Polygraphs of finite derivation type

Yves Guiraud, Philippe Malbos

POLYGRAPHS OF FINITE DERIVATION TYPE

YVES GUIRAUD PHILIPPE MALBOS

Abstract – Craig Squier proved that, if a monoid can be presented by a finite convergent string rewriting system, then it satisfies the homological finiteness condition left-FP_3. Using this result, he constructed finitely presentable monoids with a decidable word problem, but that cannot be presented by finite convergent rewriting systems. Later, he introduced the condition of finite derivation type, which is a homotopical finiteness property on the presentation complex associated to a monoid presentation. He showed that this condition is an invariant of finite presentations and he gave a constructive way to prove this finiteness property based on the computation of the critical branchings: being of finite derivation type is a necessary condition for a finitely presented monoid to admit a finite convergent presentation. This survey presents Squier's results in the contemporary language of polygraphs and higher-dimensional categories, with new proofs and relations between them.

Keywords – higher-dimensional categories, higher-dimensional rewriting, finite derivation type, low-dimensional homotopy.

M.S.C. 2000 – 68Q42, 03D05, 18D05.

1. Introduction

Given a monoid $\mathbf{M}$, a generating set Σ_1 for $\mathbf{M}$ provides a way to represent the elements of $\mathbf{M}$ in the free monoid Σ_1^*, i.e. as finite words written with the elements of Σ_1. But, in general, an element of $\mathbf{M}$ has several representatives in Σ_1^*. The *word problem* for $\mathbf{M}$ consists in finding a generating set Σ_1 and a procedure that can determine whether or not any two elements of Σ_1^* represent the same element in the monoid $\mathbf{M}$.

The word problem and convergent presentations. One way to solve the word problem is to exhibit a finite presentation $\Sigma = (\Sigma_1, \Sigma_2)$ of $\mathbf{M}$, made of a generating set Σ_1 and a set Σ_2 of directed relations with a good computational property: *convergence*. Indeed, in rewriting theory, one studies presentations where the relations in Σ_2 are not seen as equalities between the words in Σ_1^*, such as $u = v$, but, instead, as *rewriting rules* that can only be applied in one direction, like $u \Rightarrow v$, thus simulating a non-reversible computational process *reducing* the word u into the word v.

In rewriting theory, such a presentation Σ of a monoid is called a *string rewriting system* or, historically, a *semi-Thue system*; in that case, the directed relations of Σ_2 are called *rewriting rules*. A presentation Σ is *convergent* if it has the two properties of

- *termination*, i.e. all the computations end eventually, and
- *confluence*, i.e. different computations on the same input lead to the same result.

The *monoid presented by* Σ is defined as the quotient, denoted by $\overline{\Sigma}$, of the free monoid Σ_1^* over Σ_1 by the congruence generated by Σ_2. By extension, we say that Σ presents any monoid isomorphic to $\overline{\Sigma}$.

A finite and convergent presentation Σ of a monoid $\mathbf{M}$ gives a solution to the word problem, called the *normal-form procedure* and defined as follows. Given an element u of the free monoid Σ_1^*, convergence ensures that all the applications of (directed) relations to u, in every possible manner, will eventually produce a unique result: an element $\widehat{u}$ of Σ_1^* where no relation applies anymore. The word $\widehat{u}$ is called the *normal form of* u. By construction, two elements u and v of Σ_1^* represent the same element of $\mathbf{M}$ if, and only if, their normal forms are equal in Σ_1^*. Finiteness ensures that one can determine whether an element of Σ_1^* is a normal form or not, by examining all the relations.

Rewriting and polygraphs. The notion of string rewriting system comes from combinatorial algebra. It was introduced by Axel Thue in 1914 in order to solve the word problem for finitely presented semigroups [Thu14]. It is only in 1947 that the problem was shown to be undecidable, independently by Emil Post [Pos47] and Andrei Markov [Mar47a, Mar47b]. Then in 1943, Maxwell Newman gave a general setting, the abstract rewriting theory, to describe the properties of termination and confluence, and to show the first fundamental result of rewriting: Newman's lemma [New42]. Since then, rewriting theory has been mainly developed in theoretical computer science, producing numerous variants corresponding to different syntaxes of the formulas being transformed: string, terms, terms modulo, λ-terms, term-graphs, etc. Rewriting is also present in other computational formalisms such as Petri nets or logical systems.

More recently, higher-dimensional rewriting has unified several paradigms of rewriting. This approach is based on presentations by generators and relations of higher-dimensional categories, independently introduced by Albert Burroni and Ross Street under the respective names of *polygraphs* in [Bur93] and *computads* in [Str76, Str87]. Those algebraic objects have been subsequently developed in rewriting

theory, fixing the terminology to polygraph in that field [Mét03, Gui06, Laf07, Mét08, GM09, Mim10, GM11, GM12a, GM12b, GMM13, GGM].

The main useful property of polygraphs is to encapsulate, in the same globular object, an algebraic structure corresponding to the syntax of the terms and to the computations on the terms, together with a homotopical structure describing the properties of the computations. As a consequence, polygraphs provide a natural setting to formulate Squier's theory, based on the discovery of deep relations between the computational, the homological and the homotopical properties of presentations by generators and relations.

From computational to homological properties. The normal-form procedure proves that, if a monoid admits a finite convergent presentation, then it has a decidable word problem. The converse implication was still an open problem in the middle of the eighties:

> **Question. [Jan82, Jan85]** *Does every finitely presented monoid with a decidable word problem admit a finite convergent presentation?*

In [KN85], Deepak Kapur and Paliath Narendran consider Artin's presentation of the monoid $\mathbf{B}_3^+$ of positive braids on three strands

$$\langle s, t \mid sts \Rightarrow tst \rangle.$$

Kapur and Narendran proved that $\mathbf{B}_3^+$ admits no finite convergent presentation on the two generators s and t. However, they also proved one gets a finite convergent presentation of $\mathbf{B}_3^+$ by adjunction of a new generator a standing for the product st:

$$\left\langle s, t, a \;\middle|\; ta \overset{\alpha}{\Longrightarrow} as,\ st \overset{\beta}{\Longrightarrow} a,\ sas \overset{\gamma}{\Longrightarrow} aa,\ saa \overset{\delta}{\Longrightarrow} aat \right\rangle. \tag{1}$$

As a consequence, the word problem for $\mathbf{B}_3^+$ is solvable. The result of Kapur and Narendran shows that the existence of a finite convergent presentation depends on the chosen generators. Thus, to provide the awaited negative answer to the open question, one would have to exhibit a monoid with a decidable word problem but with no finite convergent presentation *for any possible set of generators*: new methods had to be introduced.

And, indeed, Craig Squier answered the question by linking the existence of a finite convergent presentation for a given monoid $\mathbf{M}$ to an invariant of the monoid: the *homological type left-*FP_3. Here, invariant is to be taken in the sense that it is independent of the choice of a presentation of $\mathbf{M}$ and, in particular, of a generating set. A monoid $\mathbf{M}$ is of homological type left-FP_3 if there exists an exact sequence

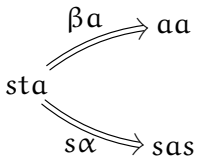
$$P_3 \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

of projective and finitely generated (left) $\mathbb{Z}\mathbf{M}$-modules, where $\mathbb{Z}$ denotes the trivial $\mathbb{Z}\mathbf{M}$-module. From a presentation Σ of a monoid $\mathbf{M}$, one can build an exact sequence of free $\mathbb{Z}\mathbf{M}$-modules

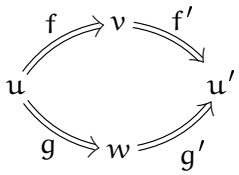
$$\mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{2}$$

where $\mathbb{Z}\mathbf{M}[\Sigma_k]$ is the free $\mathbb{Z}\mathbf{M}$-module over Σ_k. In [Squ87], Squier proved that, when Σ is convergent, its *critical branchings* form a generating set of the kernel of d_2, where a critical branching of Σ is a minimal

overlapping application of two relations on the same word of Σ_1^*. For example, the relations $\alpha : ta \Rightarrow as$ and $\beta : st \Rightarrow a$ generate a critical branching $(\beta a, s\alpha)$ on sta:

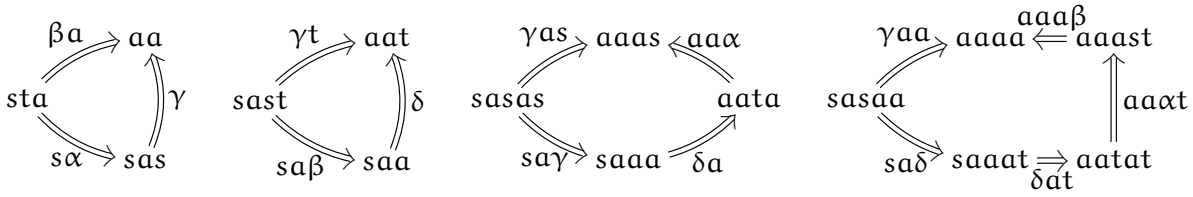


The convergence of Σ ensures that every critical branching (f, g) is confluent, that is, it can be completed by rewriting sequences f' and g' as in



(3)

For example, the presentation (1) of $\mathbf{B}_3^+$ has four critical branchings, and all of them are confluent:



Squier proved that the set Σ_3 of critical branchings of a convergent presentation Σ extends the exact sequence (2) by one step:

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0, \tag{4}$$

where the boundary map d_3 is defined on the generic branching (3) by

$$d_3(f, g) \;=\; [f] - [g] + [f'] - [g'],$$

where $[\cdot]$ satisfies

$$[\, u_1\alpha_1v_1 \Rightarrow \cdots \Rightarrow u_n\alpha_nv_n \,] \;=\; \overline{u}_1[\alpha_1] + \cdots + \overline{u}_n[\alpha_n].$$

Moreover, when the presentation Σ is finite, then its set of critical branchings is finite, yielding *Squier's homological theorem*.

5.3.5. Theorem [Squ87, Theorem 4.1] *If a monoid admits a finite convergent presentation, then it is of homological type left-*FP_3.

Finally, Squier considers in [Squ87] the family $\mathbf{S}_k$ of monoids, for $k \geqslant 2$, presented by

$$\left\langle\, a, b, t, (x_i)_{1\leqslant i\leqslant k}, (y_i)_{1\leqslant i\leqslant k} \;\middle|\; \alpha,\ (\beta_i)_{1\leqslant i\leqslant k},\ (\gamma_i)_{1\leqslant i\leqslant k},\ (\delta_i)_{1\leqslant i\leqslant k},\ (\varepsilon_i)_{1\leqslant i\leqslant k} \,\right\rangle$$

with

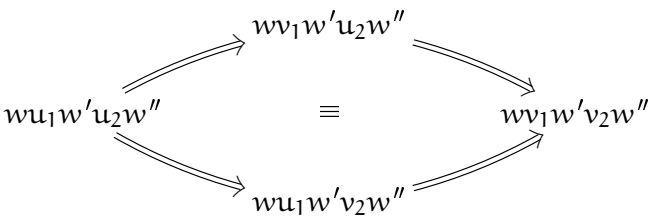
$$ab \overset{\alpha}{\Longrightarrow} 1\,, \qquad x_i a \overset{\beta_i}{\Longrightarrow} a t x_i\,, \qquad x_i t \overset{\gamma_i}{\Longrightarrow} t x_i\,, \qquad x_i b \overset{\delta_i}{\Longrightarrow} b x_i\,, \qquad x_i y_i \overset{\varepsilon_i}{\Longrightarrow} 1.$$

Each $\mathbf{S}_k$ is a finitely generated monoid with a decidable word problem. But, for $k \geqslant 2$, the monoid $\mathbf{S}_k$ is not of homological type left-FP_3 and, as a consequence, it does not admit a finite convergent presentation. Thus, Squier gave a negative answer to the open question: there exist finitely generated monoids with a decidable word problem that do not admit a finite convergent presentation (for any possible finite set of generators).

Finite derivation type. Given a monoid $\mathbf{M}$ with a presentation Σ, Squier considers in [SOK94] the presentation complex of Σ, that is a cellular complex with one 0-cell, whose 1-cells are the elements of the free monoid Σ_1^* and whose 2-cells are generated by the relations of Σ_2. More precisely, the 2-cells of the presentation complex are constructed as follows. We denote by Σ_2^- the set obtained by reversing the relations of Σ_2:

$$\Sigma_2^- \;=\; \left\{\, v \overset{\alpha^-}{\Longrightarrow} u \;\text{ such that }\; u \overset{\alpha}{\Longrightarrow} v \text{ is in } \Sigma_2 \,\right\}.$$

There is a 2-cell in the presentation complex between each pair of words with shape wuw' and wvw' such that $\Sigma_2 \amalg \Sigma_2^-$ contains a relation $u \Rightarrow v$. Then, Squier extends this 2-dimension complex with 3-cells filling all the squares formed by independent applications of relations, such as the following one, where $u_1 \Rightarrow v_1$ and $u_2 \Rightarrow v_2$ are in $\Sigma_2 \amalg \Sigma_2^-$:



We will see that, in the 2-categorical formulation of this complex that we consider, this 3-cell corresponds to the so-called exchange relations.

A *homotopy basis of* Σ is a set Σ_3 of additional 3-cells that makes Squier's complex aspherical, i.e. such that every 2-dimensional sphere can be "filled" by the 3-cells of Σ_3. The presentation Σ is of *finite derivation type* if it admits a finite homotopy basis. Squier proved that the finite derivation type property is an intrinsic property of the presented monoid:

> **4.2.3. Theorem [SOK94, Theorem 4.3]** *Let Σ and Ξ be two finite presentations of the same monoid. Then Σ has finite derivation type if and only if Ξ has finite derivation type.*

The proof given by Squier is based on Tietze transformations. In these notes, we give another proof based on a homotopy bases transfer theorem, Theorem 4.2.2. As a consequence of Theorem 4.2.3, we can define the condition of finite derivation type for monoids independently of a considered presentation: a monoid is of *finite derivation type* if its finite presentations are of finite derivation type.

From computational to homotopical properties. In [SOK94], Squier links the existence of a finite convergent presentation to the homotopical property of finite derivation type. He proves that, given a convergent presentation Σ, it is sufficient to consider one 3-cell filling the diagram (3) for each critical branching to get a homotopy basis of Σ. Such a set of 3-cells is called a *family of generating confluences of* Σ.

> **4.3.2. Theorem [SOK94, Theorem 5.2]** *Let Σ be a convergent presentation of a monoid. Every family of generating confluences of Σ is a homotopy basis.*

Moreover, if Σ is finite, the presentation Σ has finitely many critical branchings.

> **4.3.3. Theorem [SOK94, Theorem 5.3]** *If a monoid admits a finite convergent presentation, then it is of finite derivation type.*

In [SOK94], Squier used Theorem 4.3.3 to give another proof that there exist finitely generated monoids with a decidable word problem that do not admit a finite convergent presentation. Moreover, he showed that the homological finiteness condition left-FP_3 is not sufficient for a finitely presented monoid with a decidable word problem to admit a finite convergent presentation. Indeed, the monoid $\mathbf{S}_1$ given by the presentation

$$\langle\, a, b, t, x, y \;\mid\; ab \Rightarrow 1,\ xa \Rightarrow atx,\ xt \Rightarrow tx,\ xb \Rightarrow bx,\ xy \Rightarrow 1 \,\rangle$$

has a decidable word problem, admits a finite presentation and is of homological type left-FP_3, yet it is not of a finite derivation type (and, thus, it does not admit a finite convergent presentation). This example is entirely developed in Section 6.1.

Refinements of Squier's conditions. Squier's homological and homotopical finiteness conditions are related: finite derivation type implies homological type left-FP_3, as proved by several authors [CO94, Pri95, Laf95]. The converse implication is false in general, as already noted above with the monoid $\mathbf{S}_1$, but it is true in the special case of groups [CO96]. Squier has proved in [SOK94] that the invariant homological type left-FP_3 is a necessary, but not sufficient condition for a monoid to admit a finite convergent presentation, the same question being still open for the homotopical invariant finite derivation type. After Squier, various refinements of both invariants have been explored, in the quest for a complete characterisation of the existence of finite convergent presentations of monoids.

In the homological direction, thanks to the notion of abelian resolution, one defines the more restrictive conditions homological type left-FP_n, for every natural number $n > 3$, and homological type left-FP_∞: a monoid $\mathbf{M}$ has homological type left-FP_∞ if there exists a resolution of the trivial $\mathbb{Z}\mathbf{M}$-module by finitely generated and projective $\mathbb{Z}\mathbf{M}$-modules. In [Kob90], a notion of n-fold critical branching is used to complete the exact sequence (4) into a resolution, obtaining the following implication: if a monoid admits a finite convergent presentation, then it is of homological type left-FP_∞, the converse implication still being false in general. The same results are also known for associative algebras presented by a finite Gröbner basis [Ani86] and for groups [Coh92, Bro92, Gro90]. One can obtain similar implications with the properties right-FP_∞ and bi-FP_∞, defined with resolutions by right modules and bimodules, respectively. In [GM12b], the authors give a construction of a resolution involving n-fold critical branchings and based on the notion of normalisation strategy.

In the homotopical direction, the condition finite derivation type has been refined into FDT_4, a property about the existence of a finite presentation with a finite homotopy basis, itself satisfying a homotopical finiteness property [MPP05]. The condition FDT_4 is also necessary for a monoid to admit a finite convergent presentation and it is sufficient, but not necessary, for having the conditions homological type left/right/bi-FP_4. Higher-dimensional finite derivation type properties, called FDT_n, are defined in [GM12b], as a generalisation in any dimension of Squier's finite derivation type. A monoid with a finite convergent presentation is FDT_∞ and, for any n, the property FDT_n implies the homological type FP_n [GM12b].

Organisation and prerequisites. In Section 2, we consider presentations of monoids (and, more generally, of categories) by 2-polygraphs, and we explain their main rewriting properties in Section 3. In Section 4, we introduce the property of finite derivation type for categories using the structure of 3-polygraph and, in Section 4.3, we relate convergence and finite derivation type. This survey is rather self-contained, but wider categorical notions are covered in more detail by Mac Lane in [ML98] and by Barr and Wells in [BW90]. For notions of rewriting theory, one can refer to Baader and Nipkow [BN98], Terese [Ter03] and Book and Otto [BO93] for the special case of string rewriting. For extension of the finite derivation type property to higher-dimensional rewriting systems, we refer the reader to [GM09].

2. Low-dimensional categories and polygraphs

2.1. Categories and functors

2.1.1. Categories. A *category* (or 1-category) is a data $\mathbf{C}$ made of a set $\mathbf{C}_0$, whose elements are called the 0-*cells of* $\mathbf{C}$, and, for all 0-cells x and y of $\mathbf{C}$, of a set $\mathbf{C}(x,y)$, whose elements are called the 1-*cells from* x *to* y *of* $\mathbf{C}$. Those sets are equipped with the following algebraic structure:

– for all 0-cells x, y and z of $\mathbf{C}$, a map, called the *composition* (or 0-composition) of $\mathbf{C}$,

$$\gamma_{x,y,z} \,:\, \mathbf{C}(x,y) \times \mathbf{C}(y,z) \;\to\; \mathbf{C}(x,z),$$

– for every 0-cell x of $\mathbf{C}$, a specified element 1_x of $\mathbf{C}(x,x)$, called the *identity of* x.

The following relations are required to hold, where we write $u : x \to y$ to mean that u belongs to $\mathbf{C}(x,y)$:

– the composition is associative, i.e. for all 0-cells x, y, z and t and for all 1-cells $u : x \to y$, $v : y \to z$ and $w : z \to t$ of $\mathbf{C}$,

$$\gamma_{x,z,t}(\gamma_{x,y,z}(u,v),w) \;=\; \gamma_{x,y,t}(u,\gamma_{y,z,t}(v,w)),$$

– the identities are local units for the composition, i.e. for all 0-cells x and y and for every 1-cell $u : x \to y$ of $\mathbf{C}$,

$$\gamma_{x,x,y}(1_x,u) \;=\; u \;=\; \gamma_{x,y,y}(u,1_y).$$

A *groupoid* is a category where every 1-cell has an inverse, that is, for every 1-cell $u : x \to y$, there exists a (necessarily unique) 1-cell $u^- : y \to x$ such that

$$\gamma_{x,y,x}(u,u^-) \;=\; 1_x \qquad \text{and} \qquad \gamma_{y,x,y}(u^-,u) \;=\; 1_y.$$

Monoids (resp. groups) correspond exactly to categories (resp. groupoids) with only one 0-cell.

2.1.2. A globular point of view. A category $\mathbf{C}$ can also be seen as a graph

$$\mathbf{C}_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \mathbf{C}_1$$

where $\mathbf{C}_1$ is the disjoint union of all the hom-sets $\mathbf{C}(x, y)$, and the maps s_0 and t_0 send a 1-cell $u : x \to y$ to its *source* x and its *target* y, respectively. We usually simply write $s(u)$ and $t(u)$ instead of $s_0(u)$ and $t_0(u)$. The composition of $\mathbf{C}$ equips this graph with a partial function

$$\star_0 : \mathbf{C}_1 \times \mathbf{C}_1 \to \mathbf{C}_1$$

mapping a pair (u, v) of *composable* 1-cells (i.e. such that $t(u) = s(v)$) to $u \star_0 v$ (we often simply write uv). By definition, the source and target of a composite 1-cell are given by

$$s(u \star_0 v) = s(u) \qquad \text{and} \qquad t(u \star_0 v) = t(v).$$

Moreover, the associativity axiom is written as

$$(u \star_0 v) \star_0 w \;=\; u \star_0 (v \star_0 w).$$

The identities define an inclusion

$$\mathbf{C}_0 \rightarrowtail \mathbf{C}_1$$

that maps each 0-cell x to the 1-cell 1_x. By definition, the source and target of an identity 1-cell are

$$s(1_x) \;=\; x \qquad \text{and} \qquad t(1_x) \;=\; x.$$

Finally, the unit axioms become

$$1_x \star_0 u \;=\; u \;=\; u \star_0 1_y.$$

This globular definition of categories is equivalent to the original definition given in 2.1.1.

2.1.3. Functors. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. A *functor* $F : \mathbf{C} \to \mathbf{D}$ is a data made of a map $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$ and, for all 0-cells x and y of $\mathbf{C}$, a map

$$F_{x,y} \;:\; \mathbf{C}(x, y) \;\to\; \mathbf{D}(F(x), F(y)),$$

such that the following relations are satisfied:

– for all 0-cells x, y and z and all 1-cells $u : x \to y$ and $v : y \to z$ of $\mathbf{C}$,

$$F_{x,z}(u \star_0 v) \;=\; F_{x,y}(u) \star_0 F_{y,z}(v),$$

– for every 0-cell x of $\mathbf{C}$,

$$F_{x,x}(1_x) \;=\; 1_{F(x)}.$$

We often just write $F(x)$ for $F_0(x)$ and $F(u)$ for $F_{x,y}(u)$. A functor F is an *isomorphism* if the map F_0 and each map $F_{x,y}$ is a bijection. We denote by $\mathrm{Id}_{\mathbf{C}}$ the identity functor on $\mathbf{C}$. Morphisms of monoids are exactly the functors between the corresponding categories with one 0-cell.

2.2. The word problem

2.2.1. 1-polygraphs. A 1-*polygraph* is a graph Σ, i.e. a diagram of sets and maps

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1.$$

The elements of Σ_0 and Σ_1 are called the 0-*cells* and the 1-*cells* of Σ, respectively. If there is no confusion, we just write $\Sigma = (\Sigma_0, \Sigma_1)$. A 1-polygraph is *finite* if it has finitely many 0-cells and 1-cells.

2.2.2. Free categories. If Σ is a 1-polygraph, the *free category over* Σ is the category denoted by Σ^* and defined as follows:

- the 0-cells of Σ^* are the ones of Σ,
- the 1-cells of Σ^* from x to y are the finite paths of Σ, i.e. the finite sequences

  $$x \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} \cdots \xrightarrow{u_{n-1}} x_{n-1} \xrightarrow{u_n} y$$

  of 1-cells of Σ,
- the composition is given by concatenation,
- the identities are the empty paths.

If Σ has only one 0-cell, then the 1-cells of the free category Σ^* form the free monoid over the set Σ_1.

2.2.3. Generating 1-polygraph. Let $\mathbf{C}$ be a category. A 1-polygraph Σ *generates* $\mathbf{C}$ if there exists a functor

$$\pi : \Sigma^* \twoheadrightarrow \mathbf{C}$$

that is the identity on 0-cells and such that, for all 0-cells x and y of $\mathbf{C}$, the map

$$\pi_{x,y} : \Sigma^*(x,y) \twoheadrightarrow \mathbf{C}(x,y)$$

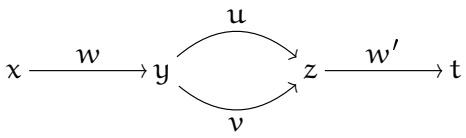
is surjective. We usually consider that π is implicitly specified for a given generating 1-polygraph Σ and, if u is a 1-cell of Σ^*, we just write $\overline{u}$ instead of $\pi(u)$. A category is *finitely generated* if it admits a finite generating 1-polygraph (in particular, the category must have finitely many 0-cells).

2.2.4. The word problem for categories. Let $\mathbf{C}$ be a category. The *word problem for* $\mathbf{C}$ is the problem of finding a generating 1-polygraph Σ for $\mathbf{C}$ together with an algorithm that decides, for any two 1-cells u and v of Σ^*, whether or not $\overline{u} = \overline{v}$ holds in $\mathbf{C}$ (that is, whether or not the 1-cells u and v represent the same 1-cell of $\mathbf{C}$). We note that, to have $\overline{u} = \overline{v}$, it is necessary for u and v to be *parallel*, i.e. they must have the same source and the same target. The word problem is undecidable in general for a given category $\mathbf{C}$, even if it is finitely generated. However, a finite convergent presentation of $\mathbf{C}$, see 3.1.7, provides a solution to the word problem for $\mathbf{C}$.

2.3. Presentations of categories

2.3.1. Spheres and cellular extensions of categories. A 1-*sphere* of a category $\mathbf{C}$ is a pair $\gamma = (u, v)$ of parallel 1-cells of $\mathbf{C}$, that is, with the same source and the same target; such a 1-sphere is denoted by $\gamma : u \Rightarrow v$. The 1-cell u is the *source of* γ and v is its *target*. A *cellular extension* of $\mathbf{C}$ is a set Γ equipped with a map from Γ to the set of 1-spheres of $\mathbf{C}$.

2.3.2. Congruences. A *congruence* on a category $\mathbf{C}$ is an equivalence relation $\equiv$ on the parallel 1-cells of $\mathbf{C}$ that is compatible with the composition of $\mathbf{C}$, that is, for all 1-cells



of $\mathbf{C}$ such that $u \equiv v$, we have $wuw' \equiv wvw'$. If Γ is a cellular extension of $\mathbf{C}$, the *congruence generated by* Γ is denoted by $\equiv_\Gamma$ and defined as the smallest congruence relation such that, if $\gamma : u \Rightarrow v$ is in Γ, then $u \equiv_\Gamma v$. The literature also calls $\equiv_\Gamma$ the *Thue congruence* generated by Γ.

2.3.3. Quotient categories. If $\mathbf{C}$ is a category and Γ is a cellular extension of $\mathbf{C}$, the *quotient of* $\mathbf{C}$ *by* Γ is the category denoted by $\mathbf{C}/\Gamma$ and defined as follows:

- the 0-cells of $\mathbf{C}/\Gamma$ are the ones of $\mathbf{C}$,
- for all 0-cells x and y of $\mathbf{C}$, the hom-set $\mathbf{C}/\Gamma(x, y)$ is the quotient of $\mathbf{C}(x, y)$ by the restriction of $\equiv_\Gamma$.

We denote by $\pi_\Gamma : \mathbf{C} \twoheadrightarrow \mathbf{C}/\Gamma$ the canonical projection. When the context is clear, we just write π for π_Γ and $\overline{u}$ for the image through π of a 1-cell u of $\mathbf{C}$.

2.3.4. 2-polygraphs. A 2-*polygraph* is a triple $\Sigma = (\Sigma_0, \Sigma_1, \Sigma_2)$ made of a 1-polygraph (Σ_0, Σ_1), often simply denoted by Σ_1, and a cellular extension Σ_2 of the free category Σ_1^*. In other terms, a 2-polygraph Σ is a 2-graph

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1^* \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Sigma_2$$

whose 0-cells and 1-cells form a free category. The elements of Σ_k are called the k-*cells of* Σ and Σ is *finite* if it has finitely many cells in every dimension.

2.3.5. Presentations of categories. If Σ is a 2-polygraph, the *category presented by* Σ is the category denoted by $\overline{\Sigma}$ and defined by

$$\overline{\Sigma} \;=\; \Sigma_1^*/\Sigma_2.$$

If $\mathbf{C}$ is a category, a *presentation of* $\mathbf{C}$ is a 2-polygraph Σ such that $\mathbf{C}$ is isomorphic to $\overline{\Sigma}$. In that case, the 1-cells of Σ are called *the generating* 1*-cells of* $\mathbf{C}$, or *the generators of* $\mathbf{C}$, and the 2-cells of Σ are called *the generating* 2*-cells of* $\mathbf{C}$, or *the relations of* $\mathbf{C}$.

2.3.6. Tietze equivalence. Two 2-polygraphs are *Tietze-equivalent* if they present the same category. It is a standard result that two (finite) 2-polygraphs are Tietze-equivalent if, and only if, they are related by a (finite) sequence of *elementary Tietze transformations*. On a 2-polygraph Σ, the elementary Tietze transformations are the following operations:

- adjunction or elimination of a generating 1-cell x and of a generating 2-cell $\alpha : u \Rightarrow x$, where u is a 1-cell of the free category over $\Sigma_1 \setminus \{x\}$,
- adjunction or elimination of a generating 2-cell $\alpha : u \Rightarrow v$ such that $u \equiv_{\Sigma_2 \setminus \{\alpha\}} v$.

2.4. Two-dimensional categories

2.4.1. 2-categories. A 2-*category* is a data $\mathcal{C}$ made of a set $\mathcal{C}_0$, whose elements are called the 0-*cells of* $\mathcal{C}$, and, for all 0-cells x and y of $\mathcal{C}$, a category $\mathcal{C}(x, y)$, whose 0-cells and 1-cells are respectively called the 1-*cells* and the 2-*cells from* x *to* y *of* $\mathcal{C}$. This data is equipped with the following algebraic structure:

- for all 0-cells x, y and z of $\mathcal{C}$, a functor
$$\gamma_{x,y,z} \ : \ \mathcal{C}(x, y) \times \mathcal{C}(y, z) \ \to \ \mathcal{C}(x, z),$$
- for every 0-cell x of $\mathcal{C}$, a specified 0-cell 1_x of the category $\mathcal{C}(x, x)$.

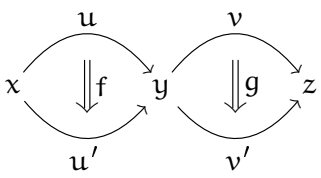
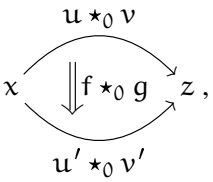
The following relations are required to hold:

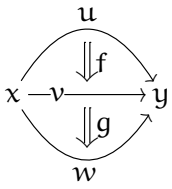
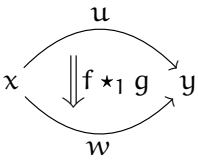
- the composition is associative, i.e. for all 0-cells x, y, z and t of $\mathcal{C}$,
$$\gamma_{x,z,t} \circ (\gamma_{x,y,z} \times \mathrm{Id}_{\mathcal{C}(z,t)}) \ = \ \gamma_{x,y,t} \circ (\mathrm{Id}_{\mathcal{C}(x,y)} \times \gamma_{y,z,t}),$$
- the identities are local units for the composition, i.e. for all 0-cells x and y of $\mathcal{C}$,
$$\gamma_{x,x,y} \circ (1_x \times \mathrm{Id}_{\mathcal{C}(x,y)}) \ = \ \mathrm{Id}_{\mathcal{C}(x,y)} \ = \ \gamma_{x,y,y} \circ (\mathrm{Id}_{\mathcal{C}(x,y)} \times 1_y).$$

This definition of 2-categories is usually stated as follows: a 2-category is a category enriched in categories. A $(2, 1)$-*category* is a 2-category whose 2-cells are invertible for the 1-composition: in other terms, it is a 2-category whose hom-categories are groupoids.

2.4.2. The globular point of view. A 2-category can, equivalently, be defined as a 2-graph
$$\mathcal{C}_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \mathcal{C}_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} \mathcal{C}_2$$
equipped with an additional algebraic structure. The definition of 2-graph requires that the source and target maps satisfy the globular relations:
$$s_0 \circ s_1 \ = \ s_0 \circ t_1 \qquad \text{and} \qquad t_0 \circ s_1 \ = \ t_0 \circ t_1.$$
The 2-graph is equipped with two compositions, the 0-*composition* $\star_0$ and the 1-*composition* $\star_1$, respectively defined on 0-composable 1-cells and 2-cells, and on 1-composable 2-cells. We also have an inclusion of $\mathcal{C}_0$ into $\mathcal{C}_1$ given by the identities of the 2-category, and an inclusion of $\mathcal{C}_1$ into $\mathcal{C}_2$ induced by the identities of the hom-categories. In details, we have the following operations:

– for all 1-cells $x \xrightarrow{u} y \xrightarrow{v} z$, a 0-composite 1-cell $x \xrightarrow{u \star_0 v} z$,

– for all 2-cells  , a 0-composite 2-cell  ,

– for all 2-cells  , a 1-composite 2-cell  ,

– for every 0-cell x, an identity 1-cell $x \xrightarrow{1_x} x$,

– for every 1-cell $x \xrightarrow{u} y$, an identity 2-cell $u \xRightarrow{1_u} u$.

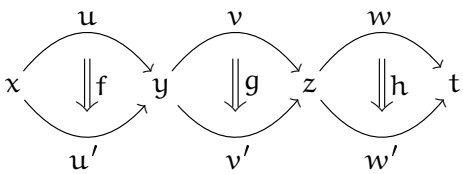
The following relations hold:

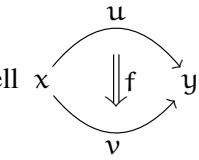
– for all 1-cells $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} t$, $(u \star_0 v) \star_0 w = u \star_0 (v \star_0 w)$,

– for every 1-cell $x \xrightarrow{u} y$, $1_x \star_0 u = u = u \star_0 1_y$,

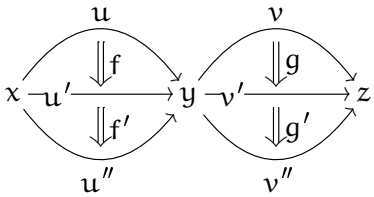
– for all 1-cells $x \xrightarrow{u} y \xrightarrow{v} z$, $1_{u \star_0 v} = 1_u \star_0 1_v$,

– for all 2-cells $u \xRightarrow{f} v \xRightarrow{g} w \xRightarrow{h} x$, $(f \star_1 g) \star_1 h = f \star_1 (g \star_1 h)$,

– for all 2-cells  , $(f \star_0 g) \star_0 h = f \star_0 (g \star_0 h)$,

– for every 2-cell  , $1_x \star_0 f = f = f \star_0 1_y$,

– for every 2-cell $u \xRightarrow{f} v$, $1_u \star_1 f = f = f \star_1 1_v$,

– for all 2-cells  , $(f \star_1 f') \star_0 (g \star_1 g') = (f \star_0 g) \star_1 (f' \star_0 g')$.

The last relation is usually called the *exchange relation* or the *interchange law* for the compositions $\star_0$ and $\star_1$. This globular definition of 2-categories is equivalent to the enriched one. In particular, the 0-composition of 2-cells with identity 2-cells defines the *whiskering* operations

– for all cells $x \xrightarrow{w} y \;\underset{v}{\overset{u}{\Downarrow f}}\; z$, the *left whiskering* $x \;\underset{w \star_0 v}{\overset{w \star_0 u}{\Downarrow w \star_0 f}}\; z$,

– for all cells $x \;\underset{v}{\overset{u}{\Downarrow f}}\; y \xrightarrow{w} z$, the *right whiskering* $x \;\underset{v \star_0 w}{\overset{u \star_0 w}{\Downarrow f \star_0 w}}\; z$,

that satisfy the following relations, implied by the exchange and associativity relations:

– for all cells $x \xrightarrow{u} y \;\underset{v''}{\overset{v}{\Downarrow f \;\; v' \;\; \Downarrow f'}}\; z$, $u \star_0 (f \star_1 f') = (u \star_0 f) \star_1 (u \star_0 f')$,

– for all cells $x \;\underset{u''}{\overset{u}{\Downarrow f \;\; u' \;\; \Downarrow f'}}\; y \xrightarrow{v} z$, $(f \star_1 f') \star_0 v = (f \star_0 v) \star_1 (f' \star_0 v)$,

– for all cells $x \xrightarrow{u} y \xrightarrow{v} z \;\underset{w'}{\overset{w}{\Downarrow f}}\; t$, $(u \star_0 v) \star_0 f = u \star_0 (v \star_0 f)$,

– for all cells $x \xrightarrow{u} y \;\underset{v'}{\overset{v}{\Downarrow f}}\; z \xrightarrow{w} t$, $(u \star_0 f) \star_0 w = u \star_0 (f \star_0 w)$,

– for all cells $x \;\underset{u'}{\overset{u}{\Downarrow f}}\; y \xrightarrow{v} z \xrightarrow{w} t$, $(f \star_0 v) \star_0 w = f \star_0 (v \star_0 w)$,

As for categories, we usually omit the $\star_0$ notation. For 2-cells, we write s and t instead of s_1 and t_1.

2.4.3. Free 2-categories. Let Σ be a 2-polygraph. The *free 2-category over* Σ is denoted by Σ^* and defined as follows:

- the 0-cells of Σ^* are the ones of Σ,
- for all 0-cells x and y of Σ, the hom-category $\Sigma^*(x,y)$ is presented by the 2-polygraph
  - whose 0-cells are the 1-cells from x to y of Σ^*,
  - whose 1-cells are the

$$x \xrightarrow{w} y \quad \overset{u}{\underset{v}{\Downarrow \alpha}} \quad z \xrightarrow{w'} t$$

    with $\alpha : u \Rightarrow v$ in Σ_2 and w and w' in Σ_1^*,
  - with one 2-cell with source $\alpha w v \star_1 u' w \beta$ and target $u w \beta \star_1 \alpha w v'$, for all $\alpha : u \Rightarrow u'$ and $\beta : v \Rightarrow v'$ in Σ_2 and w in Σ_1^*,
- for all 0-cells x, y and z of Σ the composition functor is given by the concatenation on 1-cells and, on 2-cells, by

$$\begin{aligned}\left(u_1\alpha_1 u_1' \star_1 \cdots \star_1 u_m\alpha_m u_m'\right) \star_0 \left(v_1\beta_1 v_1' \star_1 \cdots \star_1 v_n\beta_n v_n'\right) \quad = \\ u_1\alpha_1 u_1' v_1 s(\beta_1) v_1' \star_1 \cdots \star_1 u_m\alpha_m u_m' v_1 s(\beta_1) v_1' \quad \star_1 \\ u_m t(\alpha_m) u_m' v_1\beta_1 v_1' \star_1 \cdots \star_1 u_m t(\alpha_m) u_m' v_n\beta_n v_n',\end{aligned}$$

- for every 0-cell x of Σ, the identity 1-cell 1_x is the one of Σ_1^*.

By definition of the 2-category Σ^*, for all 1-cells u and v of Σ^*, we have $\overline{u} = \overline{v}$ in the quotient category $\overline{\Sigma}$ if, and only if, there exists a zigzag sequence of 2-cells of Σ^* between them:

$$u \overset{f_1}{\Longrightarrow} u_1 \overset{g_1}{\Longleftarrow} v_1 \overset{f_2}{\Longrightarrow} u_2 \Longleftarrow (\cdots) \Longrightarrow u_{n-1} \overset{g_{n-1}}{\Longleftarrow} v_{n-1} \overset{f_n}{\Longrightarrow} u_n \overset{g_n}{\Longleftarrow} v.$$

2.4.4. Free $(2,1)$-categories. If Σ is a 2-polygraph, the *free* $(2,1)$*-category over* Σ is denoted by $\Sigma^\top$ and is defined as the 2-category whose 0-cells are the ones of Σ and, for all 0-cells x and y, the hom-category $\Sigma^\top(x,y)$ is given as the quotient

$$\Sigma^\top(x,y) \;=\; \left(\Sigma \amalg \Sigma^-\right)^*(x,y)/\mathrm{Inv}(\Sigma_2),$$

where:

- the 2-polygraph Σ^- is obtained from Σ by reversing its 2-cells,
- the cellular extension $\mathrm{Inv}(\Sigma_2)$ contains the following two relations for every 2-cell α of Σ and all possible 1-cells u and v of Σ^* such that $s(u) = x$ and $t(v) = y$:

$$u\alpha v \star_1 u\alpha^- v \;\equiv\; 1_{us(\alpha)v} \qquad \text{and} \qquad u\alpha^- v \star_1 u\alpha v \;\equiv\; 1_{ut(\alpha)v}.$$

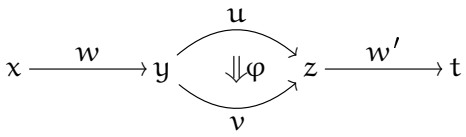
By definition of the $(2,1)$-category $\Sigma^\top$, for all 1-cells u and v of Σ^*, we have $\overline{u} = \overline{v}$ in the quotient category $\overline{\Sigma}$ if, and only if, there exists a 2-cell $f : u \Rightarrow v$ in the $(2,1)$-category $\Sigma^\top$.

3. Rewriting properties of 2-polygraphs

3.1. Convergent presentations of categories

Let us fix a 2-polygraph Σ.

3.1.1. Rewriting and normal forms. A *rewriting step of* Σ is a 2-cell of the free 2-category Σ^* with shape



where $\varphi : u \Rightarrow v$ is a 2-cell of Σ and w and w' are 1-cells of Σ^*. A *rewriting sequence of* Σ is a finite or infinite sequence

$$u_1 \overset{f_1}{\Longrightarrow} u_2 \overset{f_2}{\Longrightarrow} (\cdots) \overset{f_{n-1}}{\Longrightarrow} u_n \overset{f_n}{\Longrightarrow} (\cdots)$$

of rewriting steps. If Σ has a rewriting sequence from u to v, we say that u *rewrites into* v. Let us note that every 2-cell f of Σ^* decomposes into a finite rewriting sequence of Σ, this decomposition being unique up to exchange relations. A 1-cell u of Σ^* is a *normal form* if Σ has no rewriting step with source u, and a *normal form of* u is a 1-cell v of Σ^* that is a normal form and such that u rewrites into v.

3.1.2. Termination. We say that Σ *terminates* if it has no infinite rewriting sequence. In that case, every 1-cell of Σ^* has at least one normal form and *noetherian induction* allows definitions and proofs of properties of 1-cells of Σ^* by induction on the size of the 2-cells leading to normal forms. A *termination order on* Σ is an order relation $\leqslant$ on parallel 1-cells of Σ^* such that the following properties are satisfied:

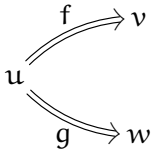
- the composition of 1-cells of Σ^* is strictly monotone in both arguments,
- every decreasing family $(u_n)_{n\in\mathbb{N}}$ of parallel 1-cells of Σ^* is stationary,
- for every 2-cell α of Σ, the strict inequality $s(\alpha) > t(\alpha)$ holds.

As a direct consequence of the definition, if Σ admits a termination order, then Σ terminates. A useful example of termination order is the *left degree-wise lexicographic order* (or *deglex* for short) generated by a given order on the 1-cells of Σ. It is defined by the following strict inequalities, where the x_is and y_js are 1-cells of Σ:

$$x_1 \cdots x_p \;<\; y_1 \cdots y_q, \qquad \text{if } p < q,$$
$$x_1 \cdots x_{k-1} x_k \cdots x_p \;<\; x_1 \cdots x_{k-1} y_k \cdots y_p, \qquad \text{if } x_k < y_k.$$

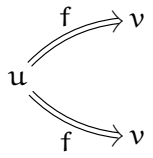
The deglex order is total if, and only if, the original order on 1-cells of Σ is total.

3.1.3. Branchings. A *branching* of Σ is a pair (f, g) of 2-cells of Σ_2^* with a common source, as in

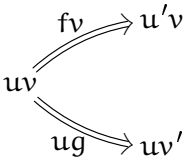


The 1-cell u is the *source* of this branching and the pair (v, w) is its *target*. We do not distinguish the branchings (f, g) and (g, f). A branching (f, g) is *local* if f and g are rewriting steps. Local branchings belong to one of the following three families:

– *aspherical* branchings, for a rewriting step $u \overset{f}{\Longrightarrow} v$:



– *Peiffer* branchings, for rewriting steps $u \overset{f}{\Longrightarrow} u'$ and $v \overset{g}{\Longrightarrow} v'$:



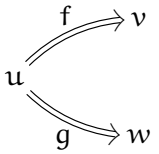
– *overlapping* branchings are the remaining local branchings.

Local branchings are compared by "inclusion", i.e. by the order $\preccurlyeq$ generated by the relations

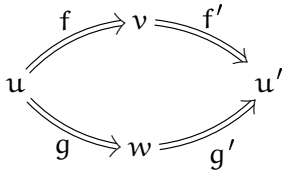
$$(f, g) \preccurlyeq (ufv, ugv)$$

given for any local branching (f, g) and any possible 1-cells u and v of Σ_1^*. An overlapping local branching that is minimal for the order $\preccurlyeq$ is called a *critical branching* (or a *critical pair*). The terms "aspherical" and "Peiffer" come from the corresponding notions for spherical diagrams in Cayley complexes associated to presentations of groups, [LS01], while "critical" is used in rewriting theory, [BO93, BN98].

3.1.4. Confluence. A branching



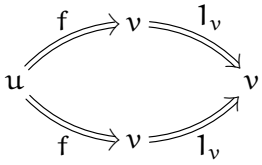
is *confluent* if there exist 2-cells f' and g' in Σ_2^*, as in the following diagram:



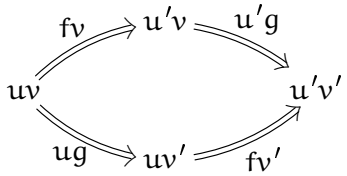
We say that Σ is *confluent* (resp. *locally confluent*) if all of its branchings (resp. local branchings) are confluent. If Σ is confluent, every 1-cell of Σ^* has at most one normal form.

3.1.5. Lemma. *A 2-polygraph is locally confluent if, and only if, all its critical branchings are confluent.*

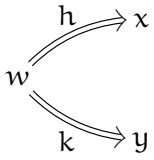
Proof. Every aspherical branching is confluent:



We also have confluence of every Peiffer local branching:



We note that, in the aspherical and Peiffer cases, the 2-cells f' and g' can be chosen in such a way that $f \star_1 f' = g \star_1 g'$ holds. Finally, in the case of an overlapping but not minimal local branching (f, g), there exist factorisations $f = uhv$ and $g = ukv$ with



a critical branching of Σ. Moreover, if the branching (h, k) is confluent, then so is (f, g):

□

The following result, also called the diamond lemma, is implied by Theorem 4.3.2.

3.1.6. Theorem (Newman's lemma [New42, Theorem 3]). *For terminating 2-polygraphs, local confluence and confluence are equivalent properties.*

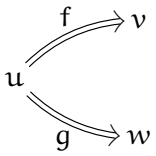
3.1.7. Convergent polygraphs. We say that Σ is *convergent* if it terminates and it is confluent. Such a Σ is called a *convergent presentation of* $\overline{\Sigma}$, and of any category that is isomorphic to $\overline{\Sigma}$. In that case, every 1-cell u of Σ_1^* has a unique normal form, denoted by $\widehat{u}$, so that we have $\overline{u} = \overline{v}$ in $\overline{\Sigma}$ if, and only if, $\widehat{u} = \widehat{v}$ holds in Σ_1^*. This defines a section $\overline{\Sigma} \rightarrowtail \Sigma_1^*$ of the canonical projection $\Sigma_1^* \twoheadrightarrow \overline{\Sigma}$, mapping a 1-cell u of $\overline{\Sigma}$ to the unique normal form of its representative 1-cells in Σ^*, still denoted by $\widehat{u}$.

As a consequence, a finite and convergent 2-polygraph Σ yields a decision procedure for the word problem of the category $\overline{\Sigma}$ it presents: the *normal-form procedure*, which takes, as input, two 1-cells u

and v of Σ^*, and decides whether $\overline{u} = \overline{v}$ holds in $\overline{\Sigma}$ or not. For that, the procedure computes the respective normal forms $\widehat{u}$ and $\widehat{v}$ of u and v. Finiteness is used to test whether a given 1-cell u is a normal form or not, by examination of all the relations and their possible applications on u. Then, the equality $\overline{u} = \overline{v}$ holds in $\overline{\Sigma}$ if, and only if, the equality $\widehat{u} = \widehat{v}$ holds in Σ^*.

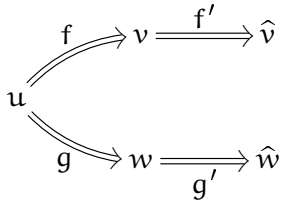
3.2. Transformations of 2-polygraphs

3.2.1. Knuth-Bendix completion. Let Σ be a terminating 2-polygraph, equipped with a total termination order $\leqslant$. A *Knuth-Bendix completion of* Σ is a 2-polygraph $\check{\Sigma}$ obtained by the following process. We start with $\check{\Sigma}$ equal to Σ and with $\mathcal{B}$ equal to the set of critical branchings of Σ. If $\mathcal{B}$ is empty, then the procedure stops. Otherwise, it picks a branching

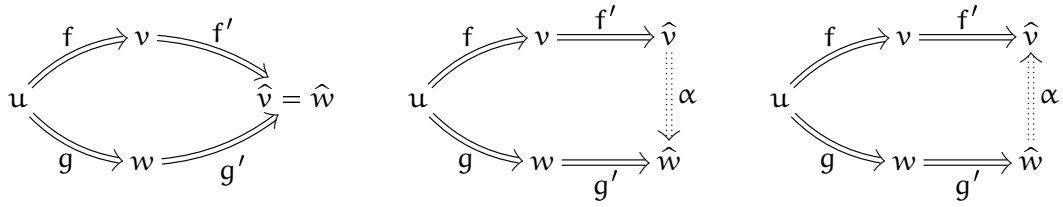


in $\mathcal{B}$ and it performs the following operations:

1. It computes 2-cells $f' : v \Rightarrow \widehat{v}$ and $g' : w \Rightarrow \widehat{w}$ of $\check{\Sigma}^*$, where $\widehat{v}$ and $\widehat{w}$ are normal forms for v and w, respectively, as in the following diagram:



2. It tests which (in)equality $\widehat{v} = \widehat{w}$ or $\widehat{v} > \widehat{w}$ or $\widehat{v} < \widehat{w}$ holds, corresponding to the following three situations, respectively:



If $\widehat{v} \neq \widehat{w}$, the procedure adds the dotted 2-cell α of the corresponding situation to $\check{\Sigma}$ and all the new critical branchings created by α to $\mathcal{B}$.

3. It removes (f, g) from $\mathcal{B}$ and restarts from the beginning.

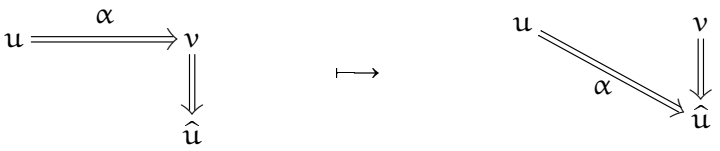
If the procedure stops, it returns the 2-polygraph $\check{\Sigma}$. Otherwise, it builds an increasing sequence of 2-polygraphs, whose limit is denoted by $\check{\Sigma}$. Note that the resulting 2-polygraph may depend on the order

of examination of the critical branchings. Also, if the starting 2-polygraph Σ is already convergent, then the Knuth-Bendix completion of Σ is Σ. By construction, the 2-polygraph $\check{\Sigma}$ is convergent and, since all the operations performed by the procedure are Tietze transformations, it is Tietze-equivalent to Σ:

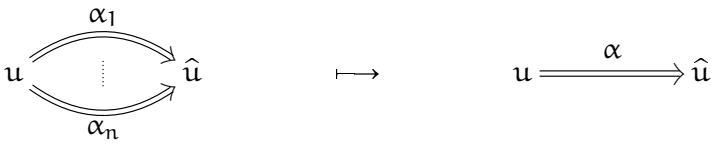
3.2.2. Theorem ([KB70]). *Any Knuth-Bendix completion $\check{\Sigma}$ of a 2-polygraph Σ, equipped with a total termination order, is a convergent presentation of the category $\overline{\Sigma}$. Moreover, the 2-polygraph $\check{\Sigma}$ is finite if, and only if, the 2-polygraph Σ is finite and the Knuth-Bendix completion procedure halts.*

3.2.3. Métivier-Squier reduction. A 2-polygraph Σ is *reduced* if, for every 2-cell $\alpha : u \Rightarrow v$ of Σ, we have that u is a normal form for $\Sigma_2 \setminus \{\alpha\}$ and that v is a normal form for Σ_2. Given a convergent 2-polygraph Σ, the *Métivier-Squier reduction of* Σ is the 2-polygraph obtained by the procedure that successively performs the following operations:

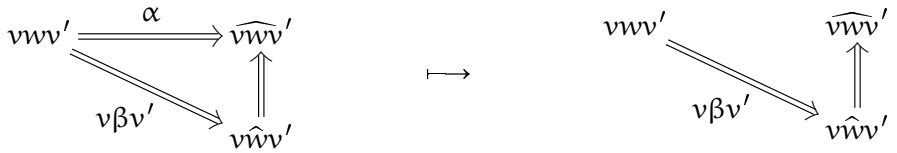
1. The procedure replaces every generating 2-cell $\alpha : u \Rightarrow v$ by $\alpha : u \Rightarrow \hat{u}$:



2. Next, if the resulting 2-polygraph contains parallel generating 2-cells, the procedure removes all but one:



3. Finally, the procedure removes every generating 2-cell α with source $v s(\beta) v'$, where β is another generating 2-cell:



By construction, we get the following result, originally obtained by Métivier for term rewriting and by Squier for string rewriting:

3.2.4. Theorem ([Mét83], [Squ87, Theorem 2.4]). *Every (finite) convergent 2-polygraph is Tietze-equivalent to a (finite) reduced convergent 2-polygraph.*

3.3. Normalisation strategies

3.3.1. Normalisation strategies. Let Σ be a 2-polygraph and let $\mathbf{C}$ denote the category presented by Σ. We consider a section $\mathbf{C} \rightarrowtail \Sigma_1^*$ of the canonical projection $\pi : \Sigma_1^* \twoheadrightarrow \mathbf{C}$, i.e. we choose, for every

1-cell u of $\mathbf{C}$, a 1-cell $\widehat{u}$ of Σ^* such that $\pi(\widehat{u}) = u$. In general, we cannot assume that the chosen section is functorial, i.e. that $\widehat{uv} = \widehat{u}\widehat{v}$ holds in Σ^*. However, we assume that $\widehat{1}_x = 1_x$ holds for every 0-cell x of $\mathbf{C}$. Given a 1-cell u of Σ^*, we simply write $\widehat{u}$ for $\widehat{\widehat{u}}$.

Such a section being fixed, a *normalisation strategy for* Σ is a map

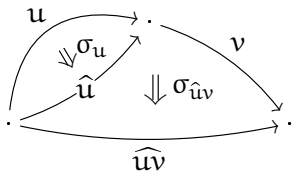
$$\sigma : \Sigma_1^* \to \Sigma_2^*$$

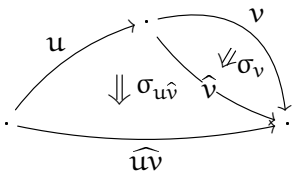
that sends every 1-cell u of Σ^* to a 2-cell

$$u \overset{\sigma_u}{\Longrightarrow} \widehat{u}$$

of $\Sigma^\top$, such that $\sigma_{\widehat{u}} = 1_{\widehat{u}}$ holds for every 1-cell u of Σ^*.

3.3.2. Left and right normalisation strategies. Let Σ be a 2-polygraph, with a chosen section. A normalisation strategy σ for Σ is a *left* one (resp. a *right* one) if it also satisfies

$\sigma_{uv} =$ 

$\big($resp. $\sigma_{uv} =$ 

$\big)$.

The 2-polygraph Σ always admits left and right normalisation strategies. For example, in the left case, let us arbitrarily choose a 2-cell $\sigma_{ua} : ua \Rightarrow \widehat{ua}$ in $\Sigma^\top$, for every 1-cell u of Σ^* and every 1-cell a of Σ, such that $\widehat{u} = u$ and ua is defined, with $\widehat{ua} \neq ua$. Then we extend σ into a left normalisation strategy for Σ by putting $\sigma_u = 1_u$ if $\widehat{u} = u$ (which implies $\sigma_{1_x} = 1_{1_x}$), and

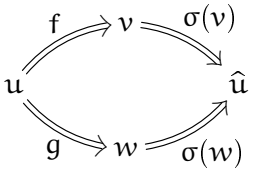
$$\sigma_u = \sigma_v a \star_1 \sigma_{\widehat{v}a}$$

if $\widehat{u} \neq u$ and $u = va$ with v in Σ^* and a in Σ.

3.3.3. Leftmost and rightmost normalisation strategies. If Σ is a reduced 2-polygraph, then, for every 1-cell u of Σ^*, the set of rewriting steps with source u can be ordered from left to right: for two rewriting steps $f = v\alpha v'$ and $g = w\beta w'$ with source u, we have $f \prec g$ if the length of v is strictly smaller than the length of w. If Σ is finite, then the order $\prec$ is total and the set of rewriting steps of source u is finite. Hence, this set contains a smallest element λ_u and a greatest element ρ_u, respectively called the *leftmost* and the *rightmost rewriting steps on* u. If, moreover, the 2-polygraph Σ terminates, the iteration of λ (resp. ρ) yields a normalisation strategy σ called the *leftmost* (resp. *rightmost*) *normalisation strategy of* Σ:

$$\sigma_u = \lambda_u \star_1 \sigma_{t(\lambda_u)} \qquad (\text{resp. } \sigma_u = \rho_u \star_1 \sigma_{t(\rho_u)}).$$

We prove, by noetherian induction, that the leftmost (resp. rightmost) normalisation strategy of Σ is a left (resp. right) normalisation strategy. The leftmost and rightmost normalisation strategies give a way to make constructive some of the results we present here. For example, they provide a deterministic choice of a confluence diagram



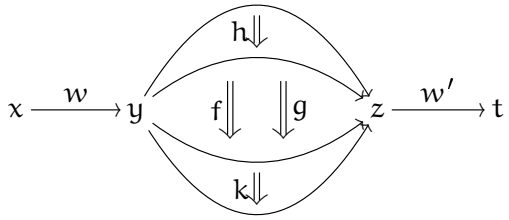
for every branching (f, g) of Σ.

4. Finite derivation type

4.1. Coherent presentations of categories

4.1.1. Cellular extensions and homotopy bases of 2-categories. Let $\mathcal{C}$ be a 2-category. A *2-sphere of* $\mathcal{C}$ is a pair $\gamma = (f, g)$ of parallel 2-cells of $\mathcal{C}$, i.e. with $s(f) = s(g)$ and $t(f) = t(g)$. We call f the *source* of γ and g its *target* and we denote such a 2-sphere by $\gamma : f \Rightarrow g$. A *cellular extension* of the 2-category $\mathcal{C}$ is a set Γ equipped with a map from Γ to the set of 2-spheres of $\mathcal{C}$.

A *congruence* on a 2-category $\mathcal{C}$ is an equivalence relation $\equiv$ on the parallel 2-cells of $\mathcal{C}$ that is compatible with the two compositions of $\mathcal{C}$, that is, for all cells



of $\mathcal{C}$ such that $f \equiv g$, we have

$$w \star_0 (h \star_1 f \star_1 k) \star_0 w' \equiv w \star_0 (h \star_1 g \star_1 k) \star_0 w'.$$

If Γ is a cellular extension of $\mathcal{C}$, the *congruence generated by* Γ is denoted by $\equiv_\Gamma$ and defined as the smallest congruence such that, if Γ contains a 3-cell $\gamma : f \Rightarrow g$, then $f \equiv_\Gamma g$. The *quotient 2-category* of a 2-category $\mathcal{C}$ by a congruence relation $\equiv$ is the 2-category, denoted by $\mathcal{C}/\equiv$, whose 0-cells and 1-cells are those of $\mathcal{C}$ and whose 2-cells are the equivalence classes of 2-cells of $\mathcal{C}$ modulo the congruence $\equiv$.

A *homotopy basis of* $\mathcal{C}$ is a cellular extension Γ of $\mathcal{C}$ such that, for every parallel 2-cells f and g of $\mathcal{C}$, we have $f \equiv_\Gamma g$, that is, the equality $\overline{f} = \overline{g}$ holds in the quotient 2-category $\mathcal{C}/\equiv_\Gamma$. For instance, the set of 2-spheres of $\mathcal{C}$ forms a homotopy basis.

4.1.2. $(3,1)$-polygraphs and coherent presentations. A $(3,1)$-*polygraph* is a pair $\Sigma = (\Sigma_2, \Sigma_3)$ made of a 2-polygraph Σ_2 and a cellular extension Σ_3 of the free $(2,1)$-category $\Sigma_2^\top$ over Σ_2, as summarised in

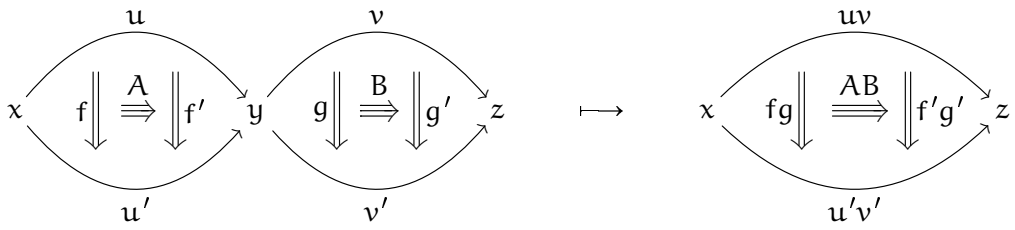
$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1^* \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Sigma_2^\top \underset{t_2}{\overset{s_2}{\leftleftarrows}} \Sigma_3$$

If $\mathbf{C}$ is a category, a *coherent presentation of* $\mathbf{C}$ is a $(3,1)$-polygraph $\Sigma = (\Sigma_2, \Sigma_3)$ such that Σ_2 is a presentation of $\mathbf{C}$ and Σ_3 is a homotopy basis of $\Sigma_2^\top$.

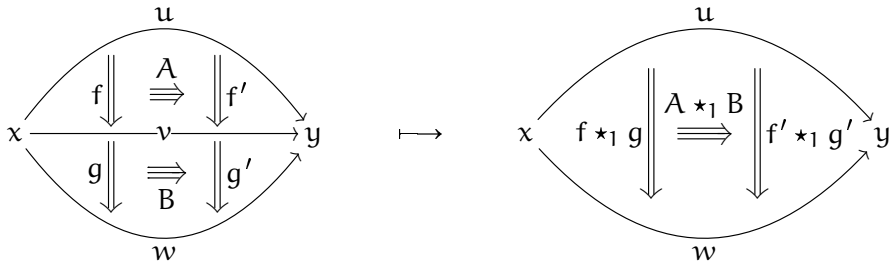
4.1.3. Finite derivation type. A 2-polygraph Σ is of *finite derivation type* if it is finite and if the $(2,1)$-category $\Sigma^\top$ admits a finite homotopy basis. A category $\mathbf{C}$ is of *finite derivation type* if it admits a finite coherent presentation.

4.1.4. 3-categories. The definition of 3-category is adapted from the one of 2-category by replacement of the hom-categories and the composition functors by hom-2-categories and composition 2-functors. In particular, in a 3-category, the 3-cells can be composed in three different ways:

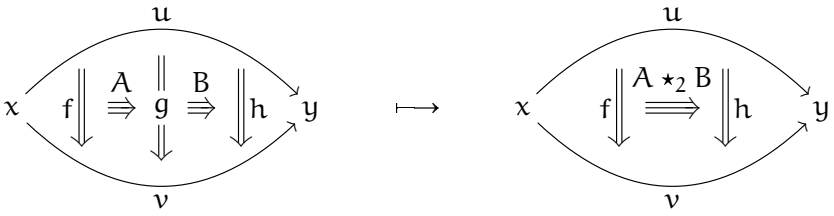
– by $\star_0$, along their 0-dimensional boundary:



– by $\star_1$, along their 1-dimensional boundary:



– by $\star_2$, along their 2-dimensional boundary:



A $(3,1)$-*category* is a 3-category whose 2-cells are invertible for the composition $\star_1$ and whose 3-cells are invertible for the composition $\star_2$. This implies that 3-cells are also invertible for the composition $\star_1$.

4.1.5. Free $(3,1)$-categories. Given a $(3,1)$-polygraph Σ, the *free* $(3,1)$-*category over* Σ is denoted by $\Sigma^\top$ and defined as follows:

– its underlying 2-category is the free $(2,1)$-category $\Sigma_2^\top$,

– its 3-cells are all the formal compositions by $\star_0$, $\star_1$ and $\star_2$ of 3-cells of Σ, of their inverses and of identities of 2-cells, up to associativity, identity, exchange and inverse relations.

In particular, we get that Σ_3 is a homotopy basis of $\Sigma_2^\top$ if, and only if, for every pair (f,g) of parallel 2-cells of $\Sigma_2^\top$, there exists a 3-cell $A : f \Rrightarrow g$ in $\Sigma^\top$.

4.2. The homotopy basis transfer theorem

The objective of this section is to prove Theorem 4.2.3: given two finite presentations of the same category, both are of finite derivation type or neither is. Towards this goal, we prove Theorem 4.2.2, that allows transfers of homotopy bases between presentations of the same category.

4.2.1. Lemma. *Let* $\mathbf{C}$ *be a category and let* Σ *and* Ξ *be presentations of* $\mathbf{C}$*. There exist 2-functors*

$$F : \Sigma^\top \to \Xi^\top \qquad \text{and} \qquad G : \Xi^\top \to \Sigma^\top$$

and, for every 1-cells u *of* $\Sigma^\top$ *and* v *of* $\Xi^\top$*, there exist 2-cells*

$$\sigma_u : GF(u) \Rightarrow u \qquad \text{and} \qquad \tau_v : FG(v) \Rightarrow v$$

in $\Sigma^\top$ *and* $\Xi^\top$*, such that the following conditions are satisfied:*

- *the 2-functors* F *and* G *induce the identity through the canonical projections onto* $\mathbf{C}$*:*

$$\begin{array}{ccc} \Sigma^\top & \xrightarrow{\pi_\Sigma} & \mathbf{C} \\ {\scriptstyle F}\downarrow & = & \downarrow{\scriptstyle \mathrm{Id}_{\mathbf{C}}} \\ \Xi^\top & \xrightarrow[\pi_\Xi]{} & \mathbf{C} \end{array} \qquad\qquad \begin{array}{ccc} \Sigma^\top & \xrightarrow{\pi_\Sigma} & \mathbf{C} \\ {\scriptstyle G}\uparrow & = & \uparrow{\scriptstyle \mathrm{Id}_{\mathbf{C}}} \\ \Xi^\top & \xrightarrow[\pi_\Xi]{} & \mathbf{C} \end{array}$$

- *the 2-cells* σ_u *and* τ_v *are functorial in* u *and* v*:*

$$\sigma_{1_x} = 1_{1_x}, \qquad \tau_{1_y} = 1_{1_y}, \qquad \sigma_{uu'} = \sigma_u \sigma_{u'} \quad \text{and} \quad \tau_{vv'} = \tau_v \tau_{v'}.$$

Proof. Let us define F, the case of G being symmetric. On a 0-cell x of Σ, we take $F(x) = x$. If $a : x \to y$ is a 1-cell of Σ, we choose, in an arbitrary way, a 1-cell $F(a) : x \to y$ in $\Xi^\top$ such that $\pi_\Xi F(a) = \pi_\Sigma(a)$. Then, we extend F to every 1-cell of $\Sigma^\top$ by functoriality. Let $\alpha : u \Rightarrow u'$ be a 2-cell of Σ. Since Σ is a presentation of $\mathbf{C}$, we have $\pi_\Sigma(u) = \pi_\Sigma(u')$, so that $\pi_\Xi F(u) = \pi_\Xi F(u')$ holds. Using the fact that Ξ is a presentation of $\mathbf{C}$, we arbitrarily choose a 2-cell $F(\alpha) : F(u) \Rightarrow F(u')$ in $\Xi^\top$. Then, we extend F to every 2-cell of $\Sigma^\top$ by functoriality.

Now, let us define σ, the case of τ being symmetric. Let a be a 1-cell of Σ. By construction of F and G, we have:

$$\pi_\Sigma GF(a) \;=\; \pi_\Xi F(a) \;=\; \pi_\Sigma(a).$$

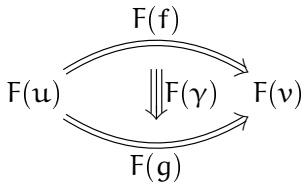
Since Σ is a presentation of $\mathbf{C}$, there exists a 2-cell $\sigma_a : GF(a) \Rightarrow a$ in $\Sigma^\top$. We extend σ to every 1-cell u of $\Sigma^\top$ by functoriality. □

4.2.2. Theorem. *Let* $\mathbf{C}$ *be a category, let* Σ *and* Ξ *be two presentations of* $\mathbf{C}$ *and let* F*,* G *and* τ *be chosen as in Lemma 4.2.1. If* Γ *is a homotopy basis of* $\Sigma^\top$*, then*

$$\Delta \;=\; F(\Gamma) \amalg \tau_\Xi$$

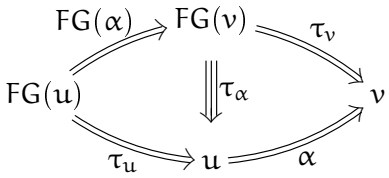
is a homotopy basis of $\Xi^\top$*, where:*

– *the cellular extension* $\mathsf{F}(\Gamma)$ *contains one* 3*-cell*



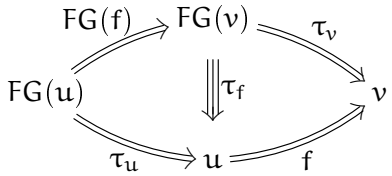
for every 3*-cell* $\gamma : \mathsf{f} \Rrightarrow \mathsf{g}$ *of* Γ*,*

– *the cellular extension* τ_Ξ *contains one* 3*-cell*



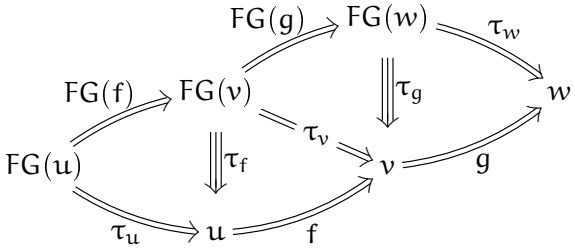
for every 2*-cell* $\alpha : \mathsf{u} \Rightarrow \mathsf{v}$ *of* Ξ*.*

Proof. Let us define, for every 2-cell f of $\Xi^\top$, a 3-cell τ_f of $\Delta^\top$ with shape



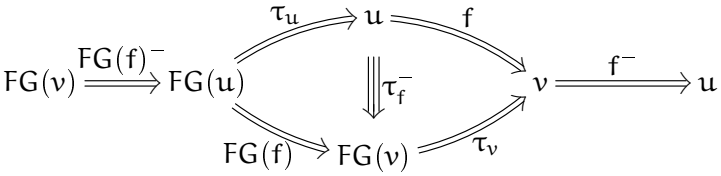
We extend the notation τ_α in a functorial way, according to the formulas $\tau_{1_\mathsf{u}} = 1_{\tau_\mathsf{u}}$, $\tau_{\mathsf{fg}} = \tau_\mathsf{f}\tau_\mathsf{g}$,

$\tau_{\mathsf{f}\star_1\mathsf{g}} =$



and

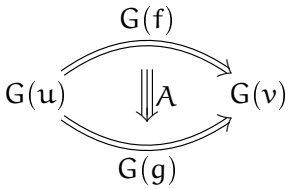
$\tau_{\mathsf{f}^-} =$



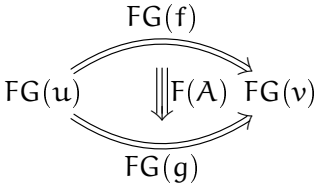
One checks that the 3-cells τ_f are well-defined, i.e. that their definition is compatible with the relations on 2-cells of $\Xi^\top$, such as the exchange relation:

$$\tau_{\mathsf{fg}\star_1\mathsf{hk}} = \tau_{(\mathsf{f}\star_1\mathsf{h})(\mathsf{g}\star_1\mathsf{k})}.$$

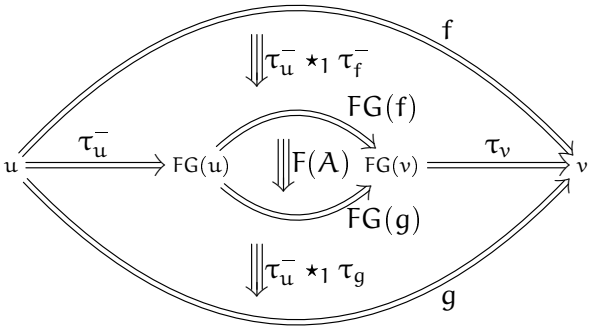
Now, let us consider parallel 2-cells $f, g : u \Rightarrow v$ of $\Xi^\top$. The 2-cells $G(f)$ and $G(g)$ of $\Sigma^\top$ are parallel so that, since Γ is a homotopy basis of $\Sigma^\top$, there exists a 3-cell



in $\Gamma^\top$. An application of F to A gives the 3-cell



of $\Xi^\top$, which, by definition of Δ and functoriality of F, is in $\Delta^\top$. Using the 3-cells $F(A)$, τ_f and τ_g, we get the following 3-cell from f to g in $\Delta^\top$:



This concludes the proof that $\Delta = F(\Gamma) \amalg \tau_\Xi$ is a homotopy basis of the $(2,1)$-category $\Xi^\top$. □

We deduce from Theorem 4.2.2 the following result.

4.2.3. Theorem ([SOK94, Theorem 4.3]). *Let Σ and Ξ be finite presentations of the same category. Then Σ is of finite derivation type if, and only if, Ξ is of finite derivation type.*

The following proposition is useful to prove that a presentation admits no finite homotopy basis.

4.2.4. Proposition. *Let Σ be a 2-polygraph and let Γ be a homotopy basis of $\Sigma^\top$. If Σ admits a finite homotopy basis, then there exists a finite subset of Γ that is a homotopy basis of $\Sigma^\top$.*

Proof. Let Δ be a finite homotopy basis of Σ and let δ be a 3-cell of Δ. Since Γ is a homotopy basis of Σ, there exists a 3-cell A_δ in $\Gamma^\top$ with boundary $(s(\delta), t(\delta))$. This induces a 3-functor

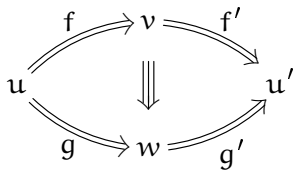
$$F : \Delta^\top \to \Gamma^\top$$

that is the identity on Σ and such that $F(\delta) = A_\delta$ for every 3-cell δ of Δ. Let Γ_Δ be the subset of Γ that contains all the generating 3-cells that appear in the 3-cells A_δ, for every δ in Δ. Since Δ is finite and each A_δ contains finitely many 3-cells of Γ, we have that Γ_Δ is finite. Finally, let us consider a 2-sphere (f, g) of $\Sigma^\top$. By hypothesis, there exists a 3-cell A in $\Delta^\top$ with boundary (f, g). By application of F, one gets a 3-cell $F(A)$ in $\Gamma^\top$ whose boundary is (f, g). Moreover, the 3-cell $F(A)$ is a composite of cells A_δ: hence, the 3-cell $F(A)$ is in $\Gamma_\Delta^\top$. As a consequence, one gets $f \equiv_{\Gamma_\Delta} g$, so that Γ_Δ is a finite homotopy basis of $\Sigma^\top$, which concludes the proof. □

4.3. Squier completion for convergent presentations

Squier completion provides a way to extend a convergent presentation of a category **C** into a coherent presentation of **C**.

4.3.1. Squier completion. For Σ a 2-polygraph, a *family of generating confluences of* Σ is a cellular extension of $\Sigma^\top$ that contains exactly one 3-cell



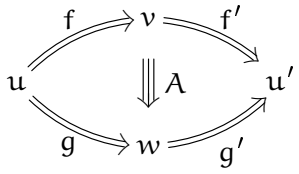
for every critical branching (f, g) of Σ. We note that, if Σ is confluent, it always admits a family of generating confluences. However, such a family is not necessarily unique, since the 3-cell can be directed in the reverse way and, for a given branching (f, g), we can have several possible 2-cells f' and g' with the required shape. Normalisation strategies provide a deterministic way to construct a family of generating confluences, see [GM12b, 4.3.2].

For a convergent 2-polygraph Σ, *Squier completion of* Σ is the $(3, 1)$-polygraph denoted by $\mathcal{S}(\Sigma)$ and defined by $\mathcal{S}(\Sigma) = (\Sigma, \Gamma)$, where Γ is a chosen family of generating confluences of Σ. By the following result, if Σ is a convergent presentation of a category **C**, then Squier completion $\mathcal{S}(\Sigma)$ is a coherent presentation of **C**.

4.3.2. Theorem ([SOK94, Theorem 5.2]). *Let Σ be a convergent 2-polygraph. Every family of generating confluences of Σ is a homotopy basis of $\Sigma^\top$.*

Proof. We fix a family of generating confluences of Σ and consider the corresponding Squier completion $\mathcal{S}(\Sigma)$. We proceed in three steps.

Step 1. We prove that, for every local branching $(f, g) : u \Rightarrow (v, w)$ of Σ, there exist 2-cells $f' : v \Rightarrow u'$ and $g' : w \Rightarrow u'$ in Σ^* and a 3-cell $A : f \star_1 f' \Rrightarrow g \star_1 g'$ in $\mathcal{S}(\Sigma)^\top$, as in the following diagram:

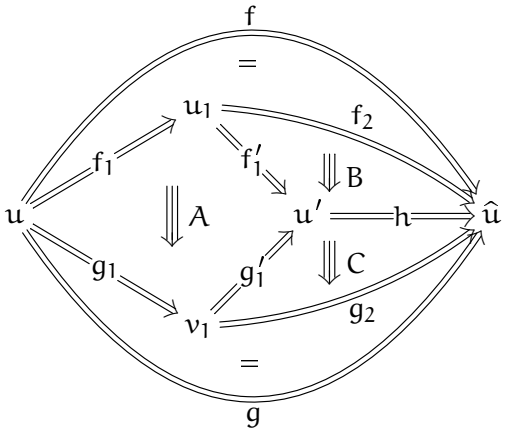


As we have seen in the study of confluence of local branchings, in the case of an aspherical or Peiffer branching, we can choose f' and g' such that $f \star_1 f' = g \star_1 g'$: an identity 3-cell is enough to link them. Moreover, if we have an overlapping branching (f, g) that is not critical, we have $(f, g) = (uhv, ukv)$ with (h, k) critical. We consider the 3-cell $\alpha : h \star_1 h' \Rrightarrow k \star_1 k'$ of $\mathcal{S}(\Sigma)$ corresponding to the critical branching (h, k) and we conclude that the following 2-cells f' and g' and 3-cell A satisfy the required conditions:

$$f' = uh'v \qquad g' = uk'v \qquad A = u\alpha v.$$

Step 2. We prove that, for every parallel 2-cells f and g of Σ^* whose common target is a normal form, there exists a 3-cell from f to g in $\mathcal{S}(\Sigma)^\top$. We proceed by noetherian induction on the common source u of f and g, using the termination of Σ. Let us assume that u is a normal form: then, by definition, both 2-cells f and g must be equal to the identity of u, so that $1_{1_u} : 1_u \Rrightarrow 1_u$ is a 3-cell of $\mathcal{S}(\Sigma)^\top$ from f to g.

Now, let us fix a 1-cell u of Σ^* with the following property: for every 1-cell v of Σ^* such that u rewrites into v in one step, and for every parallel 2-cells $f, g : v \Rightarrow \hat{v} = \hat{u}$ of Σ^*, there exists a 3-cell from f to g in $\mathcal{S}(\Sigma)^\top$. Let us consider parallel 2-cells $f, g : u \Rightarrow \hat{u}$ and let us prove the result by progressively constructing the following composite 3-cell from f to g in $\mathcal{S}(\Sigma)^\top$:



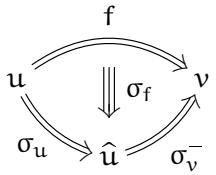
Since u is not a normal form, we can decompose $f = f_1 \star_1 f_2$ and $g = g_1 \star_1 g_2$ so that f_1 and g_1 are rewriting steps. They form a local branching (f_1, g_1) and we build the 2-cells f'_1 and g'_1 of Σ^* together with the 3-cell A of $\mathcal{S}(\Sigma)^\top$, as in the first part of the proof. Then, we consider a 2-cell $h : u' \Rightarrow \hat{u}$ in Σ^*, that must exist by confluence of Σ and since $\hat{u}$ is a normal form. We apply the induction hypothesis to the parallel 2-cells f_2 and $f'_1 \star_1 h$ in order to get B and, symmetrically, to the parallel 2-cells $g'_1 \star_1 h$ and g_2 to get C.

Step 3. We prove that every 2-sphere of $\Sigma^\top$ is the boundary of a 3-cell of $\mathcal{S}(\Sigma)^\top$. First, let us consider a 2-cell $f : u \Rightarrow v$ in Σ^*. Using the confluence of Σ, we choose 2-cells

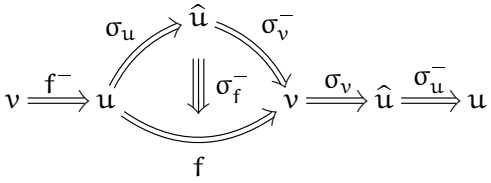
$$\sigma_u : u \Rightarrow \hat{u} \qquad \text{and} \qquad \sigma_v : v \Rightarrow \hat{v} = \hat{u}$$

in Σ^*. By construction, the 2-cells $f \star_1 \sigma_v$ and σ_u are parallel and their common target $\hat{u}$ is a normal form. Thus, there exists a 3-cell in $\mathcal{S}(\Sigma)^\top$ from $f \star_1 \sigma_v$ to σ_u or, equivalently, a 3-cell σ_f from f to $\sigma_u \star_1 \sigma_v^-$

in $\mathcal{S}(\Sigma)^\top$, as in the following diagram:



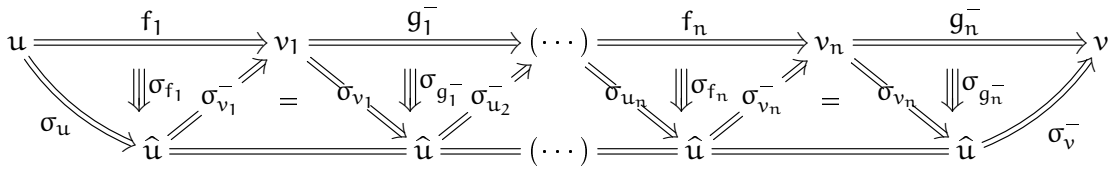
Moreover, the $(3,1)$-category $\mathcal{S}(\Sigma)^\top$ contains a 3-cell σ_{f^-} from f^- to $\sigma_v \star_1 \sigma_u^-$, given as the following composite:



Now, let us consider a general 2-cell $f : u \Rightarrow v$ of $\Sigma^\top$. By construction of $\Sigma^\top$, the 2-cell f can be decomposed (in general in a non-unique way) into a zigzag

$$u \overset{f_1}{\Longrightarrow} v_1 \overset{g_1^-}{\Longrightarrow} u_2 \overset{f_2}{\Longrightarrow} (\cdots) \overset{g_{n-1}^-}{\Longrightarrow} u_n \overset{f_n}{\Longrightarrow} v_n \overset{g_n^-}{\Longrightarrow} v$$

where each f_i and g_i is a 2-cell of Σ^*. We define σ_f as the following composite 3-cell of $\mathcal{S}(\Sigma)^\top$, with source f and target $\sigma_u \star_1 \sigma_v^-$:



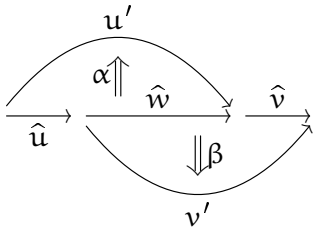
We proceed similarly for any other 2-cell $g : u \Rightarrow v$ of $\Sigma^\top$, to get a 3-cell σ_g from g to $\sigma_u \star_1 \sigma_v^-$ in $\mathcal{S}(\Sigma)^\top$. Thus, the composite $\sigma_f \star_2 \sigma_g^-$ is a 3-cell of $\mathcal{S}(\Sigma)^\top$ from f to g, concluding the proof. □

Theorem 4.3.2 is extended to higher-dimensional polygraphs in [GM09, Proposition 4.3.4]. In the special case of presentations of monoids, we recover the original result of Squier.

4.3.3. Theorem ([SOK94, Theorem 5.3]). *If a monoid admits a finite convergent presentation, then it is of finite derivation type.*

4.3.4. Generating confluences in the reduced case. Theorem 4.3.2 holds for any choice of family of generating confluences. If Σ is a reduced convergent 2-polygraph, we can construct an explicit such family as follows. Let σ be the leftmost normalisation strategy of Σ. Since Σ is reduced, every critical

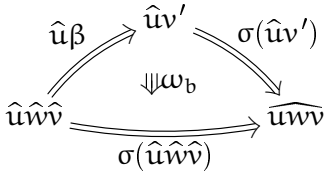
branching of Σ has the form



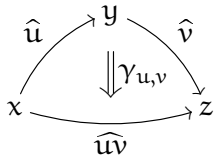
where α and β are 2-cells of Σ and where $\hat{u}$, $\hat{w}$ and $\hat{v}$ are non-identity normal forms. Let us note that $\alpha\hat{v}$ is the leftmost reduction step of $\hat{u}\hat{w}\hat{v}$ and that $\hat{u}\beta$ is its rightmost reduction step. In particular, we have

$$\sigma(\hat{u}\hat{w}\hat{v}) \;=\; \alpha\hat{v} \star_1 \sigma(u'\hat{v}).$$

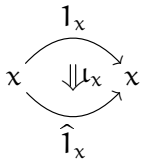
We define Σ_3 as the cellular extension of $\Sigma^\top$ made of one 3-cell with the following shape, for every critical branching $b = (\alpha\hat{v}, \hat{u}\beta)$ of Σ:



4.3.5. Example. The *standard presentation* of a category $\mathbf{C}$ is the 2-polygraph $\mathrm{Std}_2(\mathbf{C})$ defined as follows. The 0-cells and 1-cells of $\mathrm{Std}_2(\mathbf{C})$ are the ones of $\mathbf{C}$, with $\hat{u}$ denoting a 1-cell u of $\mathbf{C}$ when seen as a 1-cell of $\mathrm{Std}_2(\mathbf{C})$. The 2-polygraph $\mathrm{Std}_2(\mathbf{C})$ contains a 2-cell

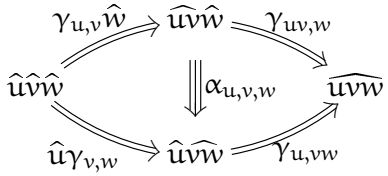


for all 1-cells $u : x \to y$ and $v : y \to z$ of $\mathbf{C}$, and a 2-cell

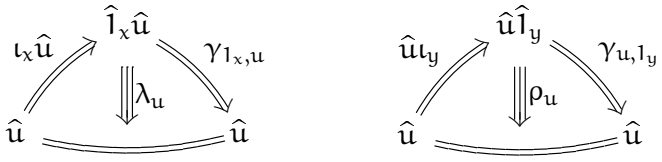


for every 0-cell x of $\mathbf{C}$. The *standard coherent presentation of* $\mathbf{C}$ is the $(3,1)$-polygraph denoted by $\mathrm{Std}_3(\mathbf{C})$ and obtained by extension of $\mathrm{Std}_2(\mathbf{C})$ with the homotopy basis made of the following 3-cells:

– for all 1-cells $u : x \to y$, $v : y \to z$ and $w : z \to t$ of $\mathbf{C}$, one 3-cell



– for every 1-cell $u : x \to y$ of $\mathbf{C}$, two 3-cells

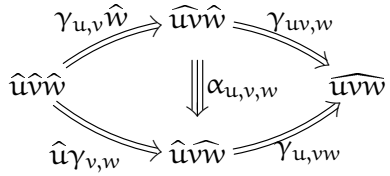


Let us prove that $\mathrm{Std}_3(\mathbf{C})$ is, indeed, a coherent presentation of $\mathbf{C}$. The standard presentation $\mathrm{Std}_2(\mathbf{C})$ is not terminating: indeed, for every 0-cell x of $\mathbf{C}$, the 2-cell ι_x creates infinite rewriting sequences

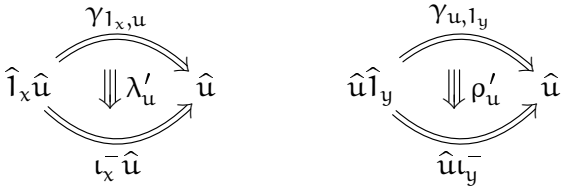
$$1_x \Rightarrow \widehat{1}_x \Rightarrow \widehat{1}_x\widehat{1}_x \Rightarrow \widehat{1}_x\widehat{1}_x\widehat{1}_x \Rightarrow \cdots$$

However, we get a convergent presentation of $\mathbf{C}$ by reversing all the 2-cells ι_x into ι_x^-. Indeed, for termination, we consider the size of the 1-cells (the number of generators they contain) and we check that each 2-cell $\gamma_{u,v}$ has source of size 2 and target of size 1, while each 2-cell ι_x^- has source of size 1 and target of size 0. As a consequence, for every non-identity 2-cell $f : u \Rightarrow v$ of the free 2-category, the size of u is strictly greater than the size of v. For confluence, we study the critical branchings, divided into three families:

– for all 1-cells $u : x \to y$, $v : y \to z$ and $w : z \to t$ of $\mathbf{C}$, one critical branching $(\gamma_{u,v}\widehat{w}, \widehat{u}\gamma_{v,w})$, giving the 3-cell



– for every 1-cell $u : x \to y$ of $\mathbf{C}$, two critical branchings $(\gamma_{1_x,u}, \iota_x^-\widehat{u})$ and $(\gamma_{u,1_y}, \widehat{u}\iota_y^-)$, producing the 3-cells



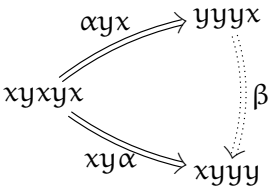
Since considering the 2-cells ι_x or ι_x^- as generators does not change the generated $(2,1)$-category, we get that those three families of 3-cells form a homotopy basis of $\mathrm{Std}_2(\mathbf{C})$. We replace λ'_u by $\lambda_u = \iota_x\widehat{u} \star_1 \lambda_u$ and ρ'_u by $\rho_u = \widehat{u}\iota_y \star_1 \rho_u$ to get the result.

4.3.6. Example. Let us consider the monoid $\mathbf{M}$ presented by the 2-polygraph

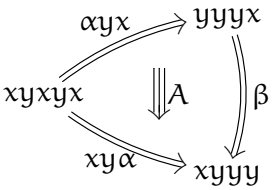
$$\Sigma = \langle\, x, y \mid xyx \overset{\alpha}{\Longrightarrow} yy \,\rangle.$$

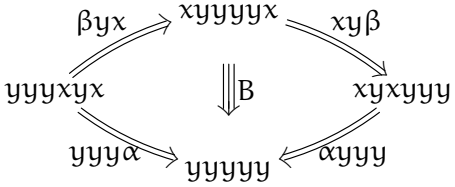
We prove that Σ terminates with the deglex order generated by $x < y$. The 2-polygraph Σ has one, non confluent critical branching $(\alpha y x, x y \alpha)$. Knuth-Bendix completion $\check{\Sigma}$ of Σ is obtained by adjunction of

the following 2-cell $\beta : yyyx \Rightarrow xyyy$:

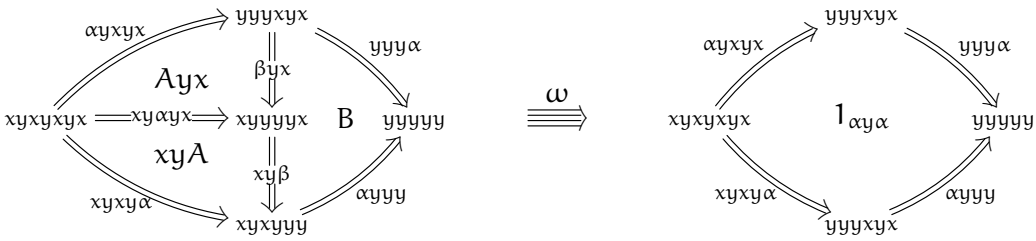


Then, Squier completion $\mathcal{S}(\check{\Sigma})$ extends $\check{\Sigma}$ with the following two 3-cells:





In fact, the 3-cell A is sufficient to get a homotopy basis of $\check{\Sigma}$, as witnessed by the following 3-sphere of the $(3,1)$-category $\mathcal{S}(\check{\Sigma})^{\top}$:



Indeed, the 3-sphere ω proves that the boundary of B is also the boundary of a 3-cell of the $(3,1)$-category $(\mathcal{S}(\check{\Sigma})\setminus\{B\})^{\top}$. This elimination mechanism, based on the study of the triple critical branchings of $\check{\Sigma}$ is part of the *homotopical reduction procedure* introduced in [GGM]. This construction coherently eliminates pairs of redundant cells of a coherent presentation. On this particular example, it yields that the $(2,1)$-category $\Sigma^{\top}$ admits an empty homotopy basis, i.e. that the $(3,1)$-polygraph $(\Sigma,\varnothing)$ is a coherent presentation of the monoid **M**.

5. A HOMOLOGICAL FINITENESS CONDITION

5.1. Monoids of finite homological type

5.1.1. Resolutions. Let **M** be a monoid. We denote by $\mathbb{Z}\mathbf{M}$ the ring generated by **M**, that is, the free abelian group over **M**, equipped with the canonical extension of the product of **M**:

$$\Big(\sum_{u\in\mathbf{M}}\lambda_u u\Big)\Big(\sum_{v\in\mathbf{M}}\lambda_v v\Big) \;=\; \sum_{u,v\in\mathbf{M}}\lambda_u\lambda_v uv \;=\; \sum_{w\in\mathbf{M}}\sum_{uv=w}\lambda_u\lambda_v w.$$

Given a (left) $\mathbb{Z}\mathbf{M}$-module M, a *resolution of* M is an exact sequence of $\mathbb{Z}\mathbf{M}$-modules

$$(\cdots) \longrightarrow M_n \xrightarrow{d_n} M_{n-1} \longrightarrow (\cdots) \longrightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M \longrightarrow 0$$

that is, a sequence $(M_n)_{n\in\mathbb{N}}$ of $\mathbb{Z}\mathbf{M}$-modules, together with a sequence $(d_n)_{n\in\mathbb{N}}$ of morphisms of $\mathbb{Z}\mathbf{M}$-modules, called the *boundary maps*, such that d_0 is surjective and

$$\operatorname{im} d_{n+1} = \ker d_n$$

holds for every natural number n. In particular, the sequence $(M_n, d_n)_{n\in\mathbb{N}}$ is a *(chain) complex of* $\mathbb{Z}\mathbf{M}$*-modules*, that is, we have the inclusion $\operatorname{im} d_{n+1} \subseteq \ker d_n$ or, equivalently, the relation $d_n d_{n+1} = 0$ holds for every natural number n. Such a resolution is called *projective* (resp. *free*) if all the modules M_n are projective (resp. free). Given a natural number n, a *partial resolution of length* n *of* $\mathbf{M}$ is defined in a similar way but with a bounded sequence $(M_k)_{0\leqslant k\leqslant n}$ of $\mathbb{Z}\mathbf{M}$-modules.

5.1.2. Contracting homotopies. Given a complex of $\mathbb{Z}\mathbf{M}$-modules

$$(\cdots) \longrightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \longrightarrow (\cdots) \longrightarrow M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M \longrightarrow 0$$

a method to prove that such a complex is a resolution of $\mathbf{M}$ is to construct a *contracting homotopy*, that is a sequence of morphisms of $\mathbb{Z}$-modules

$$(\cdots) \longleftarrow M_{n+1} \xleftarrow{i_{n+1}} M_n \xleftarrow{i_n} M_{n-1} \longleftarrow (\cdots) \longleftarrow M_1 \xleftarrow{i_1} M_0 \xleftarrow{i_0} M$$

such that $d_0 i_0 = \mathrm{Id}_M$ and, for every n, we have

$$d_{n+1} i_{n+1} + i_n d_n \;=\; \mathrm{Id}_{M_n}\,.$$

Indeed, in that case, we have that d_0 is surjective. Moreover, for every natural number n and every x in $\ker d_n$, the equality $d_{n+1} i_{n+1}(x) = x$ holds, proving that x is in $\operatorname{im} d_{n+1}$, so that $\ker d_n \subseteq \operatorname{im} d_{n+1}$ holds. As a consequence, the considered complex is a resolution of M.

5.1.3. Homological type left-FP_n. If $\mathbf{M}$ is a monoid, the *trivial* $\mathbb{Z}\mathbf{M}$*-module* is the abelian group $\mathbb{Z}$ equipped with the trivial action $un = n$, for every u in $\mathbf{M}$ and n in $\mathbb{Z}$. A monoid $\mathbf{M}$ is of *homological type left-*FP_n, for a natural number n, if there exists a partial resolution of length n of the trivial $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}$ by projective, finitely generated $\mathbb{Z}\mathbf{M}$-modules:

$$P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} (\cdots) \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0.$$

A monoid $\mathbf{M}$ is of *homological type left-*FP_∞ if there exists a resolution of $\mathbb{Z}$ by projective, finitely generated $\mathbb{Z}\mathbf{M}$-modules.

5.1.4. Lemma. *Let* $\mathbf{M}$ *be a monoid and let* n *be a natural number. The following assertions are equivalent:*

i) *The monoid* $\mathbf{M}$ *is of homological type left-*FP_n.

ii) *There exists a free, finitely generated partial resolution of the trivial* $\mathbb{Z}\mathbf{M}$*-module* $\mathbb{Z}$ *of length* n

$$F_n \longrightarrow F_{n-1} \longrightarrow (\cdots) \longrightarrow F_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

iii) *For every* $0 \leqslant k < n$ *and every projective, finitely generated partial resolution of the trivial* $\mathbb{Z}\mathbf{M}$*-module* $\mathbb{Z}$ *of length* k

$$P_k \xrightarrow{d_k} P_{k-1} \xrightarrow{d_{k-1}} (\cdots) \longrightarrow P_0 \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0,$$

the $\mathbb{Z}\mathbf{M}$*-module* $\ker d_k$ *is finitely generated.*

Lemma 5.1.4 is a consequence of the following generalisation of Schanuel's lemma. If

$$0 \longrightarrow Q \longrightarrow P_n \longrightarrow (\cdots) \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

and

$$0 \longrightarrow Q' \longrightarrow P'_n \longrightarrow (\cdots) \longrightarrow P'_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

are exact sequences of $\mathbb{Z}\mathbf{M}$-modules, with each P_k and P'_k finitely generated and projective, then the $\mathbb{Z}\mathbf{M}$-module Q is finitely generated if, and only if, the $\mathbb{Z}\mathbf{M}$-module Q' is finitely generated.

5.2. Monoids of homological type left-FP_2

5.2.1. Presentations and partial resolutions of length 2. Let $\mathbf{M}$ be a monoid and let Σ be a presentation of $\mathbf{M}$. Let us define a partial resolution of length 2 of $\mathbb{Z}$ by free $\mathbb{Z}\mathbf{M}$-modules

$$\mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

The $\mathbb{Z}\mathbf{M}$-modules $\mathbb{Z}\mathbf{M}[\Sigma_1]$ and $\mathbb{Z}\mathbf{M}[\Sigma_2]$ are the free $\mathbb{Z}\mathbf{M}$-modules over Σ_1 and Σ_2, respectively: they contain the formal sums of elements denoted by $u[x]$, where u is an element of $\mathbf{M}$ and x is a 1-cell of Σ or a 2-cell of Σ. Let us note that $\mathbb{Z}\mathbf{M}$ is isomorphic to the free $\mathbb{Z}\mathbf{M}$-module over the singleton Σ_0. The boundary maps are defined, on generators, by

$$\begin{array}{rcl} \mathbb{Z}\mathbf{M} & \xrightarrow{\varepsilon} & \mathbb{Z} \\ u & \longmapsto & 1 \end{array} \qquad \begin{array}{rcl} \mathbb{Z}\mathbf{M}[\Sigma_1] & \xrightarrow{d_1} & \mathbb{Z}\mathbf{M} \\ [x] & \longmapsto & \overline{x} - 1 \end{array} \qquad \begin{array}{rcl} \mathbb{Z}\mathbf{M}[\Sigma_2] & \xrightarrow{d_2} & \mathbb{Z}\mathbf{M}[\Sigma_1] \\ [\alpha] & \longmapsto & [s(\alpha)] - [t(\alpha)] \end{array}$$

The maps ε and d_2 are respectively called the *augmentation map* and the *Reidemester-Fox Jacobian of* Σ. In the definition of d_2, the bracket $[\cdot]$ is extended to the 1-cells of $\Sigma^\top$ thanks to the relation

$$[1] \;=\; 0 \qquad \text{and} \qquad [uv] \;=\; [u] + \overline{u}[v],$$

for all 1-cells u and v of Σ.

5.2.2. Proposition. *Let* $\mathbf{M}$ *be a monoid and let* Σ *be a presentation of* $\mathbf{M}$*. The sequence of* $\mathbb{Z}\mathbf{M}$*-modules*

$$\mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is a partial free resolution of length 2 *of* $\mathbb{Z}$*.*

Proof. We first note that the sequence is a chain complex. Indeed, the augmentation map is surjective by definition. Moreover, we have

$$\varepsilon d_1[x] \;=\; \varepsilon(\overline{x}) - \varepsilon(1) \;=\; 1 - 1 \;=\; 0,$$

for every 1-cell x of Σ. In order to check that $d_1 d_2 = 0$, we first prove, by induction on the length, that we have $d_1[u] = \overline{u} - 1$ for every 1-cell u of $\Sigma^\top$. For the unit, we have $d_1[1] = d_1(0) = 0$ and $\overline{1} - 1 = 0$. Then, for a composite 1-cell uv such that the result holds for both u and v, we get

$$d_1[uv] \;=\; d_1[u] + \overline{u} d_1[v] \;=\; \overline{u} - 1 + \overline{uv} - \overline{u} \;=\; \overline{uv} - 1.$$

As a consequence, we have

$$d_1 d_2[\alpha] \;=\; d_1[s(\alpha)] - d_1[t(\alpha)] \;=\; \overline{s(\alpha)} - \overline{t(\alpha)} \;=\; 0,$$

for every 2-cell α of Σ, where the last equality comes from $\overline{s(\alpha)} = \overline{t(\alpha)}$, that holds since Σ is a presentation of $\mathbf{M}$.

The rest of the proof consists in defining contracting homotopies i_0, i_1, i_2:

$$\mathbb{Z}\mathbf{M}[\Sigma_2] \underset{i_2}{\overset{d_2}{\rightleftarrows}} \mathbb{Z}\mathbf{M}[\Sigma_1] \underset{i_1}{\overset{d_1}{\rightleftarrows}} \mathbb{Z}\mathbf{M} \underset{i_0}{\overset{\varepsilon}{\rightleftarrows}} \mathbb{Z}$$

We choose a representative $\widehat{u}$ in Σ_1^* for every element u of $\mathbf{M}$, with $\widehat{1}_x = 1_x$ for every 0-cell x of Σ, and we fix a normalisation strategy σ for Σ. Then we define the morphisms of $\mathbb{Z}$-modules i_0, i_1 and i_2 by their values on generic elements

$$i_0(1) \;=\; 1, \qquad i_1(u) \;=\; [\widehat{u}], \qquad i_2(u[x]) \;=\; [\sigma(\widehat{u}x)],$$

where the bracket $[\cdot]$ is extended to every 2-cell of $\Sigma^\top$ thanks to the relations

$$[1_u] \;=\; 0, \qquad [ufv] \;=\; \overline{u}[f] \qquad \text{and} \qquad [f \star_1 g] \;=\; [f] + [g],$$

for all 1-cells u and v and 2-cells f and g of $\Sigma^\top$ such that the composites ufv and $f \star_1 g$ are defined.

First, we have $\varepsilon i_0 = \mathrm{Id}_{\mathbb{Z}}$. Next, for every u in $\mathbf{M}$, we have $i_0\varepsilon(u) = 1$ and

$$d_1 i_1(u) \;=\; d_1[\widehat{u}] = u - 1.$$

Thus $d_1 i_1 + i_0 \varepsilon = \mathrm{Id}_{\mathbb{Z}\mathbf{M}}$. Finally, we have, on the one hand,

$$i_1 d_1(u[x]) \;=\; i_1(u\overline{x} - u) \;=\; [\widehat{u\overline{x}}] - [\widehat{u}]$$

and, on the other hand,

$$d_2 i_2(u[x]) \;=\; d_2[\sigma(\widehat{u}x)] \;=\; [\widehat{u}x] - [\widehat{u\overline{x}}] \;=\; u[x] + [\widehat{u}] - [\widehat{u\overline{x}}].$$

For this equality, we check that $d_2[f] = [s(f)] - [t(f)]$ holds for every 2-cell f of $\Sigma^\top$ by induction on the size of f. Hence we have $d_2 i_2 + i_1 d_1 = \mathrm{Id}_{\mathbb{Z}\mathbf{M}[\Sigma_1]}$, thus concluding the proof. □

From Proposition 5.2.2, we deduce the following result:

5.2.3. Theorem. *The following properties hold.*

i) *Every monoid is of homological type left-*FP_0.

ii) *Every finitely generated monoid is of homological type left-*FP_1.

iii) *Every finitely presented monoid is of homological type left-*FP_2.

5.2.4. Examples. Let us consider the monoid $\mathbf{M}$ presented by the 2-polygraph

$$\Sigma = \langle\, a, c, t \mid at^{n+1} \overset{\alpha_n}{\Longrightarrow} ct^n\,,\ n \in \mathbb{N} \,\rangle.$$

The monoid $\mathbf{M}$ is finitely generated and, thus, it is of homological type left-FP_1. However, for every natural number n, we have

$$\begin{aligned} d_2[\alpha_{n+1}] &= [at^{n+2}] - [ct^{n+1}], \\ &= [at^{n+1}] + \overline{at^{n+1}}[t] - [ct^n] - \overline{ct^n}[t], \\ &= d_2[\alpha_n] + (\overline{at^{n+1}} - \overline{ct^n})[t]. \end{aligned}$$

The equality $\overline{at^{n+1}} = \overline{ct^n}$ holds in $\mathbf{M}$ by definition, yielding $d_2[\alpha_{n+1}] = d_2[\alpha_n]$. As a consequence, the $\mathbb{Z}\mathbf{M}$-module $\ker d_2$ is generated by the elements $[\alpha_n] - [\alpha_0]$. Since the $\mathbb{Z}\mathbf{M}$-module $\ker d_1$ is equal to $\operatorname{im} d_2$, hence isomorphic to $\mathbb{Z}\mathbf{M}[\Sigma_2]/\ker d_2$, it follows that $\ker d_1$ is generated by $[\alpha_0]$ only, so that, by Lemma 5.1.4, the monoid $\mathbf{M}$ is of homological type left-FP_2. This can also be obtained by simply observing that $\mathbf{M}$ admits the finite presentation $\langle a, c, t \mid \alpha_0 \rangle$.

Now, let us consider the monoid $\mathbf{M}$ presented by the 2-polygraph

$$\Sigma = \langle\, a, b, t \mid at^n b \overset{\alpha_n}{\Longrightarrow} 1\,,\ n \in \mathbb{N} \,\rangle.$$

The monoid $\mathbf{M}$ is of homological type left-FP_1, but not left-FP_2. This is proved by showing that $\ker d_1$ is not finitely generated as a $\mathbb{Z}\mathbf{M}$-module, which is tedious by direct computation in this case. Another way to conclude is to extend the partial resolution of Proposition 5.2.2 by one dimension: it will then be sufficient to compute $\operatorname{im} d_3$, which is trivial in this case because Σ has no critical branching, so that $\ker d_2 = 0$ and, as a consequence, $\ker d_1$ is isomorphic to $\mathbb{Z}\mathbf{M}[\Sigma_2]$. Convergent presentations provide a method to obtain such a length-three partial resolution.

5.3. Squier's homological theorem

5.3.1. Coherent presentations and partial resolutions of length 3. Let $\mathbf{M}$ be a monoid and let Σ be a coherent presentation of $\mathbf{M}$. Let us extend the partial resolution of 5.2.2 into the resolution of length 3

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where the $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}\mathbf{M}[\Sigma_3]$ is the free $\mathbb{Z}\mathbf{M}$-module over Σ_3, formed by the linear combination of elements $u[\gamma]$, with u in $\mathbf{M}$ and γ a 3-cell of Σ. The boundary map d_3 is defined, for every 3-cell γ of Σ, by

$$d_3[\gamma] \;=\; [s_2(\gamma)] - [t_2(\gamma)].$$

The bracket $[\cdot]$ is extended to 3-cells of $\Sigma^\top$ thanks to the relations

$$[uAv] \;=\; \overline{u}[A], \qquad [A \star_1 B] \;=\; [A] + [B], \qquad [A \star_2 B] \;=\; [A] + [B],$$

for all 1-cells u and v and 3-cells A and B of $\Sigma^\top$ such that the composites are defined. In particular, the latter relation implies $[1_f] = 0$ for every 2-cell f of $\Sigma^\top$. We check, by induction on the size, that $d_3[A] = [s_2(A)] - [t_2(A)]$ holds for every 3-cell A of $\Sigma^\top$.

5.3.2. Proposition. *Let* $\mathbf{M}$ *be a monoid and let* Σ *be a coherent presentation of* $\mathbf{M}$. *The sequence of* $\mathbb{Z}\mathbf{M}$*-modules*

$$\mathbb{Z}\mathbf{M}[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}\mathbf{M}[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M} \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

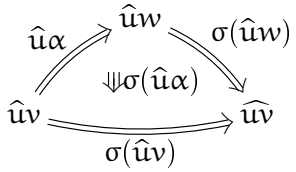
is a partial free resolution of length 3 *of* $\mathbb{Z}$.

Proof. We proceed with the same notations as the ones of the proof of Proposition 5.2.2, with the extra hypothesis that σ is a left normalisation strategy for Σ. This implies that $i_2(u[v]) = [\sigma(\widehat{u}v)]$ holds for all u in $\mathbf{M}$ and v in Σ_1^*, by induction on the length of v.

We have $d_2d_3 = 0$ because $s_1s_2 = s_1t_2$ and $t_1s_2 = t_1t_2$. Then, we define the following morphism of $\mathbb{Z}$-modules

$$\begin{array}{rcl} \mathbb{Z}\mathbf{M}[\Sigma_2] & \xrightarrow{i_3} & \mathbb{Z}\mathbf{M}[\Sigma_3] \\ u[\alpha] & \longmapsto & [\sigma(\widehat{u}\alpha)] \end{array}$$

where $\sigma(\widehat{u}\alpha)$ is a 3-cell of $\Sigma^\top$ with the following shape, with $v = s(\alpha)$ and $w = t(\alpha)$:



Let us note that such a 3-cell necessarily exists in $\Sigma^\top$ because Σ_3 is a homotopy basis of $\Sigma^\top$. Then we have, on the one hand,

$$i_2d_2(u[\alpha]) \;=\; i_2(u[v] - u[w]) \;=\; [\sigma(\widehat{u}v)] - [\sigma(\widehat{u}w)]$$

and, on the other hand,

$$\begin{aligned} d_3i_3(u[\alpha]) \;&=\; [\widehat{u}\alpha \star_1 \sigma(\widehat{u}w)] - [\sigma(\widehat{u}v)], \\ &=\; u[\alpha] + [\sigma(\widehat{u}w)] - [\sigma(\widehat{u}v)]. \end{aligned}$$

Hence $d_3i_3 + i_2d_2 = \mathrm{Id}_{\mathbb{Z}\mathbf{M}[\Sigma_2]}$, concluding the proof. □

5.3.3. Remark. The proof of Proposition 5.3.2 uses the fact that Σ_3 is a homotopy basis to produce, for every 2-cell α of Σ and every u in $\mathbf{M}$, a 3-cell $\sigma(\widehat{u}\alpha)$ with the required shape. The hypothesis on Σ_3 could thus be modified to only require the existence of such a 3-cell in $\Sigma^{\top}$: however, it is proved in [GM12b] that this implies that Σ_3 is a homotopy basis.

From Proposition 5.3.2, we deduce

5.3.4. Theorem ([CO94, Theorem 3.2], [Laf95, Theorem 3], [Pri95]). *Let* $\mathbf{M}$ *be a finitely presented monoid. If* $\mathbf{M}$ *is of finite derivation type, then it is of homological type left-*FP_3.

By Theorem 4.3.3, this implies

5.3.5. Theorem ([Squ87, Theorem 4.1]). *If a monoid admits a finite convergent presentation, then it is of homological type left-*FP_3.

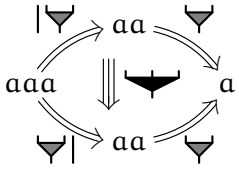
5.3.6. Example. Let us consider the monoid $\mathbf{M}$ with the convergent presentation

$$\langle a \mid aa \overset{\mu}{\Longrightarrow} a \rangle.$$

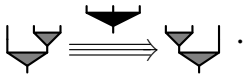
With the leftmost normalisation strategy σ, we get, writing the 2-cell μ as a string diagram :

$$\sigma(a) = 1_a \qquad \sigma(aa) = \qquad \sigma(aaa) = \mu a \star_1 \mu =$$

The presentation has exactly one critical branching, whose corresponding generating confluence can be written in the two equivalent ways



or



The $\mathbb{Z}\mathbf{M}$-module $\ker d_2$ is generated by

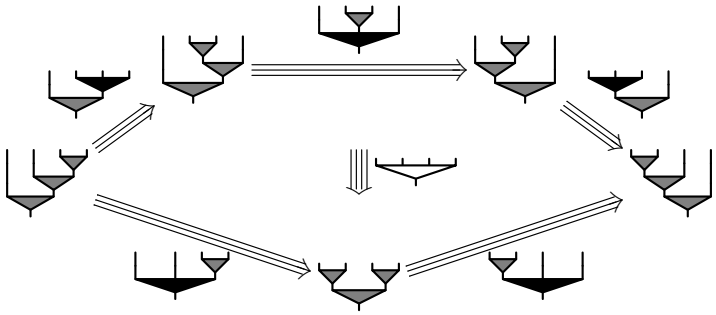
$$\begin{aligned} d_3\Big[\quad\Big] &= \Big[\quad\Big] - \Big[\quad\Big] \\ &= \Big[\quad\Big] + \Big[\quad\Big] - \Big[\quad\Big] - \Big[\quad\Big] \\ &= a\Big[\quad\Big] - \Big[\quad\Big]. \end{aligned}$$

5.3.7. Resolutions from convergent presentations. In [GM12b], the results presented here are generalised to produce a free resolution

$$(\cdots) \xrightarrow{d_{n+1}} \mathbb{Z}\mathbf{M}[\Sigma_n] \xrightarrow{d_n} \mathbb{Z}\mathbf{M}[\Sigma_{n-1}] \xrightarrow{d_{n-1}} (\cdots) \xrightarrow{d_2} \mathbb{Z}\mathbf{M}[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}\mathbf{M}[\Sigma_0] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

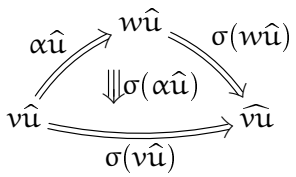
of the trivial $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}$ from a convergent presentation of a monoid $\mathbf{M}$. For $k \geqslant 4$, the $\mathbb{Z}\mathbf{M}$-module $\mathbb{Z}\mathbf{M}[\Sigma_k]$ is defined as the free $\mathbb{Z}\mathbf{M}$-module over a family Σ_k of k-cells obtained from the $(k-1)$-fold

critical branchings. For example, the 4-cell



is the only element of Σ_4 in the case of the monoid of Example 5.3.6. A related resolution is obtained in [Kob90] using antichains instead of k-fold critical branchings.

5.3.8. Other homological finiteness conditions. In the definition 5.1.3 of homological type left-FP_n for a monoid $\mathbf{M}$, the replacement of left modules by right modules, bimodules or natural systems gives the definitions of the homological types right-FP_n, bi-FP_n and FP_n, for every $0 \leqslant n \leqslant \infty$. We refer the reader to [GM12b, Section 5.2] for the relations between these different finiteness conditions. In particular, for $n = 3$, all of these homotopical conditions are consequences of the finite derivation type property. The proof is similar to the one for the left-FP_3 property in Section 5.3: for example, in the case of the right-FP_3 property, we consider right modules and, to get the contracting homotopy, we construct a right normalisation strategy σ by defining a 3-cell $\sigma(\alpha\widehat{u})$ with shape



for any generating 2-cell $\alpha : v \Rightarrow w$ and u in the monoid.

6. Squier's example and variant

6.1. Squier's example

In [Squ87], Squier defines, for every $k \geqslant 1$, the monoid $\mathbf{S}_k$ presented by

$$\left\langle a, b, t, x_1, \dots, x_k, y_1, \dots, y_k \;\middle|\; (\alpha_n)_{n\in\mathbb{N}},\ (\beta_i)_{1\leqslant i\leqslant k},\ (\gamma_i)_{1\leqslant i\leqslant k},\ (\delta_i)_{1\leqslant i\leqslant k},\ (\varepsilon_i)_{1\leqslant i\leqslant k} \right\rangle$$

with

$$at^n b \overset{\alpha_n}{\Longrightarrow} 1, \quad x_i a \overset{\beta_i}{\Longrightarrow} a t x_i, \quad x_i t \overset{\gamma_i}{\Longrightarrow} t x_i, \quad x_i b \overset{\delta_i}{\Longrightarrow} b x_i, \quad x_i y_i \overset{\varepsilon_i}{\Longrightarrow} 1.$$

In [SOK94], Squier proves the following properties for $\mathbf{S}_1$. With similar arguments, the result extends to every monoid $\mathbf{S}_k$, for $k \geqslant 1$.

6.1.1. Theorem ([SOK94, Theorem 6.7, Corollary 6.8]). *For every* $k \geqslant 1$*, the monoid* $\mathbf{S}_k$ *satisfies the following properties:*

i) *it is finitely presented,*

ii) *it has a decidable word problem,*

iii) *it is not of finite derivation type,*

iv) *it admits no finite convergent presentation.*

In [Squ87], Squier had already proved

6.1.2. Proposition ([Squ87, Example 4.5.]). *For* $k \geqslant 2$*, the monoid* $\mathbf{S}_k$ *is not of finite homological type left-*FP_3 *and, as a consequence, it does not admit a finite convergent presentation.*

Proposition 6.1.2 does not hold for $\mathbf{S}_1$. Indeed, this monoid is of homological type left-FP_∞, [Squ87, Example 4.5]. This proves that, if finite derivation type implies left-FP_3, the reverse implication does not hold for general monoids. However, in the special case of groups, the property of having finite derivation type is equivalent to the homological finiteness condition left-FP_3 [CO96]. The latter result is based on the Brown-Huebschmann isomorphism between homotopical and homological syzygies [BH82].

6.2. Proof of Theorem 6.1.1

Let us prove the result in the case of the monoid $\mathbf{S}_1$, with the following infinite presentation:

$$\mathrm{Sq} \;=\; \langle a, b, t, x, y \mid (\alpha_n)_{n\in\mathbb{N}},\ \beta,\ \gamma,\ \delta,\ \varepsilon \rangle$$

with

$$at^n b \overset{\alpha_n}{\Longrightarrow} 1, \quad xa \overset{\beta}{\Longrightarrow} atx, \quad xt \overset{\gamma}{\Longrightarrow} tx, \quad xb \overset{\delta}{\Longrightarrow} bx, \quad xy \overset{\varepsilon}{\Longrightarrow} 1.$$

In what follows, we denote by $\gamma_n : xt^n \Rightarrow t^n x$ the 2-cell of Sq^* defined by induction on n as follows:

$$\gamma_0 \;=\; 1_x \qquad \text{and} \qquad \gamma_{n+1} \;=\; \gamma t^n \star_1 t\gamma_n.$$

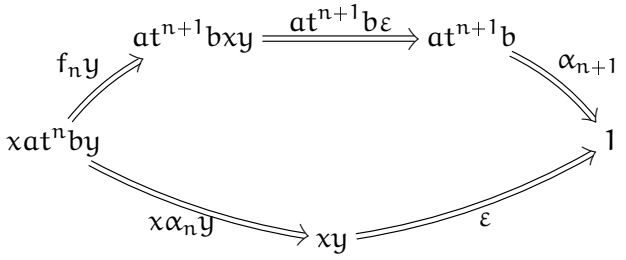
For every n, we write $f_n : xat^b \Rightarrow at^{n+1}bx$ the 2-cell of Sq^* defined as the following composite:

$$xat^n b \overset{\beta t^n b}{\Longrightarrow} atxt^n b \overset{at\gamma_n b}{\Longrightarrow} at^{n+1}xb \overset{at^{n+1}\delta}{\Longrightarrow} at^{n+1}bx.$$

We note that f_n contains no 2-cell α_k.

6.2.1. Proposition. *The monoid* $\mathbf{S}_1$ *admits the finite presentation* $\widetilde{\mathrm{Sq}} = \langle a, b, t, x, y \mid \alpha_0, \beta, \gamma, \delta, \varepsilon \rangle$.

Proof. For every natural number n, we consider the following 2-sphere of $\mathrm{Sq}^\top$:



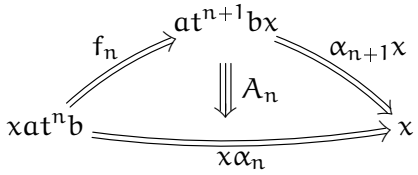
(5)

Thus, the 2-cell α_{n+1} is parallel to the composite 2-cell

$$a t^{n+1} b \varepsilon^- \star_1 f_n^- y \star_1 x \alpha_n y \star_1 \varepsilon. \tag{6}$$

Since f_n contains no α_k, this proves the result by induction on n. □

6.2.2. Proposition. *The* 2*-polygraph* Sq *is convergent and Squier completion of* Sq *contains a* 3*-cell* A_n *with shape*



for every natural number n.

Proof. Let us prove that Sq terminates. For that, we build a termination order based on derivations, similar to the method of [GM09, Theorem 4.2.1] for 3-polygraphs. We associate, to every 1-cell u of Sq^*, two maps

$$u_* : \mathbb{N} \to \mathbb{N} \qquad \text{and} \qquad \partial(u) : \mathbb{N} \to \mathbb{N}$$

as follows. First, we define them on the 1-cells of Sq:

$$x_*(n) = n + 1, \qquad a_*(n) = b_*(n) = t_*(n) = y_*(n) = n,$$

$$\partial(a)(n) = 3^n, \qquad \partial(b)(n) = \partial(y)(n) = 2^n, \qquad \partial(t)(n) = \partial(x)(n) = 0.$$

Then, we extend the mappings to every 1-cell of Sq^* thanks to the following relations:

$$1_*(n) = n, \qquad (uv)_*(n) = v_*(u_*(n)), \qquad \partial(1)(n) = 0, \qquad \partial(uv)(n) = \partial(u)(n) + \partial(v)(u_*(n)).$$

We compare parallel 1-cells of Sq^* by the order generated by $u < v$ if $u_* \leqslant v_*$ and $\partial(u) < \partial(v)$. The defining relations of $(\cdot)_*$ and ∂ imply that the composition of 1-cells of Sq^* is strictly monotone in both arguments. The natural order on $\mathbb{N}$ implies that every decreasing family of parallel 1-cells of Sq^* is stationary. To get a termination order, hence the termination of Sq, there remains to check that $u > v$ for every 2-cell $u \Rightarrow v$ of Sq. Indeed, we check that the following (in)equalities are satisfied:

$$(a t^k b)_*(n) = n = 1_*(n), \qquad (xa)_*(n) = n + 1 = (atx)_*(n), \qquad (xt)_*(n) = n + 1 = (tx)_*(n),$$

$$(xb)_*(n) = n + 1 = (bx)_*(n), \qquad (xy)_*(n) = n + 1 > n = 1_*(n),$$

and

$$\partial(a t^k b)(n) = 3^n + 2^n > 0 = \partial(1)(n), \qquad \partial(xa)(n) = 3^{n+1} > 2^n + 3^n = \partial(atx)(n),$$

$$\partial(xt)(n) = 2^{n+1} > 2^n = \partial(tx)(n), \qquad \partial(xb)(n) = 2^{n+1} > 2^n = \partial(bx)(n),$$

$$\partial(xy)(n) = 2^{n+1} > 0 = \partial(1)(n).$$

Let us prove that Sq is confluent and compute Squier completion of Sq. The 2-polygraph Sq has exactly one critical branching $(\beta t^n b, x\alpha_n)$ for every natural number n, and each of those critical branchings is confluent, yielding the 3-cell A_n. We conclude thanks to Theorem 4.3.2. □

6.2.3. Proposition. *The monoid* $\mathbf{S}_1$ *has a decidable word problem.*

Proof. The convergent presentation Sq of $\mathbf{S}_1$ is infinite, so that the normal-form procedure cannot be applied. However, the sources of the 2-cells α_n are exactly the elements of the regular language at^*b. This implies that the sources of the 2-cells of Sq form a regular language over the finite set $\{a, b, t, x, y\}$: by [OKK98, Proposition 3.6], this implies that the word problem of $\mathbf{S}_1$ is decidable. □

In order to show that $\mathbf{S}_1$ is not of finite derivation type, by Theorem 4.2.3, it is sufficient to check that the finite presentation $\widetilde{\mathrm{Sq}}$ of $\mathbf{S}_1$ given in Proposition 6.2.1 admits no finite homotopy basis. We denote by

$$\pi : \mathrm{Sq}^\top \longrightarrow \widetilde{\mathrm{Sq}}^\top$$

the projection that sends the 2-cells β, γ, δ and ε to themselves and whose value on α_n is given by induction on n, thanks to (6), i.e.

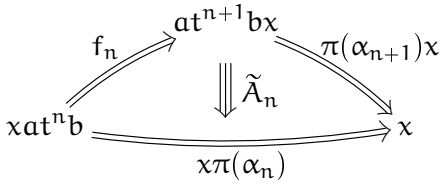
$$\pi(\alpha_0) \;=\; \alpha \qquad \text{and} \qquad \pi(\alpha_{n+1}) \;=\; g_n^- \star_1 x\pi(\alpha_n)y \star_1 \varepsilon$$

where

$$g_n \;=\; f_n y \star_1 at^{n+1}b\varepsilon.$$

By application of the homotopy basis transfer theorem 4.2.2 to Sq and $\widetilde{\mathrm{Sq}}$, with F the canonical inclusion of $\widetilde{\mathrm{Sq}}^\top$ into $\mathrm{Sq}^\top$, with $G = \pi$ and with τ mapping each 1-cell u of $\widetilde{\mathrm{Sq}}^\top$ to 1_u, we obtain

6.2.4. Lemma. *The monoid* $\mathbf{S}_1$ *admits the coherent presentation* $\langle a, b, t, x, y \mid \alpha_0, \beta, \gamma, \delta, \varepsilon \mid (\widetilde{A}_n)_{n\in\mathbb{N}}\rangle$ *where* $\widetilde{A}_n$ *is the 3-cell*



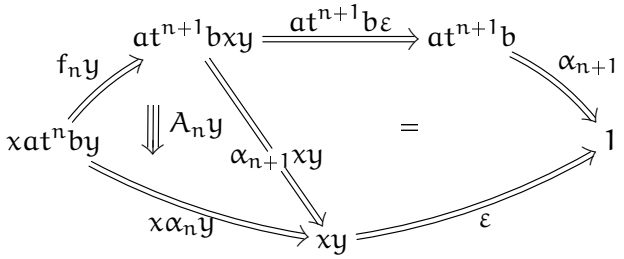
Let us now deduce that $\mathbf{S}_1$ is not of finite derivation type. A direct proof is given in [Squ87, Theorem 6.7], see also [Laf95, §6]. However, we can use an argument coming from the dimension above, to explicit as 4-cells the relations between the 3-cells. We choose this proof to incite the reader to continue the exploration of the higher dimensions of rewriting, e.g. with [Mét03, GM12b]. Precisely, we use the fact that the triple critical branchings (the minimum overlaps of three rewriting steps) of a reduced convergent 2-polygraph Σ induce a homotopy basis of Squier completion $\mathcal{S}(\Sigma)$: all the parallel 3-cells of $\mathcal{S}(\Sigma)^\top$ are equal up to the 4-cells generated by the triple critical branchings [GM12b, Proposition 4.4.4]. This implies

6.2.5. Proposition. *Let* Σ *be a reduced convergent 2-polygraph with no triple critical branching. Then all the parallel 3-cells of the free* $(3,1)$*-category* $\mathcal{S}(\Sigma)^\top$ *are equal.*

We can now conclude with

6.2.6. Proposition. *The monoid* $\mathbf{S}_1$ *is not of finite derivation type.*

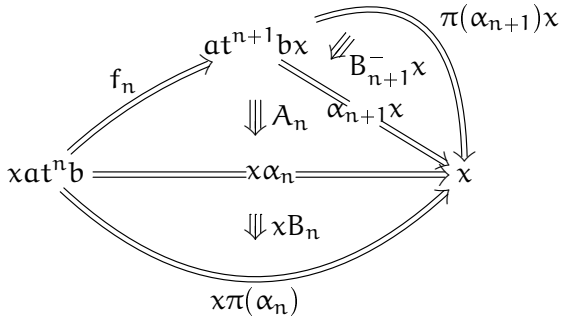
Proof. We note that the 3-cell $A_n y \star_1 \varepsilon$ of $\mathcal{S}(\mathrm{Sq})^\top$ has the 2-sphere (5) as boundary:



As a consequence, for every natural number n, the 3-cell $g_n^- \star_1 A_n y \star_1 \varepsilon$ has source α_{n+1} and target $g_n^- \star_1 x\alpha_n y \star_1 \varepsilon$. We define the 3-cell B_n of $\mathcal{S}(\mathrm{Sq})^\top$ by induction on n as

$$B_0 = 1_{\alpha_0} \qquad \text{and} \qquad B_{n+1} = \left(g_n^- \star_1 A_n y \star_1 \varepsilon\right) \star_2 B_n$$

so that B_n has source α_n and target $\pi(\alpha_n)$, by definition of π. As a consequence, the 3-cell $\widetilde{A}_n$, when seen as a 3-cell of $\mathcal{S}(\mathrm{Sq})^\top$ through the canonical inclusion, is parallel to the following composite 3-cell:



(7)

We observe that the convergent 3-polygraph is reduced and has no critical triple branching. Thus, as a consequence of Proposition 6.2.5, all the parallel 3-cells of $\mathcal{S}(\mathrm{Sq})^\top$ are equal. This implies that $\widetilde{A}_n$ is equal to the composite (7). Expanding the definition of B_{n+1}, we get:

$$\widetilde{A}_n = \left(f_n \star_1 B_n^- x\right) \star_2 \left(f_n \star_1 g_n^- x \star_1 A_n^- yx \star_1 \varepsilon x\right) \star_2 A_n \star_2 xB_n. \tag{8}$$

Now, let us assume that $\widetilde{\mathrm{Sq}}$ admits a finite homotopy basis. By Proposition 4.2.4, there exists a natural number n such that the 3-cells $\widetilde{A}_0, \ldots, \widetilde{A}_n$ form a homotopy basis of $\widetilde{\mathrm{Sq}}^\top$. In particular, the 3-cell $\widetilde{A}_{n+1}$ is parallel to a composite W of the 3-cells $\widetilde{A}_0, \ldots, \widetilde{A}_n$, hence it is equal to W in $\mathrm{Sq}^\top$.

Thus, on the one hand, by application of (8) to each of $\widetilde{A}_0, \ldots, \widetilde{A}_n$, and by definition of $B_0, \ldots, B_n$, we get that $\widetilde{A}_{n+1}$ is a composite of the 3-cells $A_0, \ldots, A_n$. But, on the other hand, the relation (8) tells us that $\widetilde{A}_{n+1}$ is equal to a composite

$$\widetilde{A}_{n+1} = C \star_2 (h \star_1 A_{n+1}^- yx \star_1 k) \star_2 A_{n+1} \star_2 D \tag{9}$$

where the 3-cells C and D contain $A_0, \ldots, A_n$ only.

To prove that this leads to a contradiction, let us consider the free right $\mathbb{Z}\mathbf{S}_1$-module $\mathbb{Z}\mathbf{S}_1[\Gamma]$ over the homotopy basis $\Gamma = \{A_n, n \in \mathbb{N}\}$. We define a map

$$[\cdot] : \mathcal{S}(\mathrm{Sq})^\top \longrightarrow \mathbb{Z}\mathbf{S}_1[\Gamma]$$

thanks to the relations

$$[uAv] \;=\; [A]\overline{v}, \qquad [A \star_1 B] \;=\; [A] + [B], \qquad [A \star_2 B] \;=\; [A] + [B],$$

for all 1-cells u and v and 3-cells A and B of $\mathcal{S}(\mathrm{Sq})^\top$ such that the composites are defined. From (9), we deduce that

$$[\widetilde{A}_{n+1}] = [C] - [A_{n+1}]\overline{yx} + [A_{n+1}] + [D] \tag{10}$$

holds in $\mathbb{Z}\mathbf{S}_1[\Gamma]$. Since the 3-cell $\widetilde{A}_{n+1}$ is a composite of the 3-cells $A_0, \dots, A_n$, we have that $[\widetilde{A}_{n+1}]$ is a linear combination of $[A_0], \dots, [A_n]$. Since the right $\mathbb{Z}\mathbf{S}_1$-module $\mathbb{Z}\mathbf{S}_1[\Gamma]$ is free, it follows from (10) that

$$\overline{yx} = 1$$

holds in $\mathbf{S}_1$. However, the 1-cells yx and 1 are distinct normal forms of the convergent presentation Sq of $\mathbf{S}_1$. This means that $\overline{yx}$ is distinct from 1 in $\mathbf{S}_1$ and, thus, leads to a contradiction, so that we conclude that $\widetilde{\mathrm{Sq}}^\top$ does not admit a finite homotopy basis. □

Finally, by Theorem 4.3.3, we get:

6.2.7. Corollary. *The monoid $\mathbf{S}_1$ admits no finite convergent presentation.*

6.3. A variant of Squier's example

Let us consider the monoid $\mathbf{M}$ presented by the following 2-polygraph from [LP91, Laf95]:

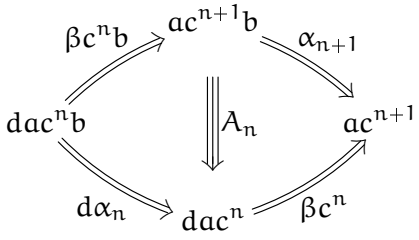
$$\mathrm{LP} \;=\; \langle\, a, b, c, d, d' \;\mid\; ab \overset{\alpha_0}{\Longrightarrow} a,\; da \overset{\beta}{\Longrightarrow} ac,\; d'a \overset{\gamma}{\Longrightarrow} ac \,\rangle.$$

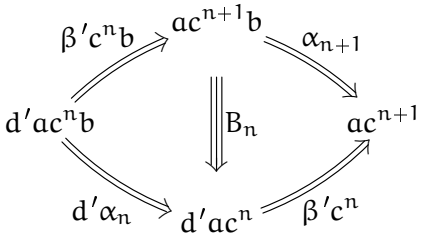
The monoid $\mathbf{M}$ has similar properties to Squier's example: it admits a finite presentation, it has a decidable word problem, yet it is not of finite derivation type and, as a consequence, it does not admit a finite convergent presentation.

To prove these facts, the 2-polygraph LP is completed, by Knuth-Bendix procedure, into the infinite convergent 2-polygraph

$$\widetilde{\mathrm{LP}} \;=\; \langle\, a, b, c, d, d' \;\mid\; (ac^n b \overset{\alpha_n}{\Longrightarrow} ac^n)_{n\in\mathbb{N}},\; da \overset{\beta}{\Longrightarrow} ac,\; d'a \overset{\beta'}{\Longrightarrow} ac \,\rangle.$$

Squier completion of $\widetilde{\mathrm{LP}}$ has two infinite families of 3-cells:





Moreover, the 2-polygraph $\widetilde{\mathrm{LP}}$ has no triple critical branching. In a similar way to the case of Squier's monoid $\mathbf{S}_1$, we get that the (finitely generated, with a decidable word problem) monoid $\mathbf{M}$ is not of finite derivation type: we prove that the 3-cells B_n induce a projection π of $\widetilde{\mathrm{LP}}^\top$ onto $\mathrm{LP}^\top$, so that the family $(\pi(A_n))_{n\in\mathbb{N}}$ is an infinite homotopy basis of $\mathrm{LP}^\top$. Then we prove that no finite subfamily of $(\pi(A_n))_{n\in\mathbb{N}}$ can be a homotopy basis of $\mathrm{LP}^\top$.

REFERENCES

[Ani86] David J. Anick, *On the homology of associative algebras*, Trans. Amer. Math. Soc. **296** (1986), no. 2, 641–659.

[BH82] Ronald Brown and Johannes Huebschmann, *Identities among relations*, Low-dimensional topology (Bangor, 1979), London Math. Soc. Lecture Note Ser., vol. 48, Cambridge Univ. Press, Cambridge, 1982, pp. 153–202.

[BN98] Franz Baader and Tobias Nipkow, *Term rewriting and all that*, Cambridge University Press, 1998.

[BO93] Ronald Book and Friedrich Otto, *String-rewriting systems*, Texts and Monographs in Computer Science, Springer-Verlag, 1993.

[Bro92] Kenneth S. Brown, *The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem*, Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989), Math. Sci. Res. Inst. Publ., vol. 23, Springer, New York, 1992, pp. 137–163.

[Bur93] Albert Burroni, *Higher-dimensional word problems with applications to equational logic*, Theoret. Comput. Sci. **115** (1993), no. 1, 43–62.

[BW90] Michael Barr and Charles Wells, *Category theory for computing science*, Prentice Hall International Series in Computer Science, Prentice Hall International, New York, 1990, Reprints in Theory and Applications of Categories, No. 22 (2012) pp. 1-538.

[CO94] Robert Cremanns and Friedrich Otto, *Finite derivation type implies the homological finiteness condition* FP_3, J. Symbolic Comput. **18** (1994), no. 2, 91–112.

[CO96] ———, *For groups the property of having finite derivation type is equivalent to the homological finiteness condition* FP_3, J. Symbolic Comput. **22** (1996), no. 2, 155–177.

[Coh92] Daniel E. Cohen, *A monoid which is right* FP_∞ *but not left* FP_1, Bull. London Math. Soc. **24** (1992), no. 4, 340–342.

[GGM] Stéphane Gaussent, Yves Guiraud, and Philippe Malbos, *Coherent presentations of Artin monoids*, Compos. Math., to appear.

[GM09] Yves Guiraud and Philippe Malbos, *Higher-dimensional categories with finite derivation type*, Theory Appl. Categ. **22** (2009), no. 18, 420–478.

[GM11] ———, *Identities among relations for higher-dimensional rewriting systems*, Semin. Congr. **26** (2011), 145–161.

[GM12a] ———, *Coherence in monoidal track categories*, Math. Structures Comput. Sci. **22** (2012), no. 6, 931–969.

[GM12b] ———, *Higher-dimensional normalisation strategies for acyclicity*, Adv. Math. **231** (2012), no. 3-4, 2294–2351.

[GMM13] Yves Guiraud, Philippe Malbos, and Samuel Mimram, *A homotopical completion procedure with applications to coherence of monoids*, Rewriting Techniques and Applications, Leibniz International Proceedings in Informatics, vol. 21, 2013, pp. 223–238.

[Gro90] John R. J. Groves, *Rewriting systems and homology of groups*, Groups – Canberra 1989, Lecture Notes in Math., vol. 1456, Springer, Berlin, 1990, pp. 114–141.

[Gui06] Yves Guiraud, *Termination orders for 3-dimensional rewriting*, J. Pure and Appl. Algebra **207** (2006), no. 2, 341–371.

[Jan82] Matthias Jantzen, *Semi Thue systems and generalized Church-Rosser properties*, Tech. report, Bericht Nr. 92, Fachbereich Informatik, Universität Hamburg, 1982.

[Jan85] _____, *A note on a special one-rule semi-Thue system*, Inform. Process. Lett. **21** (1985), no. 3, 135–140.

[KB70] Donald Knuth and Peter Bendix, *Simple word problems in universal algebras*, Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967), Pergamon, Oxford, 1970, pp. 263–297.

[KN85] Deepak Kapur and Paliath Narendran, *A finite Thue system with decidable word problem and without equivalent finite canonical system*, Theoret. Comput. Sci. **35** (1985), no. 2-3, 337–344.

[Kob90] Yuji Kobayashi, *Complete rewriting systems and homology of monoid algebras*, J. Pure Appl. Algebra **65** (1990), no. 3, 263–275.

[Laf95] Yves Lafont, *A new finiteness condition for monoids presented by complete rewriting systems (after Craig C. Squier)*, J. Pure Appl. Algebra **98** (1995), no. 3, 229–244.

[Laf07] _____, *Algebra and geometry of rewriting*, Appl. Categ. Structures **15** (2007), no. 4, 415–437.

[LP91] Yves Lafont and Alain Prouté, *Church-Rosser property and homology of monoids*, Math. Structures Comput. Sci. **1** (1991), no. 3, 297–326.

[LS01] Roger C. Lyndon and Paul E. Schupp, *Combinatorial group theory*, Classics in Mathematics, Springer-Verlag, Berlin, 2001, Reprint of the 1977 edition.

[Mar47a] Andrei Markov, *On the impossibility of certain algorithms in the theory of associative systems*, Doklady Akad. Nauk SSSR (N.S.) **55** (1947), 583–586.

[Mar47b] _____, *On the impossibility of certain algorithms in the theory of associative systems. II*, Doklady Akad. Nauk SSSR (N.S.) **58** (1947), 353–356.

[Mét83] Yves Métivier, *About the Rewriting Systems Produced by the Knuth-Bendix Completion Algorithm*, Inf. Process. Lett. **16** (1983), no. 1, 31–34.

[Mét03] François Métayer, *Resolutions by polygraphs*, Theory Appl. Categ. **11** (2003), no. 7, 148–184.

[Mét08] _____, *Cofibrant objects among higher-dimensional categories*, Homology, Homotopy Appl. **10** (2008), no. 1, 181–203.

[Mim10] Samuel Mimram, *Computing Critical Pairs in 2-Dimensional Rewriting Systems*, Rewriting Techniques and Applications, Leibniz International Proceedings in Informatics, vol. 6, 2010, pp. 227–242.

[ML98] Saunders Mac Lane, *Categories for the working mathematician*, 2nd ed., Springer, 1998.

[MPP05] Stuart McGlashan, Elton Pasku, and Stephen J. Pride, *Finiteness conditions for rewriting systems*, Internat. J. Algebra Comput. **15** (2005), no. 1, 175–205.

[New42] Maxwell Newman, *On theories with a combinatorial definition of "equivalence"*, Ann. of Math. (2) **43** (1942), no. 2, 223–243.

[OKK98] Friedrich Otto, Masashi Katsura, and Yuji Kobayashi, *Infinite convergent string-rewriting systems and cross-sections for finitely presented monoids*, J. Symbolic Comput. **26** (1998), no. 5, 621–648.

[Pos47] Emil L. Post, *Recursive unsolvability of a problem of Thue*, J. Symbolic Logic **12** (1947), 1–11.

[Pri95] Stephen J. Pride, *Low-dimensional homotopy theory for monoids*, Internat. J. Algebra Comput. **5** (1995), no. 6, 631–649.

[SOK94] Craig Squier, Friedrich Otto, and Yuji Kobayashi, *A finiteness condition for rewriting systems*, Theoret. Comput. Sci. **131** (1994), no. 2, 271–294.

[Squ87] Craig Squier, *Word problems and a homological finiteness condition for monoids*, J. Pure Appl. Algebra **49** (1987), no. 1-2, 201–217.

[Str76] Ross Street, *Limits indexed by category-valued* 2*-functors*, J. Pure Appl. Algebra **8** (1976), no. 2, 149–181.

[Str87] ———, *The algebra of oriented simplexes*, J. Pure Appl. Algebra **49** (1987), no. 3, 283–335.

[Ter03] Terese, *Term rewriting systems*, Cambridge Tracts in Theoretical Computer Science, vol. 55, Cambridge University Press, 2003.

[Thu14] Axel Thue, *Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln.*, Kristiania Vidensk. Selsk, Skr. **10** (1914), 493–524.

YVES GUIRAUD – yves.guiraud@pps.univ-paris-diderot.fr
INRIA Paris
IRIF, CNRS UMR 8243
Université Paris 7, Case 7014
75205 Paris Cedex 13, France

PHILIPPE MALBOS – malbos@math.univ-lyon1.fr
Univ Lyon, Université Claude Bernard Lyon 1
CNRS UMR 5208, Institut Camille Jordan
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne cedex, France