1 \neq x_2, t_1 > 0, t_2 > 0$ and for $i = 1, 2, r_i \in ch(x_i, s, t_i, u_0)$. Then $r_1(\theta) \neq r_2(\theta)$ for all $\theta \in (s, \min\{t_1, t_2\})$.*

From NIP, it follows that for $0 \leq s < t$,

(a). *$x \mapsto y_\pm(x, s, t, u_0)$ are non decreasing functions,*

(b). *At the points of continuity of y_+ ,*

$$y_+(x, s, t, u_0) = y_-(x, s, t, u_0),$$

and hence $ch(x, s, t, u_0) = \{r\}$, where r is given by

$$r(\theta) = \frac{x - y_+(x, s, t, u_0)}{t - s}(\theta - t) + x.$$

(c). *Let $r \in ch(x, t, u_0)$, $z = r(s)$. Let $r_1(\theta) = r(\theta)$ for $0 \leq \theta \leq s, r_2(\theta) = r(\theta)$ for $s \leq \theta \leq t$. Then $r_1 \in ch(z, s, u_0), r_2 \in ch(x, s, t, u_0)$.*

4. *Let $u(x, t) = \frac{\partial v}{\partial x}(x, t)$. Then u is the unique solution of (1.2) in $\Omega = \mathbb{R} \times \mathbb{R}_+$ with initial data u_0 and satisfying*

$$|u(x, t)| \leq \|u_0\|_\infty. \quad (4.11)$$

For a.e x , $y_-(x, t) = y_+(x, t)$ and u is given by

$$f'(u(x, t)) = \frac{x - y_+(x, t, u_0)}{t} = \frac{x - y_-(x, t, u_0)}{t}. \quad (4.12)$$

Let x be a point of differentiability of $y_\pm(x, t, u_0)$ and $y_\pm(x, t, u_0)$ is a point of differentiability of v_0 , then

$$u(x, t) = u_0(y_\pm(x, t, u_0)). \quad (4.13)$$

5. Let $u_0, w_0 \in L^\infty(\mathbb{R})$ and u, w be the solutions given in (4) with initial data u_0, w_0 respectively. Then

(a). *Monotonicity:* Let $u_0(x) \leq w_0(x)$ for $x \in \mathbb{R}$, there exists a set $N \subset \mathbb{R}$ of measure zero such that for each $t \notin N$, for a.e $x \in \mathbb{R}$,

$$u(x, t) \leq w(x, t). \quad (4.14)$$

(b). *L^1_{loc} contractivity:* Let $c = \max(\|u_0\|_\infty, \|w_0\|_\infty)$ and $I = [-c, c]$. Then there exist a $M > 0$, depending on Lipschitz constant f restricted to I such that for all $t > 0$, $a < b$,

$$\int_a^b |u(x, t) - w(x, t)| dx \leq \int_{a-Mt}^{b+Mt} |u_0(x) - w_0(x)| dx. \quad (4.15)$$

For the proofs of (1) to (4) see chapter (3) of [12] and for (5), see chapter (3) of [16].

In this sequel we follow the notations of characteristic curves as in [5]. From now onwards, we assume that $\Omega = \mathbb{R} \times (0, \infty)$, $u_0 \in L^\infty(\mathbb{R})$.

Left and right characteristic curves: Let $0 \leq s < t$, u be a solution of (1.2) with initial data u_0 and $\alpha \in \mathbb{R}$. Define the left characteristic curve $R_-(t, s, \alpha, u_0)$ and right characteristic curve $R_+(t, s, \alpha, u_0)$ and denote $R_\pm(t, \alpha, u_0) = R_\pm(t, 0, \alpha, u_0)$ by

$$R_-(t, s, \alpha, u_0) = \inf\{x; \alpha \leq y_-(x, s, t, u_0)\}, \quad (4.16)$$

$$R_+(t, s, \alpha, u_0) = \sup\{x : y_+(x, s, t, u_0) \leq \alpha\}. \quad (4.17)$$

In view of (4.10), $y_-(x, s, t, u_0) \rightarrow -\infty$ as $x \rightarrow -\infty$, $y_+(x, s, t, u_0) \rightarrow +\infty$ as $x \rightarrow +\infty$. Hence (4.16) and (4.17) are well defined. Our aim is to study the continuous dependence of $R_\pm$ on their arguments (t, α, u_0) .

For $x, y \in \mathbb{R}, t > 0$, let $r(\theta, t, x, y) \in \Gamma(x, t)$ be the line joining $(x, t), (y, 0)$ given by

$$r(\theta, t, x, y) = \left(\frac{x - y}{t} \right) (\theta - t) + x. \quad (4.18)$$

Observe that $r(0, t, x, y) = y$ and hence $r \in ch(x, t, u_0)$ if and only if y is a minimizer in (4.4). Hence define the extreme characteristic lines by

$$r_\pm(\theta, t, x) = r(\theta, t, y_\pm(x, t, u_0)). \quad (4.19)$$

Since $r_\pm(0, t, x) = y_\pm(x, t, u_0)$ and $y_-(x, t, u_0) \leq y_+(x, t, u_0)$, hence for all $\theta \in [0, t]$,

$$r_-(\theta, t, x) \leq r_+(\theta, t, x). \quad (4.20)$$

Then we have the following

LEMMA 4.1 *Let $u_0, w_0, \{u_0^k\}$ are in $L^\infty(\mathbb{R})$ and $\alpha, \{\alpha_k\}$ are in $\mathbb{R}$. Let $v, W, \{v_k\}$ be the value functions defined in (4.4) with respect to the data $u_0, w_0, \{u_0^k\}$ respectively. Let $u = \frac{\partial v}{\partial x}, w = \frac{\partial W}{\partial x}, u_k = \frac{\partial v_k}{\partial x}$ be the solutions of (1.2). Then*

1. *Let $x_1 < x_2, 0 \leq s < t$ and $\beta \in \mathbb{R}$ be a minimizer for $v(x_1, t)$ and $v(x_2, t)$ in (4.5). Then for $x_1 < x < x_2$, β is the unique minimizer for $v(x, t)$ and satisfies*

$$f'(u(x, t)) = \frac{x - \beta}{t - s}. \quad (4.21)$$

2. *Let $x_k \in \mathbb{R}, r_k \in \text{ch}(x_k, t, u_0)$ such that $\lim_{k \rightarrow \infty} (x_k, r_k(0)) = (x, \beta)$. Then $r(\cdot, t, x, \beta) \in \text{ch}(x, t, u_0)$. Furthermore*

$$\lim_{x_k \uparrow x} y_+(x_k, t, u_0) = y_-(x, t, u_0), \quad (4.22)$$

$$\lim_{x_k \downarrow x} y_-(x_k, t, u_0) = y_+(x, t, u_0). \quad (4.23)$$

In particular, y_- is left continuous and y_+ is right continuous.

3. (i). *For all $t > 0$,*

$$R_-(t, \alpha, u_0) \leq R_+(t, \alpha, u_0), \quad (4.24)$$

$$\begin{cases} y_-(R_-(t, \alpha, u_0), t, u_0) \leq \alpha \leq y_+(R_-(t, \alpha, u_0), t, u_0), \\ y_-(R_+(t, \alpha, u_0), t, u_0) \leq \alpha \leq y_+(R_+(t, \alpha, u_0), t, u_0). \end{cases} \quad (4.25)$$

Further more if $R_-(t, \alpha, u_0) < R_+(t, \alpha, u_0)$, then for all $x \in (R_-(t, \alpha, u_0), R_+(t, \alpha, u_0))$ (see Figure 7)

$$y_\pm(x, t, \alpha) = \alpha, f'(u(x, t)) = \frac{x - \alpha}{t}. \quad (4.26)$$

- (ii). *Let $0 < s < t$, then*

$$R_-(t, s, \alpha, u_0) = R_+(t, s, \alpha, u_0). \quad (4.27)$$

4. *Let $0 \leq s < t$. Then $t \mapsto R_\pm(t, \alpha, u_0)$ are Lipschitz continuous function with Lipschitz norm depends only on α and $\|u_0\|_\infty$ and satisfying*

$$\lim_{t \rightarrow 0} R_\pm(t, \alpha, u_0) = \alpha, \quad (4.28)$$

$$R_\pm(t, \alpha, u_0) = R_\pm(t, s, R_\pm(s, \alpha, u_0), u_0). \quad (4.29)$$

5. *Monotonicity: Let $u_0 \leq w_0, \alpha \leq \beta$, then*

$$R_\pm(t, \alpha, u_0) \leq R_\pm(t, \alpha, w_0), \quad (4.30)$$

$$R_\pm(t, \alpha, u_0) \leq R_\pm(t, \beta, u_0). \quad (4.31)$$

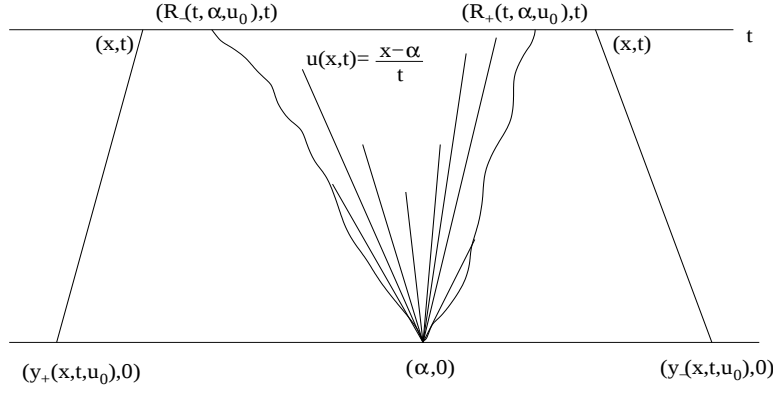


Figure 7:

6. *Continuity with respect to data:* Let $\{u_0^k\}$ be bounded in $L^\infty(\mathbb{R})$. Let $\alpha_k \rightarrow \alpha, u_0^k \rightarrow u_0$ in $L^1_{loc}(\mathbb{R})$. Then for $t > 0$.

(a). Suppose for all $k, R_-(t, \alpha_k, u_0^k) \leq R_-(t, \alpha, u_0)$, then

$$\lim_{k \rightarrow \infty} R_-(t, \alpha_k, u_0^k) = R_-(t, \alpha, u_0). \quad (4.32)$$

(b). Suppose for all $k, R_+(t, \alpha_k, u_0^k) \geq R_+(t, \alpha, u_0)$, then

$$\lim_{k \rightarrow \infty} R_+(t, \alpha_k, u_0^k) = R_+(t, \alpha, u_0). \quad (4.33)$$

(c). Suppose $R_-(t, \alpha, u_0) < \bar{R} = \lim_{k \rightarrow \infty} R_-(t, \alpha_k, u_0^k)$, then for all $x \in (R_-(t, \alpha, u_0), \bar{R})$, $y_\pm(x, t, u_0) = \alpha$ and

$$f'(u(x, t)) = \frac{x - \alpha}{t}. \quad (4.34)$$

(d). Suppose $\lim_{k \rightarrow \infty} R_+(t, \alpha_k, u_0^k) = \bar{R} < R_+(t, \alpha, u_0)$, then for all $x \in (\bar{R}, R_+(t, \alpha, u_0))$, $y_\pm(x, t, u_0) = \alpha$ and

$$f'(u(x, t)) = \frac{x - \alpha}{t}. \quad (4.35)$$

As an immediate consequence of this, if $R_-(t, \alpha, u_0) = R_+(t, \alpha, u_0)$ for $t > 0$, then $R_\pm(t, \alpha, u_0)$ is continuous at (α, u_0) .

Proof. (1). Let $x \in (x_1, x_2)$ and $r \in ch(x, s, t, u_0)$. Suppose $r(s) \neq \beta$. then r intersects one of the characteristics $(\frac{x_i - \beta}{t - s})(\theta - t) + x_i$, $i = 1, 2$, which contradicts NIP of Theorem 2.1 Hence $\beta = r(s) = y_\pm(x, s, t, u_0)$. Furthermore

$$v(x, t) = v(\beta, s) + (t - s)f^*\left(\frac{x - \beta}{t - s}\right),$$

and for a.e x ,

$$u(x, t) = \frac{\partial v}{\partial x} = f^{*'} \left(\frac{x - \beta}{t - s} \right) = (f')^{-1} \left(\frac{x - \beta}{t - s} \right).$$

This proves (1).

(2). From the continuity of v and f^* , we have

$$\begin{aligned} v(x, t) &= \lim_{k \rightarrow \infty} v(x_k, t) \\ &= \lim_{k \rightarrow \infty} \left\{ v_0(r_k(0)) + t f^* \left(\frac{x_k - r_k(0)}{t} \right) \right\} \\ &= v_0(\beta) + t f^* \left(\frac{x - \beta}{t} \right), \end{aligned}$$

and hence $r(\cdot, t, x, \beta) \in ch(x, t, u_0)$. Let $x_1 < x_2$, then from NIP, $y_+(x_1, t, u_0) \leq y_-(x_2, t, u_0)$. From monotonicity of $y_{\pm}$, we have

$$y_-(x_1, t, u_0) \leq y_+(x_1, t, u_0) \leq y_-(x_2, t, u_0) \leq y_+(x_2, t, u_0).$$

Let $x_k \uparrow x$, then from above inequality,

$$\beta = \lim_{k \rightarrow \infty} y_+(x_k, t, u_0) \leq y_-(x, t, u_0).$$

Since a subsequence of $y_+(x_k, t, u_0)$ converges to β , hence $r(\cdot, t, x, \beta) \in ch(x, t, u_0)$. Therefore $\beta \leq y_-(x, t, u_0) \leq r(0, t, x, \beta) = \beta$. This proves (4.22). Similarly (4.23) follows. This proves (2).

(3). (i). Suppose $y_-(R_-(t, \alpha, u_0), t, u_0) > \alpha$. Then from (4.22) there exist $x_0 < R_-(t, \alpha, u_0)$ such that for all $x \in (x_0, R_-(t, \alpha, u_0))$, $y_+(x, t, u_0) > \alpha$. Let x be a point of continuity of y_+ , then from (3) of theorem 2.1, $y_-(x, t, u_0) = y_+(x, t, u_0) > \alpha$ and hence $R_-(t, \alpha, u_0) \leq x < R_-(t, \alpha, u_0)$ which is a contradiction. Suppose $y_+(R_-(t, \alpha, u_0), t, u_0) < \alpha$, again from (4.23) there exist $x_0 > R_-(t, \alpha, u_0)$ such that for all $x \in (R_-(t, \alpha, u_0), x_0)$, $y_-(x, t, u_0) < \alpha$. Therefore at points x of continuity, $\alpha \leq y_+(x, t, u_0) = y_-(x, t, u_0) < \alpha$, which is a contradiction. This proves (4.25) and (4.26) follows similarly.

Suppose $R_+(t, \alpha, u_0) < R_-(t, \alpha, u_0)$, then from (4.25), $y_-(R_-(t, \alpha, u_0), \alpha, u_0) \leq \alpha \leq y_+(R_+(t, \alpha, u_0), t, u_0)$, therefore from NIP, $y_-(R_-(t, \alpha, u_0), t, u_0) = \alpha = y_+(R_+(t, \alpha, u_0), t, u_0)$. Hence from (4.21), for all $x \in (R_+(t, \alpha, u_0), R_-(t, \alpha, u_0))$, α is a minimizer for $v(x, t)$ which implies that $R_-(t, \alpha, u_0) \leq x < R_-(t, \alpha, u_0)$ which is a contradiction. This proves (4.24).

Suppose $R_-(t, \alpha, u_0) < R_+(t, \alpha, u_0)$. then from (4.24), (4.25), we have

$$\alpha \leq y_+(R_-(t, \alpha, u_0), t, u_0) \leq y_-(R_+(t, \alpha, u_0), t, u_0) \leq \alpha.$$

Therefore from (1), for all $x \in (R_-(t, \alpha, u_0), R_+(t, \alpha, u_0))$, $y_{\pm}(x, t, u_0) = \alpha$ and $f'(u(x, t)) = \frac{x - \alpha}{t}$. This proves (4.26).

(3). (ii). Let $0 < s < t$, then as in (4.24) we have $R_-(t, s, \alpha, u_0) \leq R_+(t, s, \alpha, u_0)$. Suppose $R_-(t, s, \alpha, u_0) < R_+(t, s, \alpha, u_0)$, then as in (4.26), we have for all $x \in (R_-(t, s, \alpha, u_0), R_+(t, s, \alpha, u_0))$, $f'(u(x, t)) = \frac{x-\alpha}{t-s}$. Let $R_-(t, s, \alpha, u_0) < x_1 < x_2 < R_+(t, s, \alpha, u_0)$ and $r_\pm(\cdot, t, x_1), r_\pm(\cdot, t, x_2)$ be the extreme characteristics at x_1, x_2 . Since $r_\pm(s, t, x_1) = r_\pm(s, t, x_2) = \alpha$, which contradicts NIP. This proves (ii) and hence (3).

(4). Let $0 \leq s < t, R_- = R_-(t, \alpha, u_0), y_\pm = y_\pm(R_-, t, u_0)$ and $r_\pm(\theta) = r(\theta, t, R_-, y_\pm) \in ch(R_-, t, u_0)$. Then from (3) of theorem 2.1, $r_\pm|_{(0,s)} \in ch(r_\pm(s), s, u_0)$.

Claim : $r_-(s) \leq R_-(s, \alpha, u_0) \leq r_+(s)$.

Suppose $R_-(s, \alpha, u_0) < r_-(s)$. For $x \in (R_-(s, \alpha, u_0), r_-(s))$, $y_-(x, s, \alpha) \geq \alpha$. Hence if $y_- < \alpha$ or $y_-(x, s, \alpha) > \alpha$, then the characteristics $r_-(\theta), r_-(\theta, s, x)$ intersect for some $\theta \in (0, s)$ which contradicts NIP. Therefore $\alpha = y_- = y_-(x, s, \alpha)$ and from (2) $\tilde{r}(\theta) = \tilde{r}(\theta, s, R_-(s, \alpha, u_0), \alpha) \in ch(R_-(s, \alpha, u_0), s, u_0)$. From (4.22) choose a $\xi < R_-$, $y_-(\xi, t, u_0) < \alpha$ such that the characteristic $\tilde{r}(\theta)$ and $r(\theta, t, \xi, y_+(\xi, t, u_0))$ intersect for some $\theta \in (0, s)$ which contradicts NIP.

Suppose $r_+(s) < R_-(s, \alpha, u_0)$, then for $x \in (r_+(s), R_-(s, \alpha, u_0))$, $y_-(x, s, u_0) < \alpha \leq r_+(0) = y_+$ and therefore the characteristic at (x, s) with end point $(y_-(x, s, u_0), 0)$ intersects $r_+(\theta)$ for some $\theta \in (0, s)$ contradicting NIP. This proves the claim.

From (4.10) and the claim, we have

$$R_- + \left(\frac{R_- - y_-}{t} \right) (s - t) \leq R_-(s, \alpha, u_0) \leq R_- + \left(\frac{R_- - y_+}{t} \right) (s - t)$$

that is

$$\begin{aligned} |R_- - R_-(s, \alpha, u_0)| &\leq \left(\left| \frac{R_- - y_-}{t} \right| + \left| \frac{R_- - y_+}{t} \right| \right) |s - t| \\ &\leq 2M|s - t|. \end{aligned}$$

Also from (4.10), we have $|R_- - y_\pm| = |R_- - r_\pm(0)| \leq Mt$, hence $\lim_{t \rightarrow 0} R_-(t, \alpha, u_0) = \alpha$. Similarly for $R_+(t, \alpha, u_0)$.

From (c) of (3) in Theorem 2.1, we have $r_\pm|_{[s,t]} \in ch(R_-(t, \alpha, u_0), s, t, u_0)$, hence from NIP and from the above claim we have for any $x < R_-(t, \alpha, u_0) < z$, $y_+(x, s, t, u_0) < r_-(s) \leq R_-(s, \alpha, u_0) \leq r_+(s) < y_-(z, s, t, u_0)$. Therefore from the definitions it follows that $R_-(t, \alpha, u_0) = R_-(t, s, R_-(t, s, u_0), u_0)$. Similarly for R_+ and this proves (4).

(5). From (5) of Theorem 2.1, for $t \in N$, a.e. x , $u(x, t) \leq w(x, t)$. Let $y_{1,\pm}(x) = y_\pm(x, t, u_0)$, $y_{2,\pm}(x) = y_\pm(x, t, w_0)$. Choose a dense set $D \subset \mathbb{R}$ such that for $i = 1, 2, x \in D, u(x, t) \leq w(x, t), y_{i,+}(x) = y_{i,-}(x)$. Hence from (4.12) we have for $x \in D$,

$$\frac{x - y_{1,\pm}(x)}{t} = f'(u(x, t)) \leq f'(w(x, t)) = \frac{x - y_{2,\pm}(x)}{t}.$$

This implies $y_{2,\pm}(x) \leq y_{1,\pm}(x)$. Therefore from (4.22) and (4.23),

$$\begin{aligned} R_-(t, \alpha, u_0) &= \inf\{x \in D : y_{1,-}(x) \geq \alpha\} \\ &\leq \inf\{x \in D : y_{2,-}(x) \geq \alpha\} \\ &= R_-(t, \alpha, w_0). \end{aligned}$$

$$\begin{aligned} R_+(t, \alpha, u_0) &= \sup\{x \in D, y_{1,+}(x) \leq \alpha\} \\ &\leq \sup\{x \in D : y_{2,+}(x) \leq \alpha\} \\ &= R_+(t, \alpha, w_0). \end{aligned}$$

From (4), $t \mapsto (R_\pm(t, \alpha, u_0), R_\pm(t, \alpha, w_0))$ are continuous and hence (4.30) holds for all $t > 0$.

$$\begin{aligned} R_-(t, \alpha, u_0) &= \inf\{x : y_-(x, t, u_0) \geq \alpha\} \\ &\leq \inf\{x : y_-(x, t, u_0) \geq \beta\} \\ &= R_-(t, \beta, u_0), \end{aligned}$$

and similarly for R_+ . This proves (5).

(6). From L_{loc}^1 contractivity, $u_k \rightarrow u$ in L_{loc}^1 and hence for a.e. s , $u_k(\cdot, s) \rightarrow u(\cdot, s)$ in L_{loc}^1 . Let $t > 0$ be such that for a subsequence still denoted by k such that for a.e. x

$$\lim_{k \rightarrow \infty} u_k(x, t) = u(x, t). \quad (4.36)$$

Let $y_\pm^k(x) = y_\pm(x, t, u_0^k)$, $R_\pm^k = R_\pm(t, \alpha_k, u_0^k)$. Since $\{y_\pm^k\}$ are monotone functions and $\{R_\pm^k\}$ are bounded. Hence from Helly's theorem, there exist a subsequence still denoted by k such that for a.e. x ,

$$\lim_{k \rightarrow \infty} y_\pm^k(x) = y_\pm(x) \quad (4.37)$$

$$\left(\lim_{k \rightarrow \infty} R_\pm^k, \lim_{k \rightarrow \infty} R_\pm^k \right) = \left(\bar{R}_\pm, \tilde{R}_\pm \right), \quad (4.38)$$

where u is the solution of (1.2) with $u(x, 0) = u_0(x)$. Let $D \subset \mathbb{R}$ be a dense set such that for all $x \in D$, (4.36) to (4.38) holds and further for all k ,

$$y_+^k(x) = y_-^k(x) \quad (4.39)$$

$$y_+(x, t, u_0) = y_-(x, t, u_0) \quad (4.40)$$

$$f'(u_k(x, t)) = \frac{x - y_\pm^k(x)}{t} \quad (4.41)$$

$$f'(u(x, t)) = \frac{x - y_\pm(x, t, u_0)}{t}. \quad (4.42)$$

Hence from (4.37), (4.41) and (4.42), for $x \in D$,

$$y_\pm(x) = \lim_{k \rightarrow \infty} y_\pm^k(x) = y_\pm(x, t, u_0). \quad (4.43)$$

Case (i): Let for all k , $R_-^k \leq R_-(t, \alpha, u_0)$, then $\bar{R}_- \leq R_-(t, \alpha, u_0)$. Suppose $\bar{R}_- < R_-(t, \alpha, u_0)$. Let $I = (\bar{R}_-, R_-(t, \alpha, u_0))$, $x \in D \cap I$ and choose $k_0 = k_0(x) > 0$ such that for all $k \geq k_0$, $R_-^k < x$, then

$$\alpha = \lim_{k \rightarrow \infty} \alpha_k \leq \lim_{k \rightarrow \infty} y_-^k(x) = y_-(x, t, u_0) < \alpha,$$

which is a contradiction. Hence $\bar{R}_- = R_-(t, \alpha, u_0)$.

Case (ii) : Let for all k , $R_-(t, \alpha, u_0) \leq R_-^k$, then $R_-(t, \alpha, u_0) \leq \tilde{R}_-$. Suppose $R_-(t, \alpha, u_0) < \tilde{R}_-$, then for $x \in D \cap (R_-(t, \alpha, u_0), \tilde{R}_-)$ choose $k_0 = k_0(x)$ such that for a subsequence $k > k_0$, $x < R_-^k$. Hence $\alpha \leq y_-(x, t, u_0) = \lim_{k \rightarrow \infty} y_-^k(x) \leq \alpha$ and therefore $y_-(x, t, u_0) = \alpha$. Therefore from (4.12), $f'(u(x, t)) = \frac{x-\alpha}{t}$.

Since $\{u_0^k\}$ are bounded in L^∞ and hence from (4), there exists a $C > 0$ independent of k such that for all s_1, s_2 we have

$$\begin{aligned} |R_\pm(s_1, \alpha_k, u_0^k) - R_\pm(s_2, \alpha_k, u_0^k)| &\leq |s_1 - s_2| \\ |R_\pm(s_1, \alpha, u_0) - R_\pm(s_2, \alpha, u_0)| &\leq |s_1 - s_2| \end{aligned}$$

Now suppose for $t > 0$ and for a subsequence still denoted by k such that

$$R_- = \lim_{k \rightarrow \infty} R_-(t, \alpha_k, u_0^k) < R_-(t, \alpha, u_0).$$

Therefore choose $\epsilon > 0, k_0 > 0$ such that for all $k \geq k_0$

$$R_-(t, \alpha_k, u_0^k) < R_-(t, \alpha, u_0) - 2\epsilon.$$

Let $|s - t| \leq \frac{\epsilon}{2C}$, then from the above uniform estimates we have for $k \geq k_0$

$$\begin{aligned} R_-(s, \alpha_k, u_0^k) &\leq \frac{\epsilon}{2} + R_-(t, \alpha_k, u_0^k) \\ &\leq \frac{\epsilon}{2} + R_-(t, \alpha, u_0) - 2\epsilon \\ &\leq \frac{\epsilon}{2} + R_-(s, \alpha_k, u_0^k) + \frac{\epsilon}{2} - 2\epsilon \\ &\leq R_-(s, \alpha, u_0) - \epsilon < R_-(s, \alpha, u_0). \end{aligned}$$

Now choose an $|s_0 - t| < \frac{\epsilon}{2}$ such that the previous analysis holds. Then at s_0 , we have

$$R_-(s, \alpha, u_0) - \epsilon \geq \lim_{k \rightarrow \infty} R_-(s, \alpha_k, u_0^k) = R_-(s, \alpha, u_0) < R_-$$

which is a contradiction. This proves (2.32) and similarly (2.33) holds.

Let $R_-(t, \alpha, u_0) < \bar{R} = \overline{\lim_{k \rightarrow \infty}} R_-(t, \alpha_k, u_0^k)$ and $R_-(t, \alpha, u_0) < x < \bar{R}$. Then as earlier choose an $\epsilon > 0$, a subsequence still denoted by k such that for $|s - t| < \frac{\epsilon}{2C}$ and $k \geq k_0(\epsilon)$, following holds :

$$R_-(s, \alpha, u_0) + \epsilon < x < \bar{R} - \epsilon \leq R_-(s, \alpha_k, u_0^k).$$

Now choose an $s > t$ such that $u_k(\xi, s) \rightarrow u(\xi, s)$ a.e. ξ . Hence from the previous analysis we have for all $\xi \in (R_-(s, \alpha, u_0), \bar{R} - \epsilon)$, $f'(u(\xi, s)) = \frac{\xi - \alpha}{s}$. Since $s > t$ and hence we have $f'(u(x, t)) = \frac{x - \alpha}{t}$. This proves (2.34) and similarly (2.35) follows. This proves (6) and hence the Lemma.

Next we study the characterization of $R_\pm$ and some comparison properties. For this we need some well known results which will be proved in the following Lemma.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $B(1)$ denote the unit ball in $\mathbb{R}^n$. Let $0 \leq \chi \in C_c^\infty(B(1))$ with $\int_{\mathbb{R}^n} \chi(x) dx = 1$. Let $\varepsilon > 0$ and denote $\chi_\varepsilon(x) = \frac{1}{\varepsilon^n} \chi(\frac{x}{\varepsilon})$ be the usual mollifiers. Let $u_0 \in L^1_{loc}(\mathbb{R}^n)$ and define

$$u_0^\varepsilon(x) = (\chi_\varepsilon * u_0)(x) = \int_{B(1)} \chi(y) u_0(x - \varepsilon y) dy ,$$

then

LEMMA 4.2 *Denote ess inf and ess sup by inf and sup. Then*

1. *With the above notation, for $x \in \Omega$, there exists a $\varepsilon_0 = \varepsilon_0(x) > 0$ such that for all $0 < \varepsilon < \varepsilon_0$,*

$$\inf_{y \in \Omega} u_0(y) \leq u_0^\varepsilon(x) \leq \sup_{y \in \Omega} u_0(y). \quad (4.44)$$

2. *Let $t_0, \varepsilon_0, \alpha \in \mathbb{R}$ and $\omega \in L^\infty((0, t_0))$. Let $R : (0, t_0] \rightarrow \mathbb{R}$ be a locally Lipschitz continuous function such that for a.e $t \in (0, t_0)$,*

$$\omega(t) \geq (f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) + \varepsilon_0 \quad (4.45)$$

$$\frac{dR}{dt} = \frac{f(\omega(t)) - f((f')^{-1}(\frac{R(t) - \alpha}{t}))}{\omega(t) - (f')^{-1}(\frac{R(t) - \alpha}{t})}, \quad (4.46)$$

then

$$\overline{\lim}_{t \rightarrow 0} \left| \frac{R(t) - \alpha}{t} \right| = \infty. \quad (4.47)$$

Proof. (1). Let $\Omega_\varepsilon = \{x; d(x, \Omega^c) > \varepsilon\}$. Then for $x \in \Omega$, there exists an $\varepsilon_0 > 0$, such that $x \in \Omega_\varepsilon$, for all $\varepsilon < \varepsilon_0$. Hence $x - \varepsilon y \in \Omega$, for a.e $y \in B(1)$ a.e

$$\inf_{\xi \in \Omega} u_0(\xi) \leq u_0(x - \varepsilon y) \leq \sup_{\xi \in \Omega} u_0(\xi).$$

Multiply this identity by χ and integrate over $B(1)$ gives (4.44).

(2). Suppose (4.47) is not true. That is

$$\sup_{t>0} \left| \frac{R(t) - \alpha}{t} \right| < \infty. \quad (4.48)$$

Let m be defined by

$$\begin{aligned} m &= \inf_{t \in (0, t_0)} \int_0^1 f' \left((f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) + \theta \left(w(t) - (f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) \right) \right) \\ &\quad - \left(\frac{R(t) - \alpha}{t} \right) d\theta. \end{aligned} \quad (4.49)$$

Claim : $m > 0$.

From (4.45), $w(t) - (f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) > 0$ and hence by convexity we have

$$\begin{aligned} f' \left((f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) + \theta \left(w(t) - (f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) \right) \right) &\geq f' \left((f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) \right) \\ &= \frac{R(t) - \alpha}{t}. \end{aligned}$$

Hence $m \geq 0$. Suppose $m = 0$, then there exists a sequence $t_k \rightarrow \tilde{t}$ in $[0, 1]$ such that

$$\begin{aligned} 0 &= \lim_{k \rightarrow \infty} \int_0^1 f' \left((f')^{-1} \left(\frac{R(t_k) - \alpha}{t_k} \right) + \theta \left(w(t_k) - (f')^{-1} \left(\frac{R(t_k) - \alpha}{t_k} \right) \right) \right) \\ &\quad - \left(\frac{R(t_k) - \alpha}{t_k} \right) d\theta. \end{aligned}$$

Then from (4.48), we can choose a subsequence such that

$$\frac{R(t_k) - \alpha}{t_k} \rightarrow a, \quad w(t_k) \rightarrow b \quad \text{as } k \rightarrow \infty.$$

Then from (4.45) we have $b \geq (f')^{-1}(a) + \epsilon_0$ and

$$0 = \int_0^1 [f'((f')^{-1}(a) + \theta(b - (f')^{-1}(a))) - a] d\theta$$

and hence by strict convexity

$$0 < f'((f')^{-1}(a) + \theta(b - (f')^{-1}(a))) - a = 0$$

which is a contradiction. This proves the claim. From Taylor series and the claim we have

$$\begin{aligned} \frac{dR}{dt} &= \frac{R(t) - \alpha}{t} + \int_0^1 f' \left((f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) + \theta \left(w(t) - (f')^{-1} \left(\frac{R(t) - \alpha}{t} \right) \right) \right) \\ &\quad - \left(\frac{R(t) - \alpha}{t} \right) d\theta. \\ &\geq \frac{R(t) - \alpha}{t} + m\epsilon_0 \end{aligned}$$

or

$$t \frac{d}{dt} \left(\frac{R(t) - \alpha}{t} \right) \geq m\epsilon_0.$$

For $0 < t_1 < t_0$, integrating t to t_1 to obtain

$$\frac{R(t) - \alpha}{t} \leq \frac{R(t_1) - \alpha}{t} - m\epsilon_0 \log \frac{t_1}{t} \rightarrow -\infty \text{ as } t \rightarrow 0.$$

LEMMA 4.3 *Let $T > 0, \alpha, \beta \in \mathbb{R}, u_0, v_0$ and v be as in (4.3) and (4.4). Then*

(1). *Let $x_0 \in \mathbb{R}, t > 0$ such that*

$$y_-(x_0, t, u_0) \leq \alpha \leq y_+(x_0, t, u_0), \quad (4.50)$$

then

(i). *if $x_0 \leq R_-(t, \alpha, u_0)$, then $x_0 = R_-(t, \alpha, u_0)$. If $R_-(t, \alpha, u_0) < x_0$, then for all $x \in (R_-(t, \alpha, u_0), x_0)$, $f'(u(x, t)) = \frac{x - \alpha}{t}$.*

(ii). *if $x_0 \geq R_+(t, \alpha, u_0)$, then $x_0 = R_+(t, \alpha, u_0)$. If $x_0 < R_+(t, \alpha, u_0)$, then for all $x \in (x_0, R_+(t, \alpha, u_0))$, $f'(u(x, t)) = \frac{x - \alpha}{t}$.*

(2). (i). *Let $x \geq R_-(t, \alpha, u_0)$, then*

$$v(x, t) = \inf_{y \geq \alpha} \left\{ v_0(y) + t f^* \left(\frac{x - y}{t} \right) \right\}. \quad (4.51)$$

(ii). *Let $x \leq R_+(t, \alpha, u_0)$, then*

$$v(x, t) = \inf_{y \leq \alpha} \left\{ v_0(y) + t f^* \left(\frac{x - y}{t} \right) \right\}. \quad (4.52)$$

(iii). *Let $\alpha < \beta$ and for $0 < t < T$ assume that*

$$R_+(t, \alpha, u_0) < R_-(t, \beta, u_0),$$

then for $R_+(t, \alpha, u_0) < x < R_-(t, \beta, u_0)$,

$$v(x, t) = \inf_{\alpha \leq y \leq \beta} \left\{ v_0(y) + t f^* \left(\frac{x - y}{t} \right) \right\}, \quad (4.53)$$

$$m = \inf_{y \in [\alpha, \beta]} u_0(y) \leq u(x, t) \leq \sup_{y \in [\alpha, \beta]} u_0(y) = M. \quad (4.54)$$

$$f'(m) \leq \frac{x - y_+(x, t, u_0)}{t} \leq f'(M). \quad (4.55)$$

(3). Let $L(t, \alpha, u_0) \in \{R_{\pm}(t, \alpha, u_0)\}$, $R(t, \beta, u_0) \in \{R_{\pm}(t, \beta, u_0)\}$. Suppose at $t = T$,

$$L(T, \alpha, u_0) = R(T, \beta, u_0), \quad (4.56)$$

then for all $t \geq T$, (see Figure 9).

$$L(t, \alpha, u_0) = R(t, \beta, u_0). \quad (4.57)$$

Furthermore, let $\{u_0^k\}$ and u_0 are in $L^\infty(\mathbb{R})$ with $\sup_k \|u_0^k\|_\infty < \infty$. Let $(\alpha_k, \beta_k, u_0^k) \rightarrow (\alpha, \beta, u_0)$ as $k \rightarrow \infty$ in $\mathbb{R}^2 \times L^1_{loc}(\mathbb{R})$ and $T_k \rightarrow T$ in $\mathbb{R}$ such that

$$\begin{aligned} R_-(T, \alpha, u_0) &= R_+(T, \beta, u_0) \\ R_-(T_k, \alpha_k, u_0^k) &= R_+(T_k, \beta_k, u_0^k). \end{aligned} \quad (4.58)$$

Then for $t > T$,

$$\begin{aligned} \lim_{k \rightarrow \infty} R_+(t, \alpha_k, u_0^k) &= \lim_{k \rightarrow \infty} R_-(t, \beta_k, u_0^k) \\ &= R_+(t, \alpha, u_0) \\ &= R_-(t, \alpha, u_0). \end{aligned} \quad (4.59)$$

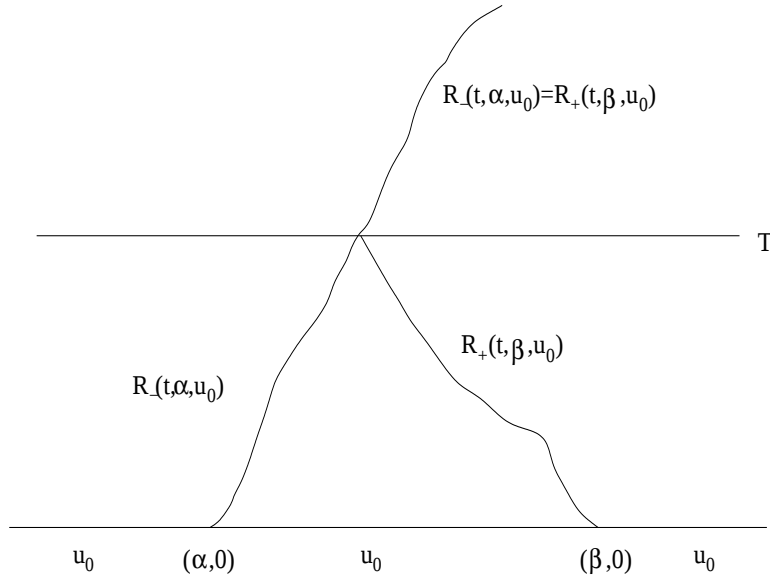


Figure 8:

Proof. (1). It is enough to prove (i) and (ii) follows similarly. Let $C = R_-(t, \alpha, u_0)$, then from (4.22) $y_-(C, t, u_0) \leq \alpha$. Suppose $x_0 < C$, then from (4.50), the characteristic line joining $(C, t), (y_-(C, t, u_0), 0)$ and $(x_0, t), (y_+(x_0, t, u_0), 0)$ intersect if

$y_+(x_0, t, u_0) > \alpha$ or $y_-(C, t, u_0) < \alpha$, which contradicts NIP. Hence $y_+(x_0, t, u_0) = y_-(C, t, u_0) = \alpha$. Therefore from (4.21), for $x_0 < x < C$, $f'(u(x, t)) = \frac{x-\alpha}{t}$. This implies that $C = R_-(t, \alpha, u_0) < x < C$, which is a contradiction. Hence $x_0 = R_-(t, \alpha, u_0)$. Suppose $C < x_0$, then from the definition and (4.50), we have $y_-(x_0, t, u_0) \leq \alpha \leq y_-(x_0, t, u_0)$ and hence $y_-(x_0, t, u_0) = \alpha$ and from (4.21), $f'(u(x, t)) = \frac{x-\alpha}{t}$ for all $C < x < x_0$. This proves (1).

(2). It is enough to prove (i) and (ii) follows similarly. Let $x \geq R_-(t, \alpha, u_0)$, then from (4.25), $y_+(x, t, u_0) \geq \alpha$. Therefore

$$\begin{aligned} & \inf \left\{ \inf_{y \geq \alpha} \{v_0(y) + t f^* \left(\frac{x-y}{t} \right)\}, \inf_{y < \alpha} \{v_0(y) + t f^* \left(\frac{x-y}{t} \right)\} \right\} \\ &= v(x, t) \\ &= v_0(y_+(x, t, u_0)) + t f^* \left(\frac{x - y_+(x, t, u_0)}{t} \right). \end{aligned}$$

Hence

$$v(x, t) = \inf_{y \geq \alpha} \{v_0(y) + t f^* \left(\frac{x-y}{t} \right)\}.$$

(iii). (4.53) follows from (4.51) and (4.52). Let $\varepsilon > 0$, $u_0^\varepsilon = \chi_\varepsilon * u_0$ and $v_0^\varepsilon, v_\varepsilon$ be as in (4.3), (4.4) respectively. Let $u^\varepsilon = \frac{\partial v^\varepsilon}{\partial x}$ be the solution of (1.2) in $\Omega = \mathbb{R} \times \mathbb{R}_+$. Since v_0^ε is differentiable and hence for a.e x and from (4.13), $u^\varepsilon(x, t) = u_0^\varepsilon(y_+(x, t, u_0))$. Since $u_0^\varepsilon \rightarrow u_0$ in L^1_{loc} and hence $u^\varepsilon \rightarrow u$ in L^1_{loc} . Therefore from (4.32) to (4.35), we have for $0 < t < T$,

$$\overline{\lim}_{\varepsilon \rightarrow 0} R_+(t, \alpha, u_0^\varepsilon) \leq R_+(t, \alpha, u_0) < R_-(t, \beta, u_0) \leq \underline{\lim}_{\varepsilon \rightarrow 0} R_-(t, \beta, u_0^\varepsilon).$$

Let $\varepsilon_k \rightarrow 0$ and choose a dense set $D \subset (R_+(t, \alpha, u_0), R_-(t, \beta, u_0))$ such that for all $x \in D$,

$$\begin{aligned} \lim_{k \rightarrow \infty} u^{\varepsilon_k}(x, t) &= u(x, t) \\ y(x) &= y_+(x, t, u_0) = y_-(x, t, u_0) \\ y_k(x) &= y_+(x, t, u_0^{\varepsilon_k}) = y_-(x, t, u_0^{\varepsilon_k}). \end{aligned}$$

For $x \in D$, choose $k_0(x)$ such that for all $k \geq k_0(x)$, $x \in (R_+(t, \alpha, u_0^{\varepsilon_k}), R_-(t, \beta, u_0^{\varepsilon_k}))$. Then from ((4.53)), $y_k \in [\alpha, \beta]$. Since $u^{\varepsilon_k}(x, t) = u_0^{\varepsilon_k}(y_k(x))$, hence from (4.44),

$$m \leq u_0^{\varepsilon_k}(y_k(x)) = u^{\varepsilon_k}(x, t) \leq M.$$

Letting $k \rightarrow \infty$ to obtain (4.54). From (4.12),

$$f'(u^{\varepsilon_k}(x, t)) = \frac{x - y_k(x)}{t},$$

letting $k \rightarrow \infty$ to obtain

$$\begin{aligned} \frac{x - y(x)}{t} &= f'(u(x, t)) = \lim_{k \rightarrow \infty} f'(u^{\varepsilon_k}(x, t)) \\ &= \lim_{k \rightarrow \infty} \frac{x - y_k(x)}{t}. \end{aligned}$$

Hence $\lim_{k \rightarrow \infty} y_k(x) = y(x)$,

$$f'(m) \leq f'(u_0^{\varepsilon_k}(y_k(x))) = \frac{x - y_k(x)}{t} \leq f'(M),$$

Now letting $k \rightarrow \infty$ to obtain

$$f'(m) \leq \frac{x - y(x)}{t} \leq f'(M),$$

For $x \notin D$, choose $x_k \uparrow x$, $y_+(x_k, t, u_0) = y_-(x_k, t, u_0)$. Then from (4.22), $y_+(x_k, t, u_0) \rightarrow y_-(x, t, u_0)$. Now apply the inequalities for x_k and let $k \rightarrow \infty$ to obtain (4.54), (4.55). This proves (2).

(3). Without loss of generality we can take $L(t, \alpha, u_0) = R_-(t, \alpha, u_0)$ and $R(t, \beta, u_0) = R_+(t, \beta, u_0)$. Similar proof follows in all other cases. Let $C = R_-(T, \alpha, u_0) = R_+(T, \beta, u_0)$ and $t > T$. Then from (4.27) and (4.29) we have

$$\begin{aligned} R_-(t, \alpha, u_0) &= R_-(t, T, C, u_0) \\ &= R_+(t, T, C, u_0) \\ &= R_+(t, \beta, u_0). \end{aligned} \tag{4.60}$$

This proves (4.57).

Let $t > T$, then choose $k_0 = k_0(t)$ such that $t > T_k$, for all $k > k_0$. Then from (4.57) we have

$$\begin{aligned} R_k(t) &= R_-(t, \alpha_k, u_0^k) = R_+(t, \alpha_k, u_0^k), \\ R(t) &= R_-(t, \alpha, u_0) = R_+(t, \alpha, u_0). \end{aligned}$$

Hence from (6) of Lemma 4.2,

$$\overline{\lim}_{k \rightarrow \infty} R_k(t) \leq R_+(t, \alpha, u_0) = R_-(t, \alpha, u_0) \leq \underline{\lim}_{k \rightarrow \infty} R_k(t). \tag{4.61}$$

This proves (4.59) and hence the Lemma.

Next we give a criteria under which $R_+ = R_-$. Let $\beta < \gamma$ and $I_1 = [\beta, \gamma]$, Define

$$m = \inf_{y \in I_1} u_0(y), \quad M = \sup_{y \in I_1} u_0(y), \quad I_2 = [f'(m), f'(M)].$$

Let

$$a_0 = \max\{f^*(q) - Mq; q \in I_2\}, \quad f'(q_0) = \max\{q; f^*(q) - Mq \leq a_0\}.$$

Then we have the following.

LEMMA 4.4 Let $\alpha < \beta < \gamma, \varepsilon_0 > 0$. Let $u_0 \in L^\infty(\mathbb{R})$, a_0 and q_0 as above. Suppose

$$\inf_{[\alpha, \beta]} u_0(y) \geq q_0 + \varepsilon_0, \quad (4.62)$$

then for all $t > 0$,

$$R_+(t, \beta, u_0) = R_-(t, \beta, u_0).$$

Proof. Suppose for some $T > 0$, $R_+(T, \beta, u_0) > R_-(T, \beta, u_0)$, then from (4.57), for $0 < t < T$,

$$R_-(t, \beta, u_0) < R_+(t, \beta, u_0)$$

and from (4.26) for $R_-(t, \beta, u_0) < x < R_+(t, \beta, u_0)$.

$$f'(u(x, t)) = \frac{x - \beta}{t}. \quad (4.63)$$

From (4.28) we can choose T sufficiently small such that for all $0 < t \leq T$,

$$R_+(t, \alpha, u_0) < R_-(t, \beta, u_0) < R_+(t, \beta, u_0) < R_-(t, \gamma, u_0). \quad (4.64)$$

Claim: Let $L(t) = R_-(t, \beta, u_0)$, then for $0 < t \leq T$

$$f'(u(L(t)+, t)) \leq \frac{R_+(t, \beta, u_0) - \beta}{t} \leq f'(q_0). \quad (4.65)$$

Let $x_k > R_+(t, \beta, u_0)$ be such that $y_+(x_k, t, u_0) = y_-(x_k, t, u_0)$ and $\lim_{k \rightarrow \infty} x_k = R_+(t, \beta, u_0)$. Then from (4.55)

$$f'(m) \leq \frac{x - y_-(x_k, t, u_0)}{t} \leq f'(M).$$

Letting $k \rightarrow \infty$ and from (4.23) we have

$$f'(m) \leq \frac{R_+(t, \beta, u_0) - y_+(R_+(t, \beta, u_0), t, u_0)}{t} \leq f'(M). \quad (4.66)$$

Let $v_0(y) = \int_{\beta}^y u_0(\theta) d\theta$, hence $v_0(\beta) = 0$. Denote $R(t) = R_+(t, \beta, u_0)$, $y_{\pm}(t) = y_{\pm}(R_+(t, \beta, u_0), t, u_0)$, then from (4.63), $y_-(t) = \beta$ and from (4.4) we have

$$\begin{aligned} tf^*\left(\frac{R(t) - \beta}{t}\right) &= v_0(y_-(t)) + tf^*\left(\frac{R(t) - y_-(t)}{t}\right) \\ &= v_0(y_+(t)) + tf^*\left(\frac{R(t) - y_+(t)}{t}\right) \\ &\leq M(y_+(t) - \beta) + tf^*\left(\frac{R(t) - y_+(t)}{t}\right) \\ &\leq M(y_+(t) - R(t)) + M(R(t) - \beta) + tf^*\left(\frac{R(t) - y_+(t)}{t}\right), \end{aligned}$$

and hence

$$f^* \left(\frac{R(t) - \beta}{t} \right) - M \left(\frac{R(t) - \beta}{t} \right) \leq f^* \left(\frac{R(t) - y_+(t)}{t} \right) - M \left(\frac{R(t) - y_+(t)}{t} \right).$$

From (4.66) it follows that

$$f^* \left(\frac{R(t) - \beta}{t} \right) - M \left(\frac{R(t) - \beta}{t} \right) \leq a_0,$$

Letting x tends to $L(t)$ in (4.63) to obtain

$$f'(u(L(t)+, t)) = \frac{L(t) - \beta}{t} \leq \frac{R_+(t, \beta, u_0) - \beta}{t} \leq f'(q_0). \quad (4.67)$$

This proves (4.65) and hence the claim.

From (4.54), for $R_+(t, \alpha, u_0) < x < R_-(t, \beta, u_0) = L(t)$, $u(x, t) \geq \inf_{y \in [\alpha, \beta]} u_0(y)$, hence from (4.62) and (4.67), we have

$$\begin{aligned} u(L(t)-, t) &\geq \inf_{y \in [\alpha, \beta]} u_0(y) \geq q_0 + \varepsilon_0 \\ &\geq u(L(t)+, t) + \varepsilon_0 \\ &= f^{*'} \left(\frac{L(t) - \beta}{t} \right) + \varepsilon_0. \end{aligned} \quad (4.68)$$

From RH condition across $L(t)$ gives

$$\frac{dL}{dt} = \frac{f(u(L(t)-, t)) - f \left(f^{*'} \left(\frac{L(t) - \beta}{t} \right) \right)}{u(L(t)-, t) - f^{*'} \left(\frac{L(t) - \beta}{t} \right)}. \quad (4.69)$$

Therefore $L(t)$ satisfies the hypothesis (2) of Lemma 2.3 and hence from (4.47)

$$\lim_{t \rightarrow 0}^- \left| \frac{L(t) - \beta}{t} \right| = \infty,$$

which contradicts the uniform Lipschitz continuity of L from (4) of Lemma 2.2. Hence $R_-(t, \beta, u_0) = R_+(t, \beta, u_0)$, for all t , and this proves the Lemma.

REMARK 4.1 *Observe that q_0 entirely depends on the bounds of u_0 in $[\beta, \gamma]$.*

LEMMA 4.5 *Let u be the solution of (1.2) with*

$$\bar{u}_0(x) = u(x, 0) = \begin{cases} a & \text{if } x < \alpha, \\ u_0(x) & \text{if } x > \alpha. \end{cases}$$

Then for $x < R_-(t, \alpha, \bar{u}_0)$,

$$u(x, t) = a, \quad (4.70)$$

$$f'(a) = \frac{R_-(t, \alpha, \bar{u}_0) - y_-(R_-(t, \alpha, \bar{u}_0), t, \bar{u}_0)}{t}. \quad (4.71)$$

Proof. Since $\bar{v}_0(x) = \int_0^x \bar{u}_0(\theta) d\theta$ is differentiable for $x < \alpha$ and hence from (4.13), for a.e. $x < \alpha$, $u(x, t) = \bar{u}_0^\alpha(y_+(x, t, \bar{u}_0)) = a$ and

$$f'(a) = \frac{x - y_+(x, t, \bar{u}_0)}{t}.$$

From (4.22) and letting $x \uparrow R_-(t, \alpha, \bar{u}_0)$ to obtain (4.71). This proves the Lemma.

Analysis of initial value problem with data taking three values : Consider the following initial value problem taking three values. Let $a, \lambda, m \in \mathbb{R}, \alpha < \beta$ and consider

$$u_0^\lambda(x) = \begin{cases} a & \text{if } x < \alpha, \\ \lambda & \text{if } \alpha < x < \beta, \\ m & \text{if } x > \beta. \end{cases} \quad (4.72)$$

and denote

$$v_0^\lambda(x) = \int_\beta^x u_0^\lambda(\theta) d\theta, \quad (4.73)$$

and v^λ be as in (4.4). Let $u^\lambda = \frac{\partial v^\lambda}{\partial x}$ be the entropy solution of (1.2) in $\Omega = \mathbb{R} \times \mathbb{R}_+$ with initial data u_0^λ . Assume that

$$\lambda > \max(a, m), \quad (4.74)$$

then α is a point of rarefaction and β is the shock point.

Let

$$\begin{aligned} L_1(t) &= \alpha + f'(a)t, \\ L_2^\lambda(t) &= \alpha + f'(\lambda)t, \\ S^\lambda(t) &= \beta + \left(\frac{f(\lambda) - f(m)}{\lambda - m} \right) t. \end{aligned}$$

Let $(x_0(\lambda), T_0(\lambda))$ be the point of intersection of L_2^λ and S^λ given by

$$T_0(\lambda) = \frac{\beta - \alpha}{f'(\lambda) - \left(\frac{f(\lambda) - f(m)}{\lambda - m} \right)},$$

$$x_0(\lambda) = \alpha + \frac{(\beta - \alpha)f'(\lambda)}{f'(\lambda) - \left(\frac{f(\lambda) - f(m)}{\lambda - m}\right)}.$$

Since β is the point of shock and hence from (4.26) we have

$$R_+(t, \beta, u_0^\lambda) = R_-(t, \beta, u_0^\lambda) = R^\lambda(t) \text{ (def)}. \quad (4.75)$$

Then the solution u^λ for $t \leq T_0(\lambda)$ is given by

$$R^\lambda(t) = S^\lambda(t). \quad (4.76)$$

$$u^\lambda(x, t) = \begin{cases} m & \text{if } x > S^\lambda(t), \\ \lambda & \text{if } L_2^\lambda(t) < x < S^\lambda(t), \\ (f')^{-1}\left(\frac{x-\alpha}{t}\right) & \text{if } L_1(t) < x < L_2^\lambda(t), \\ a & \text{if } x < L_1(t). \end{cases} \quad (4.77)$$

Define $T_1(\lambda) > T_0(\lambda)$ be the first point of intersection of L_2^λ and R^λ . If they do not meet, then define $T_1(\lambda) = \infty$. Next Lemma describes the behavior of u^λ for $t > T_0(\lambda)$.

LEMMA 4.6 *Let λ satisfies (4.74). Then u^λ is given by (see Figure 9).*

(i). For $T_0(\lambda) < t < T_1(\lambda)$, $y_\pm(L_1(t), t, u_0^\lambda) = a$ and

$$u^\lambda(x, t) = \begin{cases} m & \text{if } x > R^\lambda(t), \\ f'^{-1}\left(\frac{x-\alpha}{t}\right) & \text{if } L_1(t) < x < R^\lambda(t), \\ a & \text{if } x < L_1(t). \end{cases} \quad (4.78)$$

(ii). $t > T_1(\lambda)$, then u^λ is the solution of (1.2) with initial data

$$u^\lambda(x, T_1(\lambda)) = \begin{cases} a & \text{if } x < R^\lambda(T_1(\lambda)), \\ m & \text{if } x > R^\lambda(T_1(\lambda)). \end{cases} \quad (4.79)$$

Furthermore for any compact sets K_1 and K_2 of $\mathbb{R}$ with $K = K_1 \times K_2$, $\eta > 0$, $T \leq T_1(\lambda)$ be bounded, then

$$\lim_{\lambda \rightarrow \infty} \inf_{(a, m) \in K} T_1(\lambda) = \infty, \quad (4.80)$$

$$f'(u^\lambda(R^\lambda(t)-, t)) = \begin{cases} f'(\lambda) & \text{if } 0 < t < T_0(\lambda), \\ \frac{R^\lambda(t) - \alpha}{t} & \text{if } T_0(\lambda) < t < T_1(\lambda). \end{cases} \quad (4.81)$$

$$\lim_{\lambda \rightarrow \infty} \inf_{\substack{(a, m) \in K \\ T_0(\lambda) \leq t \leq T}} u^\lambda(R^\lambda(t)-, t) = \infty. \quad (4.82)$$

$$\lim_{\lambda \rightarrow \infty} \inf_{\eta \leq t \leq T} R^\lambda(t) = \infty, \quad (4.83)$$

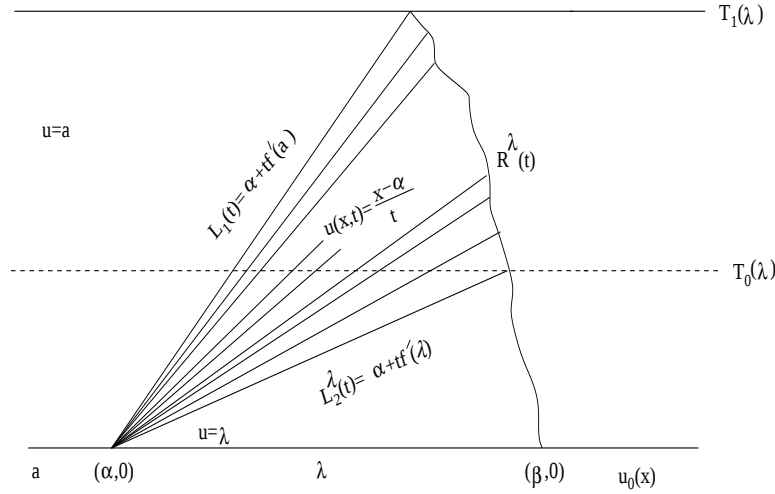


Figure 9:

Proof. Let $T_0(\lambda) < t \leq T_1(\lambda)$. Since $v_0^\lambda(x)$ is differentiable for $x > \beta$ and hence from (4.13) and (4.51), $u^\lambda(x, t) = u_0(y_+(x, t, u_0^\lambda)) = m$ if $x > R_+(t, \beta, u_0^\lambda) = R^\lambda(t)$. Next we show that for $L^1(t) < x < R^\lambda(t)$, $y_\pm(x, t, u_0^\lambda) = \alpha$.

$L_1(t) < x < R^\lambda(t)$. Then $y_+(x, t, u_0^\lambda) < \beta$. Suppose for some $x_0 \in (L_1(t), R^\lambda(t))$, $y_-(x, t, u_0^\lambda) < \alpha$, then for all $x \in (L_1(t), x_0)$, $y_-(x, t, u_0^\lambda) < \alpha$, $u^\lambda(x, t) = u_0(y_-(x, t, u_0^\lambda)) = a$ and

$$\frac{L_1(t) - \alpha}{t} = f'(a) = f'(u^\lambda(x, t)) = \frac{x - y_-(x, t, u_0^\lambda)}{t} > \frac{L_1(t) - \alpha}{t},$$

which is a contradiction. Suppose $y_+(x_0, t, u_0^\lambda) > \alpha$, then for all $x_0 < x < R^\lambda(t)$, $\alpha < y_+(x, t, u_0^\lambda) < \beta$. Since u_0^λ is differentiable in (α, β) and hence from (4.13), for a.e $x \in (x_0, R^\lambda(t))$,

$$u^\lambda(x, t) = u_0^\lambda(y_+(x, t, u_0^\lambda)) = \lambda, f'(\lambda) = f'(u^\lambda(x, t)) = \frac{x - y_+(x, t, u_0^\lambda)}{t}.$$

Suppose $x_0 < L_2^\lambda(t)$, then for $x_0 < x < \min(L_2^\lambda(t), R^\lambda(t))$,

$$f'(\lambda) = \frac{x - y_+(x, t, u_0^\lambda)}{t} < \frac{L_2^\lambda(t) - \alpha}{t} = f'(\lambda),$$

which is a contradiction. Suppose $L_2^\lambda(t) < x_0 < R^\lambda(t)$, then for $x \in (x_0, R^\lambda(t))$, characteristic γ at (x, t) given by $\gamma(\theta) = y_+(x, t, u_0^\lambda) + f'(\lambda)\theta$ intersects S^λ at t_0 , where

$$t_0 = \frac{\beta - y_+(x, t, u_0^\lambda)}{f'(\lambda) - \frac{f(\lambda) - f(m)}{\lambda - m}} < \frac{\beta - \alpha}{f'(\lambda) - \frac{f(\lambda) - f(m)}{\lambda - m}} = T_0(\lambda),$$

which contradicts NIP, since $S^\lambda(t)$ is a characteristic for $0 < t < T_0(\lambda)$. Hence for $L_1(t) < x < R^\lambda(t)$, $y_+(x, t, u_0^\lambda) = y_-(x, t, u_0^\lambda) = \alpha$ and from (4.21), we have

$$f'(u^\lambda(x, t)) = \frac{x - \alpha}{t}. \quad (4.84)$$

Now letting $x \downarrow L_1(t)$ and from (4.22) to obtain $y_+(L_1(t), t, u_0^\lambda) = \alpha$ and $f'(u^\lambda(L_1(t) +, t)) = \frac{L_1(t) - \alpha}{t} = f'(a)$. This implies $u^\lambda(L_1(t) +, t) = a$. From RH condition across $L_1(t)$ implies that $u^\lambda(L_1(t) -, t) = a$. Therefore from (4.12), (4.22), (4.23) $y_\pm(L_1(t), t, u_0^\lambda) = a$. This implies for $x < L_1(t)$, $y_+(x, t, u_0^\lambda) < \alpha$ and hence from (4.13), $u^\lambda(x, t) = u_0(y_+(x, t, u_0^\lambda)) = a$. This proves (4.78) and hence (4.79).

Let

$$\begin{aligned} y_\pm(t, \lambda) &= y_\pm(R^\lambda(t), t, u_0^\lambda), \\ y_\pm(\lambda) &= y_\pm(R^\lambda(T_1(\lambda)), T_1(\lambda), u_0^\lambda), \\ R^\lambda &= R^\lambda(T_1(\lambda)). \end{aligned}$$

Let $T_0(\lambda) < t \leq T_1(\lambda)$ and letting $x \uparrow R^\lambda(t)$ in (4.84) to obtain

$$\frac{R^\lambda(t) - y_-(t, \lambda)}{t} = f'(u^\lambda(R^\lambda(t) -, t)) = \frac{R^\lambda(t) - \alpha}{t}. \quad (4.85)$$

Hence $y_-(t, \lambda) = \alpha$. Also at $t = T_1(\lambda)$,

$$f'(a) = \frac{R^\lambda - \alpha}{T_1(\lambda)} = \frac{R^\lambda - y_-}{T_1(\lambda)}. \quad (4.86)$$

$$\frac{R^\lambda(t) - y_+(t, \lambda)}{t} = \lim_{x \downarrow R^\lambda(t)} \frac{x - y_-(x, t, u_0^\lambda)}{t} \quad (4.87)$$

$$= \lim_{x \downarrow R^\lambda(t)} f'(u^\lambda(x, t)) = f'(m). \quad (4.88)$$

From (4.85) to (4.88) we can evaluate $v^\lambda(R^\lambda(t), t)$ by

$$\begin{aligned} -(\beta - \alpha)\lambda + tf^*\left(\frac{R^\lambda(t) - \alpha}{t}\right) &= (y_+(t, \lambda) - \beta)m + tf^*\left(\frac{R^\lambda(t) - y_+(t, \lambda)}{t}\right) \\ &= m\left(\frac{y_+(t, \lambda) - R^\lambda(t)}{t}t + R^\lambda(t) - \beta\right) \\ &+ tf^*(f'(m)) \\ &= -tmf'(m) + m(R^\lambda(t) - \alpha) + m(\alpha - \beta) \\ &+ tf^*(f'(m)). \end{aligned}$$

$$\frac{(\beta - \alpha)(\lambda - m)}{t} = f^*\left(\frac{R^\lambda(t) - \alpha}{t}\right) - \frac{R^\lambda(t) - \alpha}{t} - f^*(f'(m) + mf'(m)). \quad (4.89)$$

Let $t = T_1(\lambda)$ then $\frac{R^\lambda - \alpha}{T_1(\lambda)} = f'(a)$ and hence the right hand side of (4.89) is bounded uniformly for $(a, m) \in K$ and hence as $\lambda \rightarrow \infty, T_1(\lambda) \rightarrow \infty$. This proves (4.80).

Observe that $R_+(t, \alpha, u_0^\lambda) = L_2^\lambda(t)$ and $L_2^\lambda(t) < R^\lambda(t)$ for $0 < t < T_0(\lambda)$. Hence for a.e $x \in (L_2^\lambda(t), R^\lambda(t))$, $y_+(x, t, u_0^\lambda) = y_-(x, t, u_0^\lambda) \in (\alpha, \beta)$ and from (4.13),

$$u^\lambda(x, t) = u_0^\lambda(y_+(x, t, u_0^\lambda)) = \lambda.$$

From this and (4.85), (4.81) follows. Let $T_0(\lambda) < t \leq T$, then from superlinearity of f^* , (4.82) follows from (4.85), (4.89). Suppose $\lim_{\lambda \rightarrow \infty} T_0(\lambda) = 0$, and then (4.83) follows from (4.81), (4.82). Hence assume that $\lim_{\lambda \rightarrow \infty} T_0(\lambda) > 0$, then if $\eta < T_0(\lambda)$, then (4.83) follows from (4.89). This proves the Lemma.

Next we generalize the above Lemma by replacing m by u_0 . More precisely let

$$u_0^\lambda(x) = \begin{cases} a & \text{if } x < \alpha, \\ \lambda & \text{if } \alpha < x < \beta, \\ u_0(x) & \text{if } x > \beta, \end{cases} \quad (4.90)$$

and u^λ be the solution of (1.2) with initial data u_0^λ . Let

$$m_1 = \inf_{x \geq \alpha} u_0(x), m_2 = \sup_{x \geq \alpha} u_0(x). \quad (4.91)$$

For $i = 1, 2$, define $u_0^{i,\lambda}$ by

$$u_0^{i,\lambda}(x) = \begin{cases} a & \text{if } x < \alpha, \\ \lambda & \text{if } \alpha < x < \beta, \\ m_i & \text{if } \beta < x, \end{cases} \quad (4.92)$$

and let u_i^λ be the solution of (1.2) with initial data $u_0^{i,\lambda}$. Let $L_1(t), L_2^\lambda(t)$ be as defined earlier, then

LEMMA 4.7 *Let $T > 0$ be fixed, then there exist $\lambda_0 = \lambda_0(m_1, m_2, a, t)$ such that for $\lambda \geq \lambda_0, 0 < t \leq T$,*

$$R_-(t, \beta, u_0^\lambda) = R_+(t, \beta, u_0^\lambda). \quad (4.93)$$

and denote $R(\lambda, t) = R_-(t, \beta, u_0^\lambda)$, then

(i). $t \rightarrow R(\lambda, t)$ is a strictly increasing function.

(ii). $\lambda \rightarrow R(\lambda, t)$ is a strictly increasing function. Let $T_1(\lambda)$ be the first point of intersection of $L_1(t)$ and $R(t, \lambda)$. Then for any fixed $T > 0$

$$\lim_{t \rightarrow \infty} T_1(\lambda) = \infty, \quad \lim_{\lambda \rightarrow \infty} R(\lambda, T) = \infty. \quad (4.94)$$

Let $T_0(\lambda)$ be the first point of intersection of $L_2^\lambda(t)$ and $R(\lambda, t)$. Then

$$u^\lambda(x, t) = \begin{cases} a & \text{if } x < L_1(t) \\ \lambda & \text{if } L_2^\lambda(t) < x < R(\lambda, t), \quad 0 < t < T_0(\lambda). \end{cases} \quad (4.95)$$

Proof. Let q_0 be as in (4.62), then for $\lambda > q_0$, from Lemma 4.4 we have for $i = 1, 2$,

$$R_-(t, \beta, u_0^\lambda) = R_+(t, \beta, u_0^\lambda), \quad (4.96)$$

$$R_-(t, \beta, u_0^{i, \lambda}) = R_+(t, \beta, u_0^{i, \lambda}) \quad (4.97)$$

and denote $R_i(\lambda, t) = R_-(t, \beta, u_0^{i, \lambda})$, $T_{11}(\lambda), T_1(\lambda), T_{21}(\lambda)$ the first points of intersection of $L_1(t)$ with $R_1(\lambda, t), R(\lambda, t), R_2(\lambda, t)$ respectively. Since $u_0^{1, \lambda} \leq u_0^\lambda \leq u_0^{2, \lambda}$, hence from (4.30)

$$R_1(\lambda, t) \leq R(\lambda, t) \leq R_2(\lambda, t), \quad T_{11}(\lambda) \leq T_1(\lambda) \leq T_{2,1}(\lambda). \quad (4.98)$$

Then from (4.80), it follows that

$$\lim_{\lambda \rightarrow \infty} T_1(\lambda) = \infty.$$

Next we obtain a bound on $u^\lambda(R(\lambda, t), t)$. For this let $\bar{u}(x, t)$ be the solution of (1.2) with initial data $\bar{u}_0(x)$ defined by

$$\bar{u}_0(x) = \begin{cases} \min(a, m_1) & \text{if } x < \beta, \\ u_0(x) & \text{if } x > \beta, \end{cases}$$

then for $\lambda > m$, $\bar{u}_0(x) \leq u_0^{1, \lambda}(x) \leq u_0^\lambda(x)$ and hence $\bar{u}(x, t) \leq u^\lambda(x, t)$ and $R_+(t, \beta, \bar{u}_0) \leq R(\lambda, t)$. Since for $y > \beta$, $\int_\beta^y \bar{u}_0(\theta) d\theta = \int_\beta^y u_0^\lambda(\theta) d\theta$ and hence from (4.51) we have for $x > R(\lambda, t)$,

$$\begin{aligned} V^\lambda(x, t) &= \inf_{y \geq \beta} \left\{ \int_\beta^y u_0^\lambda(\theta) d\theta + t f^* \left(\frac{x - y}{t} \right) \right\} \\ &= \inf_{y \geq \beta} \left\{ \int_\beta^y \bar{u}_0(\theta) d\theta + t f^* \left(\frac{x - y}{t} \right) \right\} \\ &= V(x, t), \end{aligned}$$

where $u^\lambda = \frac{\partial V^\lambda}{\partial x}$ and $\bar{u} = \frac{\partial V}{\partial x}$. Hence for $x > R(\lambda, t)$,

$$u^\lambda(x, t) = \bar{u}(x, t). \quad (4.99)$$

Therefore

$$\begin{aligned} |u^\lambda(R(\lambda, t)+, t)| &\leq \|\bar{u}\|_\infty \\ &\leq \max(m_2, a). \end{aligned} \quad (4.100)$$

For $i = 1, 2$, let $T_{i,0}(\lambda)$ be the first intersection point of $L_2^\lambda(t)$ and $R_i(\lambda, t)$ and $T_{i,1}(\lambda) > T_{i,0}(\lambda)$ be the points of intersections of $L_1(t)$ and $R_i(\lambda, t)$. Then from Lemma 4.8, we can choose $\lambda_0 \geq q_0 + \|\bar{u}\|_\infty$ such that for all $\lambda \geq \lambda_0$, $f'(\lambda) > 0$, $f(\lambda) > f(\|\bar{u}\|_\infty)$ and

$$T_{1,1}(\lambda) > T, \quad R_1(\lambda, T) > L_1(t). \quad (4.101)$$

$$\inf_{T_{1,0}(\lambda) \leq t \leq T} f^{*'} \left(\frac{R_1(\lambda, t) - \alpha}{t} \right) = \inf_{T_{1,0}(\lambda) \leq t \leq T} u^{1,\lambda}(R_1(\lambda, t)-, t) > \lambda_0. \quad (4.102)$$

From (4.83) and (4.98) we have

$$\lim_{\lambda \rightarrow \infty} R(\lambda, T) \geq \lim_{\lambda \rightarrow \infty} R_1(\lambda, T) = \infty.$$

This proves (4.94).

Next imitating the proof as in Lemma 4.6 and from (4.99) we have for $0 < t < T$,

$$u^\lambda(x, t) = \begin{cases} \bar{u}(x, t) & \text{if } x > R(\lambda, t), \\ (f')^{-1} \left(\frac{x-\alpha}{t} \right) & \text{if } t > T_0(\lambda), L_1(t) < x < R(\lambda, t), \\ \lambda & \text{if } 0 < t < T_0(\lambda), L_2^\lambda(t) < x < R(\lambda), \\ a & \text{if } x < L_1(t). \end{cases} \quad (4.103)$$

Let $0 < t < T_0(\lambda)$ then from (4.100) and the choice of λ_0 , we have for a.e. t ,

$$\begin{aligned} \frac{d}{dt} R(\lambda, t) &= \frac{f(u^\lambda(R(\lambda, t)-, t)) - f(u^\lambda(R(\lambda, t)+, t))}{u^\lambda(R(\lambda, t)-, t) - u^\lambda(R(\lambda, t)+, t)} \\ &= \frac{f(\lambda) - f(u^\lambda(R(\lambda, t)+, t))}{\lambda - u^\lambda(R(\lambda, t)+, t)} > 0. \end{aligned}$$

Let $T_0(\lambda) < t \leq T$, then from (4.98), $T_{1,0}(\lambda) \leq T_0(\lambda)$. Hence from (4.103), (4.102)

$$\begin{aligned} u^\lambda(R(\lambda, t)-, t) &= f^{*'} \left(\frac{R(\lambda, t) - \alpha}{t} \right) \\ &\geq f^{*'} \left(\frac{R_1(\lambda, t) - \alpha}{t} \right) \\ &= u^{1,\lambda}(R_1(\lambda, t)-, t) \\ &> \lambda_0. \end{aligned}$$

Since $f'(\lambda) > 0$ for $\lambda \geq \lambda_0$, hence

$$f(u^\lambda(R(\lambda, t)-, t)) \geq f(\lambda_0) > f(\|\bar{u}\|_\infty).$$

Therefore from (4.99), (4.100) we have for $T_0(\lambda) < t \leq T$.

$$\begin{aligned} \frac{d}{dt}R(\lambda, t) &= \frac{f(u^\lambda(R(\lambda, t)-, t)) - f(u^\lambda(R(\lambda, t)+, t))}{u^\lambda(R(\lambda, t)-, t) - u^\lambda(R(\lambda, t)+, t)} \\ &= \frac{f(u^\lambda(R(\lambda, t)-, t)) - f(\bar{u}(R(\lambda, t)+, t))}{u^\lambda(R(\lambda, t)-, t) - \bar{u}(R(\lambda, t)+, t)} \\ &> 0. \end{aligned}$$

This proves that $t \rightarrow R(\lambda, t)$ is a strictly increasing function.

Claim: $R(\lambda, t) \leq L_2^\lambda(t)$ for $t > T_0(\lambda)$.

Suppose for some $t_0 > T_0(\lambda)$, $R(\lambda, t_0) > L_2^\lambda(t_0)$, then for a.e $x \in (L_2^\lambda(t_0), R(\lambda, t_0))$, $y_+(x, t, u_0^\lambda) \in (\alpha, \beta)$ and hence from (4.13) and differentiability of u_0^λ in (α, β) gives $u(x, t_0) = \lambda$ and $f'(\lambda) = \frac{x - y_+(x, t_0, u_0^\lambda)}{t_0}$. Hence the characteristic line $r(\theta)$ at (x, t_0) is parallel to L_2^λ and $r(\theta) \geq L_2^\lambda(\theta)$ for $\theta \in [0, t_0]$. Since $t \rightarrow R(\lambda, t)$ is an increasing function for $t \in (0, T_1(\lambda))$ and $T_0(\lambda) < t_0$, hence $R(\lambda, T_0(\lambda)) < x$. Furthermore $y_+(R(\lambda, T_0(\lambda)), T_0(\lambda), u_0^\lambda) \geq \beta$. Hence the characteristic line at $(R(\lambda, T_0(\lambda)), T_0(\lambda))$ intersect r which contradicts NIP. This proves the claim.

Hence for $t \geq T_0(\lambda)$,

$$\frac{R(\lambda, t) - \alpha}{t} \leq \frac{L_2^\lambda(t) - \alpha}{t} = f'(\lambda). \quad (4.104)$$

Let $\lambda_0 \leq \lambda_1 < \lambda_2$, then $u_0^{\lambda_1} \leq u_0^{\lambda_2}$ and hence $R(\lambda_1, t) \leq R(\lambda_2, t)$ and for a.e. x , $y_\pm(x, t, u_0^{\lambda_1}) \geq y_\pm(x, t, u_0^{\lambda_2})$. Suppose for some $0 < t_0 < T$, $R = R(\lambda_1, t_0) = R(\lambda_2, t_0)$. From (4.99) at $x = R$, we have $\alpha \leq y_-(R, t_0, u_0^{\lambda_2}) \leq y_-(R, t_0, u_0^{\lambda_1}) < \beta$, and $u^{\lambda_1}(R+, t_0) = \bar{u}(R+, t_0) = u^{\lambda_2}(R+, t_0)$. Hence from (4.23) $y_+(R, t_0, u_0^{\lambda_1}) = y_+(R, t_0, u_0^{\lambda_2})$.

Let for $i = 1, 2$, $y = y_+(R, t_0, u_0^{\lambda_i})$, $y_i = y_-(R, t_0, u_0^{\lambda_i})$ and $V_0^{\lambda_i}(y) = \int_{\beta}^y u_0^{\lambda_i}(\theta) d\theta$,

then $V_0^{\lambda_1}(y) = V_0^{\lambda_2}(y)$ for $y \geq \beta$. Hence from (4.4) we have

$$\begin{aligned}
\lambda_2(y_2 - \beta) + t_0 f^* \left(\frac{R - y_2}{t_0} \right) &= V_0^{\lambda_2}(y_2) + t f^* \left(\frac{R - y_2}{t_0} \right) \\
&= V_0^{\lambda_2}(y) + t f^* \left(\frac{R - y}{t_0} \right) \\
&= V_0^{\lambda_1}(y) + t f^* \left(\frac{R - y}{t_0} \right) \\
&= V_0^{\lambda_1}(y_1) + t f^* \left(\frac{R - y_1}{t_0} \right) \\
&= \lambda_1(y_1 - \beta) + t_0 f^* \left(\frac{R - y_1}{t_0} \right).
\end{aligned}$$

Let $f'(\theta_i) = \frac{R - y_i}{t_0}$, then $R = f'(\theta_i)t_0 + y_i$ and since $y_2 \leq y_1$ implies that $f'(\theta_2) \geq f'(\theta_1)$, hence $\theta_2 \geq \theta_1$. Substituting this in the above expression and using $f^*(f'(p)) = pf'(p) - f(p)$ to obtain

$$(R - t_0 f'(\theta_2))\lambda_2 + t_0 f^*(f'(\theta_2)) = (R - t_0 f'(\theta_1))\lambda_1 + t_0 f^*(f'(\theta_1)) + \beta(\lambda_2 - \lambda_1)$$

$$R = \beta + \frac{t_0}{(\lambda_2 - \lambda_1)} [(\lambda_2 - \theta_2)f'(\theta_2) - (\lambda_1 - \theta_1)f'(\theta_1)] + \left(\frac{f(\theta_2) - f(\theta_1)}{\lambda_2 - \lambda_1} \right) t_0.$$

That is for $i = 1, 2$,

$$y_i = \beta + \frac{t_0}{(\lambda_2 - \lambda_1)} [(\lambda_2 - \theta_2)f'(\theta_2) - (\lambda_1 - \theta_1)f'(\theta_1)] + t_0 \left[\frac{f(\theta_2) - f(\theta_1)}{\lambda_2 - \lambda_1} - f'(\theta_i) \right]. \quad (4.105)$$

Case (i) : Let $y_2 = y_1$. Then $\theta_2 = \theta_1$ and hence from (4.105), $\beta = y_1 < \beta$ which is a contradiction.

Case (ii): Let $\alpha < y_2 < y_1$.

Since $V_0^{\lambda_i}$ is differentiable for $y \in (\alpha, \beta)$ and hence from (4.13) , (4.23), we have $f'(\lambda_i) = \frac{R - y_i}{t_0}$. Therefore from (4.105) and from strict convexity of f we have

$$y_1 = \beta + t_0 \left[\frac{f(\lambda_2) - f(\lambda_1)}{\lambda_2 - \lambda_1} - f'(\lambda_1) \right] > \beta.$$

which is a contradiction.

Case(iii): Let $\alpha = y_2 < y_1$.

Since $y_1 > \alpha$, hence $f'(\theta_1) = \frac{R - y_1}{t_0} = f'(\lambda_1)$ and $\frac{R - \alpha}{t_0} = f'(\theta_2)$. From (4.104), $f'(\theta_2) \leq f'(\lambda_2)$ and hence $\lambda_2 \geq \theta_2$. Since $\lambda_1 \geq \lambda_0$ and hence $f'(\theta_2) \geq f'(\lambda_1) > 0$.

From (4.105), $\theta_1 = \lambda_1$ and convexity of f we have

$$\begin{aligned} 0 > \frac{(\lambda_2 - \lambda_1)(y_1 - \beta)}{t_0} &= (\lambda_2 - \theta_2)(f'(\theta_2) - f'(\lambda_1)) + f(\theta_2) - f(\lambda_1) \\ &\geq (\theta_2 - \lambda_1)f'(\lambda_1) \\ &> 0, \end{aligned}$$

which is a contradiction. This proves $\lambda \rightarrow R(\lambda, t)$ is a strictly increasing function for $\lambda \geq \lambda_0$ and $0 < t \leq T$. This proves the Lemma.

Next we consider the variation from the right, Let u^μ be the solution of (1.2) with initial data u_0^μ given by

$$u_0^\mu = \begin{cases} u_0(x) & \text{if } x < \alpha, \\ \mu & \text{if } \alpha < x < \beta, \\ a & \text{if } x > \beta. \end{cases}$$

We state the following Lemma without proof since the proof follows exactly as that of Lemma 4.7.

Define

$$L_1(t) = \beta + f'(a)t, L_2^\mu(t) = \beta + f'(\mu)t.$$

LEMMA 4.8 *There exist $\mu_1 = \mu_1(m_1, m_2, a)$ such that for $\mu < \mu_1, t > 0$,*

$$R_-(t, \alpha, u_0^\mu) = R_+(t, \alpha, u_0^\mu), \quad (4.106)$$

and denote $R(\mu, t) = R_-(t, \alpha, u_0^\mu)$. Let $T_0(\mu) > 0$ be the first point of intersection of $R(\mu, t)$ and $L_2^\mu(t)$ and $T_1(\mu) > T_0(\mu)$ be the first point of intersection of $R(\mu, t)$ and $L_1(t)$. Then

$$\lim_{\mu \rightarrow -\infty} T_1(\mu) = \infty. \quad (4.107)$$

For $0 < t < T_1(\mu)$,

$$u^\mu(x, t) = \begin{cases} a & \text{if } x > L_1(t), \\ (f')^{-1}\left(\frac{x-\beta}{t}\right) & \text{if } T_0(\mu) < t < T_1(\mu), \\ & R(\mu, t) < x < L_1(t), \\ \mu & \text{if } 0 < t < T_0(\mu), \\ & R(\mu, t) < x < L_2^\mu(t). \end{cases} \quad (4.108)$$

Furthermore let $T > 0$ be fixed, then there exist $\mu_0 = \mu_0(T, \mu_1) < \mu_1$ such that

(i). $\mu \rightarrow R(\mu, t)$ is a strictly increasing function for $0 < t \leq T$ and

$$\lim_{\mu \rightarrow -\infty} R(\mu, t) = -\infty. \quad (4.109)$$

(ii). For $0 < t < T_1(\mu)$, $t \rightarrow R(\mu, t)$ is a strictly decreasing function of t .

Interaction of $R_\pm$ with initial data: We study the interaction of $R_\pm$ with varying parameters in the data. For this first we need the following elementary results.

Let $B_1, B_2, \mu_0 < \lambda_0$, $L \in C(\mathbb{R}_+ \times [\lambda_0, \infty))$, $R \in C(\mathbb{R}_+ \times (-\infty, \mu_0])$ be given and for $\lambda \geq \lambda_0, \mu \leq \mu_0$, L and R satisfies the following hypothesis,

(H₁). $\lambda \mapsto L(t, \lambda)$, $\mu \mapsto R(t, \mu)$ are strictly increasing functions such that for all $\lambda \geq \lambda_0, \mu \leq \mu_0$,

$$L(0, \lambda) = B_1, \quad R(0, \mu) = B_2, \quad (4.110)$$

and for any $0 < \alpha < \beta$,

$$\lim_{\lambda \rightarrow \infty} \inf_{t \in [\alpha, \beta]} L(t, \lambda) = \infty, \quad \lim_{\mu \rightarrow -\infty} \sup_{t \in [\alpha, \beta]} R(t, \mu) = -\infty. \quad (4.111)$$

(H₂). For $\lambda \geq \lambda_0, \mu \leq \mu_0$, $t \mapsto L(t, \lambda)$, is a strictly increasing function and $t \mapsto R(t, \mu)$ is a strictly decreasing function.

Let $I = [\lambda_0, \infty) \times (-\infty, \mu_0]$ and define $x_0(t), y_0(t), \lambda(x, t), \mu(y, t), \delta(\lambda, \mu), c(\lambda, \mu)$ as follows:

$$x_0(t) = L(t, \lambda_0), \quad y_0(t) = R(t, \mu_0) \quad (4.112)$$

$$L(t, \lambda(x, t)) = x, \quad R(t, \mu(y, t)) = y \quad (4.113)$$

$$L(\delta(\lambda, \mu), \lambda) = R(\delta(\lambda, \mu), \mu) = c(\lambda, \mu), \quad (4.114)$$

then we have the following

LEMMA 4.9 1. x_0 is a strictly increasing continuous and y_0 is a strictly decreasing function satisfying

$$(x_0(0), y_0(0)) = (B_1, B_2). \quad (4.115)$$

2. For $x \geq x_0(t), y \leq y_0(t), (\lambda(x, t), \mu(y, t)) \in I, x \mapsto \lambda(x, t), t \rightarrow \mu(y, t)$ are strictly increasing functions and $t \mapsto \lambda(x, t), y \mapsto \mu(y, t)$ are strictly decreasing continuous functions in $(0, \infty)$. Also for $x > B_1, y < B_2$

$$\lim_{t \rightarrow 0} (\lambda(x, t), \mu(y, t)) = (\infty, -\infty). \quad (4.116)$$

3. Let $B_1 < B_2$ and $(\lambda, \mu) \in I$. Then $\delta(\lambda, \mu)$ exist and is a continuous function. Furthermore $\lambda \rightarrow \delta(\lambda, \mu)$ is a decreasing function and $\mu \mapsto \delta(\lambda, \mu)$ is an increasing function and

$$\lim_{\lambda \rightarrow \infty} \delta(\lambda, \mu) = \lim_{\mu \rightarrow -\infty} \delta(\lambda, \mu) = 0 \quad (4.117)$$

$$\lim_{\mu \rightarrow -\infty} c(\lambda, \mu) = B_1, \quad \lim_{\lambda \rightarrow \infty} c(\lambda, \mu) = B_2. \quad (4.118)$$

Proof.

1. Follows from (H_1) .
2. From (4.111) for $t > 0$, $L(t, \cdot) : [\lambda_0, \infty) \rightarrow [x_0(t), \infty)$ is a homeomorphism and hence $\lambda(x, t)$ exist and $x \mapsto \lambda(x, t)$ is a strictly increasing function. Let $t_1 < t_2$ and suppose $\lambda(x, t_1) \leq \lambda(x, t_2)$, then

$$x = L(t_1, \lambda(x, t_1)) \leq L(t_1, \lambda(x, t_2)) < L(t_2, \lambda(x, t_2)) = x,$$

which is a contradiction. Hence $t \mapsto \lambda(x, t)$ is a strictly decreasing function. Let $(x_n, t_n) \rightarrow (x, t)$, $\lambda(x_n, t_n) \rightarrow \lambda$, then

$$x = \lim_{n \rightarrow \infty} L(t_n, \lambda(x_n, t_n)) = L(t, \lambda),$$

and hence $\lambda = \lambda(x, t)$. This proves the continuity of $\lambda(x, t)$. Suppose as $t_n \rightarrow 0$, $\{\lambda(x, t_n)\}$ is bounded. Then for a subsequence still denote by n such that $\lambda(x, t_n) \rightarrow \lambda$ as $n \mapsto \infty$. Therefore by continuity of L and (4.110)

$$B_1 < x = \lim_{n \rightarrow \infty} L(t_n, \lambda(x, t_n)) = L(0, \lambda) = B_1,$$

which is a contradiction. Hence $\lambda(x, t) \rightarrow \infty$ as $t \rightarrow 0$. Similarly for $\mu(y, t)$ and this proves (2).

3. For $(\lambda, \mu) \in I$, $t \mapsto L(t, \lambda) \geq B_1$ and is a strictly increasing function and $t \mapsto R(t, \mu) \leq B_2$ is a strictly decreasing function. Hence there exist a unique $\delta(\lambda, \mu)$ satisfying (4.114) and $B_1 \leq c(\lambda, \mu) \leq B_2$ and continuity follows from the uniqueness of $\delta(\lambda, \mu)$.

Let $\lambda_1 < \lambda_2$ and $\delta(\lambda_1, \mu) \leq \delta(\lambda_2, \mu)$. Then

$$\begin{aligned} R(\delta(\lambda_1, \mu), \mu) &= L(\delta(\lambda_1, \mu), \lambda_1) \leq L(\delta(\lambda_2, \mu), \lambda_1) \\ &< L(\delta(\lambda_2, \mu), \lambda_2) \\ &= R(\delta(\lambda_2, \mu), \mu) \end{aligned}$$

and hence $\delta(\lambda_2, \mu) < \delta(\lambda_1, \mu)$ which is a contradiction. Suppose $\lim_{\lambda \rightarrow \infty} \delta(\lambda, \mu) = \delta_0 > 0$, then from (4.111),

$$\infty = \lim_{\lambda \rightarrow \infty} L(\delta(\lambda, \mu), \lambda) = \lim_{\delta(\lambda, \mu) \rightarrow \delta_0} R(\delta(\lambda, \mu), \mu) = R(\delta_0, \mu) < \infty,$$

which is a contradiction hence $\delta_0 = 0$ and

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} c(\lambda, \mu) &= \lim_{\lambda \rightarrow \infty} L(\delta(\lambda, \mu), \lambda) \\ &= \lim_{\delta(\lambda, \mu) \rightarrow 0} R(\delta(\lambda, \mu), \mu) \\ &= B_2, \end{aligned}$$

similarly for $\mu \rightarrow \delta(\lambda, \mu)$. This proves (3) and hence the Lemma.

COROLLARY 4.1 *Let $\delta_0 > 0$, then there exist $\lambda_1 \geq \lambda_0, \mu_1 \leq \mu_0$ such that for all $\lambda \geq \lambda_1, \mu \leq \mu_1$,*

$$\delta(\lambda, \mu) \leq \delta_0. \quad (4.119)$$

Proof. Since $\delta(\lambda, \mu_0) \rightarrow 0$ as $\lambda \rightarrow \infty$, hence choose $\lambda_1 \geq \lambda_0$ such that $\delta(\lambda_1, \mu_0) \leq \delta_0$. Let $\mu_1 = \mu_0$, then for $\lambda \geq \lambda_1, \mu \leq \mu_1$, we have,

$$\delta(\lambda, \mu) \leq \delta(\lambda_1, \mu) \leq \delta(\lambda_1, \mu_1) \leq \delta_0.$$

This proves the Corollary.

Let $T > 0$ and $A_1 < B_1 \leq C \leq B_2 < A_2$ and for $i = 1, 2$, define $a_i, l_i, 0 < \delta_0 < T$ by

$$\begin{aligned} f'(a_i) &= \frac{C - A_i}{T} \\ l_i(t) &= A_i + t f'(a_i) \\ \delta_0 &= \min\{l_1(B_1), l_2(B_2)\}. \end{aligned}$$

Let u_1^λ and u_2^μ be solutions of (1.2) with respective initial data u_0^λ, u_0^μ given by

$$u_0^{1,\lambda}(x) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \theta_f & \text{if } x > B_2. \end{cases} \quad (4.120)$$

$$u_0^{2,\mu}(x) = \begin{cases} \theta_f & \text{if } x < B_1, \\ u_0(x) & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ a_2 & \text{if } x > A_2. \end{cases} \quad (4.121)$$

From Lemma 4.7 and 4.8 we can choose $\lambda_0 = \lambda_0(\|u_0\|_\infty)$, $\mu_0 = \mu_0(\|u_0\|_\infty)$ such that for all $\lambda \geq \lambda_0, \mu \leq \mu_0, t > 0$,

$$L(t, \lambda) = R_-(t, B_1, u_0^{1,\lambda}) = R_+(t, B_1, u_0^{1,\lambda}) \quad (4.122)$$

$$R(t, \mu) = R_-(t, B_2, u_0^{2,\mu}) = R_+(t, B_2, u_0^{2,\mu}), \quad (4.123)$$

and for $0 < t \leq T$, L and R satisfies the hypothesis $(H_1), (H_2)$ of Lemma 4.9. Let $(c(\lambda, \mu), \delta(\lambda, \mu))$ be the point of intersection of $L(t, \lambda)$ and $R(t, \mu)$ as defined in (4.114). From Corollary 4.1, choose $\lambda_1 \geq \lambda_0, \mu_1 \leq \mu_0$ such that for all $\lambda \geq \lambda_1, \mu \leq \mu_1$

$$\delta(\lambda, \mu) < \delta_0. \quad (4.124)$$

LEMMA 4.10 *With the above notation and let $u(x, t, \lambda, \mu)$ be the solution of (1.2) with initial condition $u_0^{\lambda, \mu}$ given by*

$$u_0^{\lambda, \mu}(x) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < B_1, \\ u_0 & \text{if } B_1 < x < B_2, \\ \mu & \text{if } B_2 < x < A_2, \\ a_2 & \text{if } x > A_2. \end{cases} \quad (4.125)$$

then for $0 < t < \delta(\lambda, \mu)$,

$$u_1^\lambda(x, t) = u_2^\mu(x, t) \quad \text{if } L(t, \lambda) < x < R(t, \mu), \quad (4.126)$$

$$u(x, t, \lambda, \mu) = \begin{cases} u_1^\lambda(x, t) & \text{if } x < L(t, \lambda), \\ u_1^\lambda(x, t) & \text{if } L(t, \lambda) < x < R(t, \mu), \\ u_2^\mu(x, t) & \text{if } x > R(t, \mu). \end{cases} \quad (4.127)$$

Proof. Let $\gamma = \frac{B_1+B_2}{2}$ and define $v_0^{1,\lambda}(x) = \int_\gamma^x u_0^{1,\lambda}(\theta) d\theta$, $v_0^{2,\mu}(x) = \int_\gamma^x u_0^{2,\mu}(\theta) d\theta$,
 $v_0^{\lambda, \mu}(x) = \int_\gamma^x u_0^{\lambda, \mu}(\theta) d\theta$. Then for $x \in [B_1, B_2]$,

$$v_0^{1,\lambda}(x) = v_0^{2,\mu}(x) = \int_{\frac{B_1+B_2}{2}}^x u_0(\theta) d\theta. \quad (4.128)$$

$$v_0^{\lambda, \mu}(x) = \begin{cases} v_0^{1,\lambda}(x) & \text{if } x < B_1, \\ v_0^{2,\mu}(x) & \text{if } x > B_2. \end{cases} \quad (4.129)$$

Claim: Let $v^{1,\lambda}, v^{2,\mu}$ be the corresponding value functions associated to $v_0^{1,\lambda}, v_0^{2,\mu}$ defined in (4.4). Then

$$v^{1,\lambda}(x, t) = \inf_{y \in [B_1, B_2]} \left\{ v_0^{1,\lambda}(y) + t f^* \left(\frac{x-y}{t} \right) \right\}, \quad \text{if } L(t, \lambda) < x < B_2 \quad (4.130)$$

$$v^{2,\mu}(x, t) = \inf_{y \in [B_1, B_2]} \left\{ v_0^{2,\mu}(y) + t f^* \left(\frac{x-y}{t} \right) \right\}, \quad \text{if } B_1 < x < R(t, \lambda). \quad (4.131)$$

Let $L(t, \lambda) < x < B_2$, then from (4.122) $y_\pm(x, t, u_0^{1,\lambda}) > B_1$. Suppose for some $x_0 \in (L(t, \lambda), B_2)$, $y_+(x_0, t, u_0^{1,\lambda}) > B_2$. Since $v_0^{1,\lambda}$ is differentiable in (B_2, ∞) and hence from (4.13) for a.e. $x \in (x_0, B_2)$, $u^{1,\lambda}(x, t) = \frac{\partial v^{1,\lambda}}{\partial x}(x, t) = \theta_f$ and $0 = f'(\theta_f) = \frac{x - y_+(x, t, u_0^{1,\lambda})}{t}$. Hence $B_2 > x = y_+(x, t, u_0^{1,\lambda}) > B_2$, which is a contradiction. Therefore $y_\pm(x, t, u_0^{1,\lambda}) \in [B_1, B_2]$ and hence (4.130) follows. Similarly (4.131) holds. This proves the claim.

From (4.128), (4.130), (4.131), for $L(t, \lambda) < x < R(t, \mu)$, $v^{1,\lambda}(x, t) = v^{2,\mu}(x, t)$ and hence for a.e. x , $u^{1,\lambda}(x, t) = \frac{\partial v^{1,\lambda}}{\partial x}(x, t) = \frac{\partial v^{2,\mu}}{\partial x}(x, t) = u^{2,\mu}(x, t)$. This proves (4.126). In view of (4.126), RHS of (4.127) is a solution of (1.2) with initial data $u_0^{\lambda,\mu}$. Hence from uniqueness of solutions (4.127) follows. This proves the Lemma.

As an immediate consequence of Lemma 4.10 and (4.27), (4.122), (4.123) we have

COROLLARY 4.2 *Let $\lambda \geq \lambda_1, \mu \leq \mu_1$, then*

$$\begin{aligned} R_{\pm}(t, B_1, u_0^{\lambda,\mu}) &= L(t, \lambda) \quad 0 < t < \delta(\lambda, \mu), \\ R_{\pm}(t, B_2, u_0^{\lambda,\mu}) &= R(t, \mu) \quad 0 < t < \delta(\lambda, \mu), \\ R_{\pm}(t, B_1, u_0^{\lambda,\mu}) &= R_{\pm}(t, B_2, u_0^{\lambda,\mu}), \quad t \geq \delta(\lambda, \mu). \end{aligned}$$

Furthermore, denote $S(t, \lambda, \mu) = R_+(t, B_1, u_0^{\lambda,\mu})$ for $t > \delta(\lambda, \mu)$, then $(t, \lambda, \mu) \mapsto S(t, \lambda, \mu)$ is continuous and

$$u(x, t, \lambda, \mu) = \begin{cases} u^{1,\lambda}(x, t) & \text{if } x < S(t, \lambda, \mu), \\ u^{2,\mu}(x, t) & \text{if } x > S(t, \lambda, \mu), \end{cases} \quad (4.132)$$

Proof. Let $(t_k, \lambda_k, \mu_k) \rightarrow (t, \lambda, \mu)$. From Lemma 4.9, $\delta(\lambda_k, \mu_k) \rightarrow \delta(\lambda, \mu)$ and hence for $t > \delta(\lambda, \mu)$

$$\begin{aligned} |S(t_k, \lambda_k, \mu_k) - S(t, \lambda, \mu)| &\leq |S(t_k, \lambda_k, \mu_k) - S(t, \lambda_k, \mu_k)| \\ &\quad + |S(t, \lambda_k, \mu_k) - S(t, \lambda, \mu)|. \end{aligned}$$

From (4) of Lemma 4.1 and from (3) of Lemma 4.3, the right hand side tends to zero as $k \rightarrow \infty$. Let $v^{\lambda,\mu}$ be the cost function associated to $v_0^{\lambda,\mu}$ defined in (4.4). For $x < S(t, \lambda, \mu)$, $y_{\pm}(x, t, u_0^{\lambda,\mu}) < B_1$ and hence from (4.129), $v^{\lambda,\mu}(x, t) = v^{1,\lambda}(x, t)$. Hence $u(x, t, \lambda, \mu) = \frac{\partial v^{1,\lambda}}{\partial x}(x, t) = u^{1,\lambda}(x, t)$. Similarly for $x > S(t, \lambda, \mu)$, $u(x, t, \lambda, \mu) = u^{2,\mu}(x, t)$, this proves (4.127) and hence the Lemma.

LEMMA 4.11 *Let $\lambda \geq \lambda_1, \mu \leq \mu_1$ and $\delta(\lambda, \mu) < t_0 \leq T$, then*

(i). *Suppose $l_1(t_0) = S(t_0, \lambda, \mu)$. Then for all $t_0 < t < T$,*

$$S(t, \lambda, \mu) < l_1(t). \quad (4.133)$$

$$u(x, t, \lambda, \mu) = \begin{cases} a_2 & \text{if } 0 < t < T, \ x > l_2(t) \\ a_1 & \text{if } x < \min(l_1(t), S(t, \lambda, \mu)). \end{cases} \quad (4.134)$$

(ii). Suppose $l_2(t_0) = S(t_0, \lambda, \mu)$. Then for all $t_0 < t < T$,

$$S(t, \lambda, \mu) > l_2(t). \quad (4.135)$$

$$u(x, t, \lambda, \mu) = \begin{cases} a_1 & \text{if } 0 < t < T, \ x < l_1(t), \\ a_2 & \text{if } x > \max(l_2(t), S(t, \lambda, \mu)). \end{cases} \quad (4.136)$$

Furthermore there exist λ_2 and μ_2 such that $S(T, \lambda_2, \mu_2) = C$ and for $0 < t < T$, u satisfies

$$u(x, t, \lambda_2, \mu_2) = \begin{cases} a_1 & \text{if } x < l_1(t), \\ a_2 & \text{if } x > l_2(t). \end{cases} \quad (4.137)$$

Proof. (See Figure 10) Let $g(t) = \min(l_1(t), S(t, \lambda, \mu))$. Then we claim that for all $x < g(t)$,

$$u(x, t, \lambda, \mu) = a_1. \quad (4.138)$$

Suppose $x < l_1(t) \leq S(t, \lambda, \mu)$, then from (4.132), (4.95) we have $u(x, t, \lambda, \mu) = u^{1,\lambda}(x, t) = a_1$. Hence assume that $S(t, \lambda, \mu) < l_1(t)$. Suppose there exist $x_0 < S(t, \lambda, \mu)$ such that $y_+(x_0, t, u_0^{\lambda, \mu}) > A_1$, then for all $x \in (x_0, S(t, \lambda, \mu))$, $A_1 < y_+(x, t, u_0^{\lambda, \mu}) < B_1$. Since $u_0^{\lambda, \mu}$ is differentiable in (A_1, B_1) and hence from (4.13), for a.e. $x \in (x_0, S(t, \lambda, \mu))$

$$\begin{aligned} f'(\lambda) = f'(u(x, t, \lambda, \mu)) &= \frac{x - y_+(x, t, u_0^{\lambda, \mu})}{t} \\ &< \frac{l_1(t) - A_1}{t} = f'(a_1), \end{aligned}$$

which is a contradiction since $\lambda > a_1$. Hence $y_+(x, t, u_0^{\lambda, \mu}) \leq A_1$ for all $x \in (x_0, S(t, \lambda, \mu))$. Suppose $y_+(x_0, t, u_0^{\lambda, \mu}) = A_1$. Then from (4.21) $f'(u(x, t)) = \frac{x - A_1}{t}$ for $x \in (x_0, S(t, \lambda, \mu))$. Let $\gamma_x(\theta) = A_1 + \theta \left(\frac{x - A_1}{t} \right) < l_1(\theta)$ be the characteristic at (x, t) , then from (c) of (3) in Theorem 2.1, γ_x is also a characteristic at $(\gamma_x(s), s)$ for $0 < s < t$ and $f'(u(\gamma_x(s), s, \lambda, \mu)) = \frac{\gamma_x(s) - A_1}{s} < \frac{l_1(s) - A_1}{s} = f'(a_1)$. Let $s < \delta(\lambda, \mu)$, then $l_1(s) < L(s, \lambda)$ and hence $f'(a_1) > f'(u(\gamma_x(s), s, \lambda, \mu)) = f'(a_1)$ which is a contradiction. This proves the claim.

Let $t_0 > \delta(\lambda, \mu)$ such that $l(t_0) = S(t_0, \lambda, \mu)$. From (4.132) and Lemma 4.8 for $x > S(t_0, \lambda, \mu)$,

$$a_2 = \max(\mu, a_2) \geq u(x, t_0, \lambda, \mu). \quad (4.139)$$

Let $t_0 < t < T$ and w be the solution of (1.2) with initial data w_0 at t_0 is given by

$$w_0(x) = \begin{cases} a_1 & \text{if } x < S(t_0, \lambda, \mu) = l_1(t_0), \\ a_2 & \text{if } x > S(t_0, \lambda, \mu) = l_1(t_0). \end{cases}$$

Then w admits a shock at $l_1(t_0)$ and for $t > t_0$ is given by

$$\begin{aligned}
\eta(t) &= l_1(t_0) + \frac{f(a_1) - f(a_2)}{a_1 - a_2}(t - t_0) \\
&< l_1(t_0) + f'(a_1)(t - t_0) \\
&= A_1 + f'(a_1)t \\
&= l_1(t),
\end{aligned} \tag{4.140}$$

since f is strictly convex and $f'(a_1) > 0 > f'(a_2)$. From (4.138) and (4.139), $w_0(x) \geq$

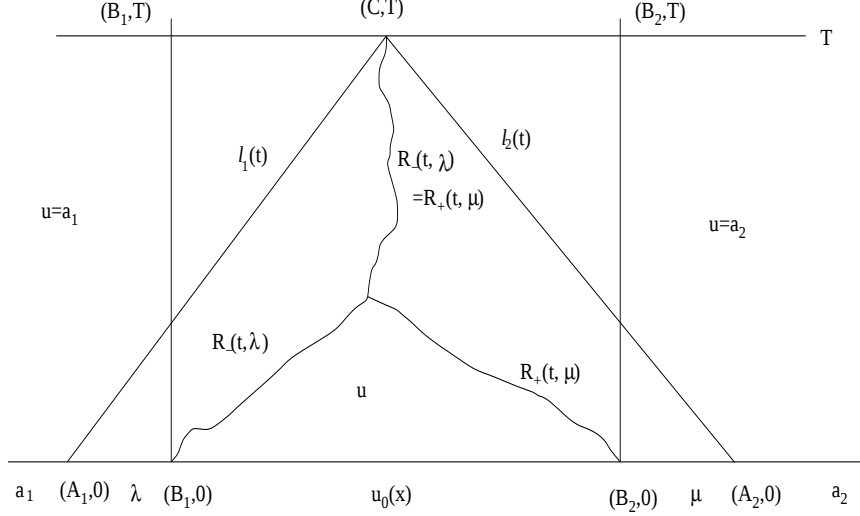


Figure 10:

$u(x, t_0, \lambda, \mu)$ and therefore from (4.29) and (4.30) we have for $t > t_0$, $l_1(t) > \eta(t) \geq S(t, \lambda, \mu)$. This proves (4.133).

From (3) of Lemma 2.4, $(\lambda, \mu) \rightarrow S(T, \lambda, \mu)$ is a continuous function for $\lambda \geq \lambda_1$ and $\mu \leq \mu_1$. From (4.94), choose a $\tilde{\lambda}_1 > \lambda_1$ such that $S(T, \tilde{\lambda}_1, \mu_1) > T$ and from (4.109) choose $\tilde{\mu}_1 < \mu_1$ such that $S(T, \lambda_1, \tilde{\mu}_1) < T$. From Corollary 4.1, S is continuous in $[\lambda_1, \tilde{\lambda}_1] \times [\mu_1, \tilde{\mu}_1]$ and therefore there exist a $(\lambda_2, \mu_2) \in [\lambda_1, \tilde{\lambda}_1] \times [\mu_1, \tilde{\mu}_1]$ such that $S(T, \lambda_2, \mu_2) = C$. Hence (4.137) follows from (4.136). This proves the Lemma.

Proof of Lemma 2.3. In Lemma 4.7, take $A = \alpha$, $B = \beta$, $l(t) = L_1(t)$. Then from (4.94), choose a λ_0 large such that for all $0 < t \leq T$ and for all $\lambda \geq \lambda_0$, $l(t) < R(\lambda, t)$. Then (2.23) follows from (4.95) and from Rankine-Hugoniot condition across $l(t)$. Similarly (2.24) follows from Lemma 4.8 and (4.107) and (4.108).

Proof of Lemma 2.4 This follows from Lemma 4.11 and (4.136) and Rankine-Hugoniot conditions across $l_1(t)$ and $l_2(t)$.

EXAMPLE 4.1 (Counter Example) : Let $\alpha = 0, x_k < 0, \lim_{k \rightarrow \infty} x_k = 0, \lambda > \theta_f$ and define u_0, u_0^k by

$$u_0(x) = \begin{cases} \theta_f & \text{if } x < 0, \\ \lambda & \text{if } x > 0. \end{cases}$$

$$u_0^k(x) = \begin{cases} \theta_f & \text{if } x < x_k, \\ \lambda & \text{if } x > x_k. \end{cases}$$

Then the solution u and u_k with respective initial datas u_0 and u_0^k are given by

$$u(x, t) = \begin{cases} \theta_f & \text{if } x < 0, t > 0, \\ (f')^{-1}(\frac{x}{t}) & \text{if } 0 < x \leq f'(\lambda)t, \\ \lambda & \text{if } x > f'(\lambda)t, \end{cases}$$

then

$$R_-(t, 0, u_0) = 0.$$

$$u_k(x, t) = \begin{cases} \theta_f & \text{if } x < x_k, t > 0, \\ (f')^{-1}(\frac{x-x_k}{t}) & \text{if } x_k < x < f'(\lambda)t + x_k, \\ \lambda & \text{if } x > f'(\lambda)t + x_k, \end{cases}$$

then

$$R_-(t, 0, u_0^k) = f'(\lambda)t,$$

$$\int_{\mathbb{R}} |u_0(x) - u_0^k(x)| dx = \int_{x_k}^0 (\lambda - \theta_f) = (\lambda - \theta_f)|x_k| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

But

$$\lim_{k \rightarrow \infty} R_-(t, 0, u_0^k) = f'(\lambda)t > 0 = R_-(t, 0, u_0).$$

EXAMPLE 4.2 (Counter Example) : Let $A_1 = B_1 = C_1, A_2 = B_2 = C_2, \rho(x) = x$ for $x \in (C_1, C_2)$ and

$$u_0(x) = \begin{cases} a_2 & \text{if } x > B_2 \\ a_1 & \text{if } x < B_1, \end{cases}$$

where $a_2 < \theta_f$ and θ_f is the point of minima of f .

Suppose there exists a solution $(u, \bar{u}_0)$ to problem (I), then by Lax-Oleinik formula we have

$$\bar{u}_0(x) = \theta_f \quad \text{if } x \in (B_1, B_2)$$

$$u(x, t) = \theta_f \quad \text{if } (x, t) \in (B_1, B_2) \times (0, T).$$

On the otherhand, since $a_2 < \theta_f$ there is a shock wave entering the region $(B_1, B_2) \times (0, T)$ at $(B_2, 0)$ which is a contradiction because the solution $u = \theta_f$ in this region.

5 Extensions:

PROPOSITION 5.1 (*Controllability of constant states*):

1. In theorem 1.1, $g(x) = m$ a constant if and only if m satisfies

$$\frac{C_2 - A_2}{T} \leq f'(m) \leq \frac{C_1 - A_1}{T}. \quad (5.1)$$

2. In theorem 1.2, $g(x_0) = m$ a constant if and only m satisfies

$$f'(m) \geq \frac{C}{T - \delta}. \quad (5.2)$$

3. In theorem 1.3, $g_1(x) = m_1, g_2(x) = m_2$ are constants. then g_1, g_2 is controllable if and only if m_1, m_2 satisfies

$$f'(m_1) \geq \frac{C - A_1}{T - \delta}, f'(m_2) \leq \frac{A_2 - C}{T - \delta}. \quad (5.3)$$

Proof. (1). $g(x) = m$ if and only if $\rho(x) = x - Tf'(m)$ for all $x \in (C_1, C_2)$. Hence from (1.22) we have $A_1 \leq \rho(x) \leq A_2$ implies that $\frac{x-A_2}{T} \leq f'(m) \leq \frac{x-A_1}{T}$ and hence (5.1) holds.

(2). From (1.23), $g(x) = m$ if and only if $\delta \leq \rho(x) \leq T$ and hence $\delta \leq x - Tf'(m) \leq T$. This implies (5.2). Similarly (5.3) follows from (1.25) and (1.26). This proves the theorem.

(3). Follows similarly.

5.1 Controllability on the boundary

As mentioned in the introduction problems (I) and (III) deal with the controllability at time $t = T$. What about the controllability at $x = A_2$. More precisely

Problem (IV): Let $T > 0$ and $A_1 < A_2$. Given $u_0 \in L^\infty(\mathbb{R})$, $g \in L^\infty(0, T)$ find $\bar{u}_0 \in L^\infty((A_1, A_2))$ and u a solution of (1.2) in $\Omega = (-\infty, A_2) \times (0, T)$ such that

$$f'(u(A_2, t)) = g(t) \quad \text{if } 0 < t \leq T, \quad (5.4)$$

and

$$u(x, 0) = \begin{cases} u_0(x) & \text{if } x < A_1 \\ \bar{u}_0(x) & \text{if } A_1 < x < A_2. \end{cases} \quad (5.5)$$

Then we have the following

THEOREM 5.1 : Let $A_1 < B < A_2, \wedge > 0$ and $\rho : [0, T] \rightarrow [B, A_2]$ be a non increasing left continuous function such that for all $t \in [0, T]$,

$$\left| \frac{A_2 - \rho(t)}{t} \right| \leq \wedge, \quad (5.6)$$

and $f'(g(t)) = \frac{A_2 - \rho(t)}{t}$. Then there exist $(u, \bar{u}_0)$ satisfying (5.4) and (5.5).

Proof. Proof follows on the same lines as in theorem (1.2) and hence only sketch the main idea of the proof.

Step 1. This step is analogous to Lemma 2.1. Frist assume that ρ is discrete. That is there exist a partition $0 = t_n \leq t_{n-1} \leq \dots \leq t_0 = T$ and $B = x_0 < x_1 < \dots < x_n = A_2$. Define a_i and b_i by

$$\begin{aligned} f'(a_i) &= \frac{A_2 - x_i}{t_i}, f'(b_i) = \frac{A_2 - x_{i-1}}{t_i} \\ s_i(t) &= A_2 + (t - t_i) \frac{f(a_i) - f(b_i)}{a_i - b_i} \\ f'(a_i) &= \frac{A_2 - x_i}{t_i} > \frac{A_2 - x_{i-1}}{t_i} = f'(b_i). \end{aligned}$$

Then

$$f'(b_i) = \frac{A_2 - x_{i-1}}{t_i} < \frac{A_2 - x_{i-1}}{t_{i-1}} = f'(a_{i-1}).$$

Hence $a_i > b_i, a_{i+1} > b_i$ and from convexity. $f'(a_i) > \frac{f(a_i) - f(b_i)}{a_i - b_i} > f'(b_i)$. Therefore

$$\begin{aligned} x_i = A_2 - t f'(a_i) &< A_2 - t \frac{f(a_i) - f(b_i)}{a_i - b_i} \\ &= s_i(0) < A_2 - t f'(b_i) \\ &= x_{i+1}. \end{aligned}$$

Hence for $0 \leq t \leq T$,

$$l_i(t) \leq s_i(t) \leq m_i(t),$$

where $l_i(t) = x_i + f'(a_i)t, m_i(t) = x_{i-1} + f'(b_i)t$. Define ρ_n and g_n by

$$\begin{aligned} \rho_n(t) &= x_0 \chi_{[T, t_1]} + \sum_{i=1}^n x_i \chi_{(t_i, t_{i+1}]}(t) \\ f'(g_n(t)) &= \frac{A_2 - \rho_n(t)}{t}. \end{aligned}$$

Define u_n in $\Omega = (-\infty, A_2) \times (0, T)$ by

$$f'(u_n(x, t)) = \begin{cases} a_n & \text{if } x \leq l_n(t), \\ a_i & \text{if } l_i(t) \leq x < s_i(t), \\ b_i & \text{if } s_i(t) < x \leq m_i(t), \\ (f')^{-1}\left(\frac{A_2 - x_i}{t}\right) & \text{if } m_i(t) \leq x \leq l_{i-1}(t), \end{cases}$$

then u_n is a solution of (1.2) in Ω satisfying

$$\begin{aligned} f'(u_n(x, t)) &= a_n \quad \text{if } x \leq l_n(t) = B + t\left(\frac{A_2 - B}{T}\right) \\ f'(u_n(A_2, t)) &= \frac{x - \rho_n(t)}{t}. \end{aligned}$$

Let $\bar{u}_{n,0}(x) = u_n(x, 0)$ for $B \leq x \leq A_2$, then as in the proof of Lemma 2.1 and from (5.6), for a subsequence $u_n \rightarrow \tilde{u}$ in $L^1_{loc}(\Omega)$, $u_n(\cdot, 0) \rightarrow \tilde{u}_0$ in $L^1((B, A_2))$, $\rho_n \rightarrow \rho$ a. e. such that u satisfies (1.2) and for a.e. t ,

$$f'(\tilde{u}(A_2, t)) = \frac{A_2 - \rho(t)}{t} \quad \text{if } t \in (0, T), \quad (5.7)$$

$$\tilde{u}(x, 0) = \tilde{u}_0(x) \quad \text{if } x \in (B, A_2), \quad (5.8)$$

$$f'(\tilde{u}(x, t)) = \frac{A_2 - B}{T} \quad \text{if } x \leq l_0(t). \quad (5.9)$$

Step 2. From Lemma 4.8 there exists a μ and a solution u_1 of (1.2) in Ω satisfying

$$f'(u_1(x, t)) = a_0 \quad \text{if } x > l_0(t), \quad 0 \leq t < T \quad (5.10)$$

$$u_1(x, 0) = \begin{cases} a_0 & \text{if } x > B, \\ \mu & \text{if } A_1 < x < B, \\ u_0(x) & \text{if } x < A_1. \end{cases}$$

Now define $(u, \bar{u}_0)$ in Ω by

$$u(x, t) = \begin{cases} \tilde{u}(x, t) & \text{if } x > l_1(t), \\ u_1(x, t) & \text{if } x < l_0(t), \end{cases}$$

$$\bar{u}_0(x) = \begin{cases} u_0(x) & \text{if } x < A_1, \\ \mu & \text{if } A_1 < x < B, \\ \tilde{u}_0(x) & \text{if } x \in (B, A_2). \end{cases}$$

Then $(u, \bar{u}_0)$ is the required solution to problem (IV).

5.2 Controllability of initial and boundary values:

All three problems deals with finding either initial data or purely boundary data. In fact one can combine both and is as follows.

Problem V: Let $u_0 \in L^\infty$, $T > 0$, $0 < C_1 < C_2$, $0 < A$, Let $\rho_1 : [0, C_1] \rightarrow [0, T]$, $\rho_2 : [C_1, C_2] \rightarrow [0, A]$ be such that

- (i) ρ_1 is a non increasing right continuous function.
- (ii) ρ_2 is a non decreasing function.

Define g_1 and g_2 by

$$f'(g_1(x)) = \frac{x}{T - \rho_1(x)} \quad \text{if } x \in [0, C_1]$$

$$f'(g_2(x)) = \frac{x - \rho_2(x)}{T} \quad \text{if } x \in [C_1, C_2].$$

Then the problem is to find $b \in L^\infty(0, T)$ and $\bar{u}_0 \in L^\infty(0, A)$ such that a solution u of (1.2) in $\mathbb{R} \times (0, T)$ satisfying the following initial boundary data

$$u(0, t) = b(t) \quad \text{if } 0 < t < T. \quad (5.11)$$

$$u(x, 0) = \begin{cases} \bar{u}_0(x) & \text{if } x \in (0, A), \\ u_0(x) & \text{if } x \in (A, 0), \end{cases} \quad (5.12)$$

and

$$f'(u(x, t)) = \begin{cases} g_1(x) & \text{if } x \in (0, C_1), \\ g_2(x) & \text{if } x \in (C_1, C_2). \end{cases} \quad (5.13)$$

THEOREM 5.2 *Let $\lambda > 0, 0 < A_1 < A$ be given. Let ρ_1 and ρ_2, g_1 and g_2 be as above and satisfying*

$$0 \leq \rho_2(x) \leq A_1, \quad \left| \frac{x}{T - \rho_1(x)} \right| \leq \wedge \quad (5.14)$$

then problem (V) admits a solution.

Idea of the proof. First get a free region by choosing λ large such that the solution u_λ of (1.2) in $\mathbb{R} \times (0, \infty)$ satisfying for $0 < t < T$,

$$u_\lambda(x, t) = a_1 = \frac{C_2 - A_1}{T}, \quad \text{if } x < A_1 + t f'(a_1) = l_1(t),$$

$$u_\lambda(x, 0) = \begin{cases} a_1 & \text{if } x < A_1, \\ \lambda & \text{if } A_1 < x < A, \\ u_0(x) & \text{if } x > A. \end{cases}$$

Existence of u_λ is guaranteed from Lemma 4.7. Let $f'(a_0) = \frac{C_1}{T}$ and for $0 < t < T$ define the free region F_1 and F_2 by

$$F_1 = \{(x, t) : 0 < x < l_0(t) = t f'(a_0)\}, F_2 = \{(x, t) : l_0(t) < x < l_1(t) = A_1 + t f'(a_1)\}.$$

Since $0 \leq \rho_1(x) \leq T$ for $x \in (0, C_1)$ and satisfying (5.14), therefore from Lemma 4.1, there exist a solution u_1 of (1.2) in F_1 and $b \in L^\infty(0, T)$ such that

$$\begin{aligned} u_1(0, t) &= b(t) \\ u_1(x, T) &= g_1(x) \quad \text{if } x \in (0, C_1) \\ u_1(l_0(t)-, t) &= a_0. \end{aligned}$$

From Lemma 4.2, there exist a solution u_2 of (1.2) in F_2 and $\tilde{u}_0 \in L^\infty(0, A_1)$ such that

$$\begin{aligned} u_2(x, T) &= g_2(x) \quad \text{if } x \in (C_1, C_2) \\ u_2(x, 0) &= \tilde{u}_0(x) \quad \text{if } x \in (0, A_1) \\ u_2(l_0(t)+, t) &= a_0, \quad u_2(l_1(t)-, t) = a_1. \end{aligned}$$

From RH conditions, glue u_1, u_2, u_λ to a single solution u of (1.2) in $(0, \infty) \times (0, T)$ by

$$u(x, t) = \begin{cases} u_1(x, t) & \text{if } (x, t) \in F_1, \\ u_2(x, t) & \text{if } (x, t) \in F_2, \\ u_\lambda(x, t) & \text{if } x > l_1(t), \end{cases}$$

and

$$u(x, 0) = \begin{cases} \tilde{u}_0(x) & \text{if } x \in (0, A_1), \\ \lambda & \text{if } x \in (A_1, A), \\ u_0(x) & \text{if } x \in (A, \infty). \end{cases}$$

Then $(u, u(x, 0), b)$ is the required solution to problem (V) The same method allows to generalize problem III also.

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