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Automatic semigroups *vs* automaton semigroups *

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Abstract

We develop an effective and natural approach to interpret any semigroup admitting a special language of greedy normal forms as an automaton semigroup, namely the semigroup generated by a Mealy automaton encoding the behaviour of such a language of greedy normal forms under one-sided multiplication. The framework embraces many of the well-known classes of (automatic) semigroups: finite monoids, free semigroups, free commutative monoids, trace or divisibility monoids, braid or Artin-Tits or Krammer or Garside monoids, Baumslag-Solitar semigroups, etc. Like plactic monoids or Chinese monoids, some neither left- nor right-cancellative automatic semigroups are also investigated, as well as some residually finite variations of the bicyclic monoid. It provides what appears to be the first known connection from a class of automatic semigroups to a class of automaton semigroups. It is worthwhile noting that, in all these cases, "being an automatic semigroup" and "being an automaton semigroup" become dual properties in a very automata-theoretical sense. Quadratic rewriting systems and associated tilings appear as a cornerstone of our construction.

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1 Introduction

The half century long history of the interactions between (semi)group theory and automata theory went through a pivotal decade from the mid-eighties to the mid-nineties. Concomitantly and independently, two new theories truly started to develop and thrive: automaton (semi)groups on the one hand with the works of Aleshin [2, 3] and Grigorchuk [24, 25] and the book [43], and automatic (semi)groups on the other hand with the work of Cannon and Thurston and the book [21]. We refer to [49] for a clear and short survey on the known interactions groups/automata. A deeper and more extended survey by Bartholdi and Silva can be found out in two chapters [5, 6] of the forthcoming AutoMathA handbook. We can refer to [12, 41] for automaton semigroups and to [15, 29] for automatic semigroups.

As their very name indicates, automaton (semi)groups and automatic (semi)groups share a same defining object: the *automaton* or the letter-to-letter transducer in this case. Beyond this common origin, these two topics until now happened to remain largely distant both in terms of community and in terms of tools or results. Typically, any paper on one or the other topic used to contain a sentence like "it should be emphasized that, despite their similar names, the notions of automaton (semi)groups are entirely separate from the notions of

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automatic (semi)groups". This was best evidenced by the above-mentioned valuable handbook chapter [5] which splits into exactly two sections (automatic groups and automaton groups) without any reference between one and the other appearing explicitly.

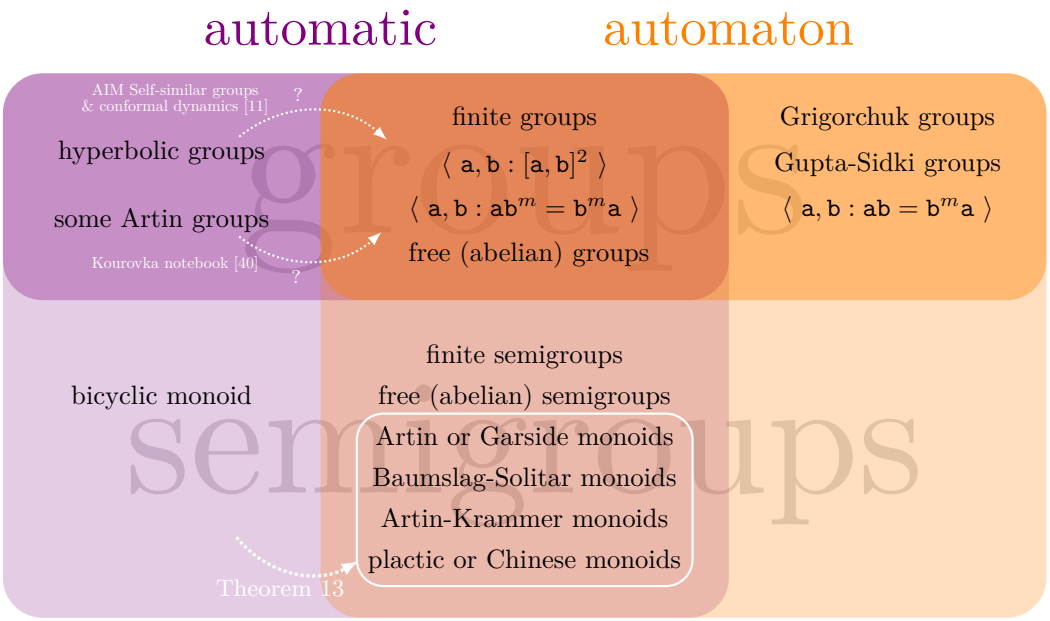


Figure 1 The big picture: comparing the classes of automatic *vs* automaton (semi)groups.

A significant problem is to recognize whether a given (semi)group is self-similar, that is, an automaton (semi)group. Amongst the unsolved problems in group theory of the Kourovka Notebook [40], there is the one (with number 16.84) asked by Sushchanskii (see also [36]):

> Can the n-strand braid group $\mathbf{B}_n$ act faithfully on a regular rooted tree by finite-state automorphisms?

This amounts to ask whether the braid group is a subgroup of some automaton group. Amongst the thirty-odd listed problems from [11], we can pick the one with the number 1.1:

> It seems quite difficult to show whether a given group is self-similar.
> - Are Gromov hyperbolic groups self-similar?
> - Find obstructions to self-similarity.

All these questions can be meaningfully rephrased in terms of semigroups or monoids.

Our aim here is to establish a possible connection between being an automatic semigroup and being an automaton semigroup. Preliminary observations are that these classes intersect non trivially and that none is included in the other (see Figure 1). Like the Grigorchuk group for instance, many automaton groups are infinite torsion groups, hence cannot be automatic groups. By contrast, it is an open question whether every automatic group is an automaton group. The latter is related to the question whether every automatic group is residually finite, which remains open despite the works by Wise [51] and Elder [20]. Like the bicyclic monoid, some automatic semigroups are not residually finite, hence cannot be automaton semigroups (see [12] for instance).

As for the intersection, we know that at least finite semigroups, free semigroups (of rank at least 2, see [9, 10]), free abelian semigroups happen to be both automatic semigroups and automaton semigroups. We propose here a new and natural way to interpret algorithmically each semigroup from a wide class of automatic semigroups—encompassing all the above-mentioned classes—as an automaton semigroup (Theorem 13). Furthermore, it is worthwhile noting that, in all these cases, "being an automatic semigroup" and "being an automaton semigroup" become dual properties in a very automata-theoretical sense (Corollary 16).

The structure of the paper is as follows. As a simple preliminary, Section 2 illustrates in a deliberately informal manner how a single Mealy automaton can be used in order to define both selfsimilar structures and automatic structures (via a principle of duality). In Section 3, we set up the notations for Mealy automata and recall necessary notions of dual automaton, cross-diagram, and selfsimilar structure. In Section 4, we recall basics about normal forms and automatic structures, and we give necessary notions of quadratic normalisation, square-diagram, and Garside family. Section 5 is devoted to our main result (Theorem 13 and Corollary 16), while Section 6 finally gathers several carefully selected examples and counterexamples.

2 A preliminary example

As their very name indicates, automaton (semi)groups and automatic (semi)groups share a same defining object. In both cases, a *Mealy automaton* (see Definition 1) basically allows to transform words into words.

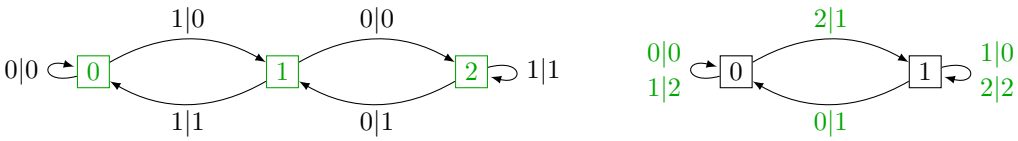


Figure 2 Two (dual) Mealy automata: the left-hand one computes the division by 3 in base 2 (most significant digit first) *vs* the right-hand one computes the multiplication by 2 in base 3 (least significant digit first).

The Mealy automaton displayed on Figure 2 (left) became one of the most classical examples of a transducer (see [47] for a delightfully alternate history): when starting from the state 0 and reading any binary word $\mathbf{u}$ (most significant digit first), it computes the division by 3 in base 2 by outputting the (quotient) binary word $\mathbf{v}$ (most significant digit first) satisfying

$$(\mathbf{u})_2 = 3 \times (\mathbf{v})_2 + f \tag{1}$$

where $f \in \{0, 1, 2\}$ corresponds to the arrival state of the run, and where $(\mathbf{w})_b$ conventionally denotes the number that is represented by $\mathbf{w}$ in base b.

For the current preliminary section, let us now focus on this basic example and consider the two different viewpoints described as follows. On the one hand, it seems natural to consider the set of those functions (from binary words to binary words) thus associated with each state, then to compose them with each other, and finally to study the (semi)group which is generated by such functions.

For instance, the function $\mathbf{u} \mapsto \mathbf{v}$ associated with the state 0 can be squared, cubed, and so on, to obtain functions, which can be again interpreted as the division by $9, 27, \ldots$ (in base 2 with most significant digit first). One can show here that the generated semigroup is the rank 3 free semigroup $\{0, 1, 2\}^+$ (provided that the three states and their induced functions are identified).
This simple idea coincides with the notion of automaton (semi)groups or selfsimilar structures (see Definition 2). With this crucial standpoint, we can compute (semi)group operations by manipulating the corresponding Mealy automaton (see [4, 5, 31, 42]), and hopefully foresee some combinatorial and dynamical properties by examining its shape (see [7, 8, 16, 22, 23, 30, 33, 34, 50] for instance).

On the other hand, it may be also natural to simply iterate the runs. The starting language is again over the (input/output) alphabet, now the images of the transformations are some languages over the stateset.
For instance, restarting again from the state 0, the previously output word $\mathbf{v}$ (satisfying Equation (1)) can be read in turn, and so on. The successive arrival states can be then collected and concatenated in order to obtain here the decomposition of $(\mathbf{u})_2$ in base 3 (least significant digit first).
This second idea coincides with the fundamental notion of automatic (semi)groups (see Definition 3).

To conclude this preliminary section, let us mention that states and letters of any Mealy automaton play a symmetric role, and that several properties can be beneficially derived from the so-called *dual (Mealy) automaton*, obtained by exchanging the stateset and the alphabet (see [32] for an overview).
For instance, Figure 2 displays a pair of dual automata. While the left-hand automaton allows to compute the division by 3 in base 2 (most significant digit first) as we have seen just above, its dual automaton (right) essentially computes the multiplication by 2 in base 3 (least significant digit first). More precisely, its state 0 induces the function $x \mapsto 2 \times x$, while its state 1 induces the function $x \mapsto 2 \times x + 1$: they together generate the rank 2 free semigroup. Now, the induced functions happen to be invertible and to generate a group which is isomorphic with the so-called Baumslag-Solitar group

$$\mathbf{BS}(2, 1) = \langle\ \alpha, \delta : \delta\alpha^2 = \alpha\delta\ \rangle$$

where δ denotes the *doubling* action induced by the state 0 and α is defined as the product of the inverse of δ and the action induced by the state 1, hence α corresponds to the *adding* function $x \mapsto x + 1$ (see [7] for further details).

Besides, such a Mealy automaton (right) can be used to compute the base 2 representation from the base 3 representation of the fractional part of any non-negative rational number, by iterating runs as explained above. For instance, iterated runs from the state 0 and the initial word 12 produce the word $(110001)^\omega$, both words representing (the fractional part of) the real $\left(\frac{7}{9}\right)_{10}$ in base 3 (least significant digit first) and in base 2 (most significant digit first) respectively.

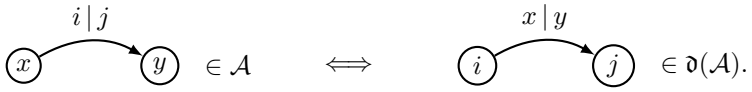
This innocuous example allows to illustrate the quite simple machineries associated both with automaton semigroups and with automatic semigroups. It also aims to give an informal glimpse on their behaviours through the duality principle: for instance, division *vs* multiplication, factor *vs* base, least *vs* most significant digit first, integer part *vs* fractional part.

3 Mealy automata and selfsimilar structures

We first recall the formal definition of an automaton.

▶ **Definition 1.** A *(finite, deterministic, and complete) automaton* is a triple $\big(Q, \Sigma, \tau = (\tau_i \colon Q \to Q)_{i\in\Sigma}\big)$, where the *stateset* Q and the *alphabet* Σ are non-empty finite sets, and where the τ_i's are functions.
A *Mealy automaton* is a quadruple $\big(Q, \Sigma, \tau = (\tau_i \colon Q \to Q)_{i\in\Sigma}, \sigma = (\sigma_x \colon \Sigma \to \Sigma)_{x\in Q}\big)$ such that both (Q, Σ, τ) and (Σ, Q, σ) are automata.

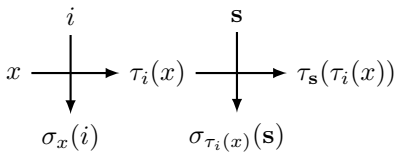
In other terms, a Mealy automaton is a complete, deterministic, letter-to-letter transducer with the same input and output alphabet.
The graphical representation of a Mealy automaton is standard, see Figures 2, 8, and 11.
In a Mealy automaton $\mathcal{A} = (Q, \Sigma, \tau, \sigma)$, the sets Q and Σ play dual roles. So we may consider the *dual (Mealy) automaton* defined by $\mathfrak{d}(\mathcal{A}) = (\Sigma, Q, \sigma, \tau)$:



We view $\mathcal{A} = (Q, \Sigma, \tau, \sigma)$ as an automaton with an input and an output tape, thus defining mappings from input words over Σ to output words over Σ. Formally, for $x \in Q$, the map $\sigma_x \colon \Sigma^* \to \Sigma^*$, extending $\sigma_x \colon \Sigma \to \Sigma$, is defined recursively by:

$$\forall i \in \Sigma,\ \forall \mathbf{s} \in \Sigma^*, \qquad \sigma_x(i\mathbf{s}) = \sigma_x(i)\sigma_{\tau_i(x)}(\mathbf{s})\,. \tag{2}$$

Equation (2) can be easier to understood if depicted by a *cross-diagram* (see [1]):



By convention, the image of the empty word is itself. The mapping σ_x for each $x \in Q$ is length-preserving and prefix-preserving. We say that σ_x is the *production function* associated with $(\mathcal{A}, x)$. For $\mathbf{x} = x_1 \cdots x_n \in Q^n$ with $n > 0$, set $\sigma_{\mathbf{x}} \colon \Sigma^* \to \Sigma^*, \sigma_{\mathbf{x}} = \sigma_{x_n} \circ \cdots \circ \sigma_{x_1}$. Denote dually by $\tau_i \colon Q^* \to Q^*, i \in \Sigma$, the production functions associated with the dual automaton $\mathfrak{d}(\mathcal{A})$. For $\mathbf{s} = s_1 \cdots s_n \in \Sigma^n$ with $n > 0$, set $\tau_{\mathbf{s}} \colon Q^* \to Q^*,\ \tau_{\mathbf{s}} = \tau_{s_n} \circ \cdots \circ \tau_{s_1}$.

▶ **Definition 2.** The semigroup of mappings from Σ^* to Σ^* generated by $\{\sigma_x, x \in Q\}$ is called the *semigroup generated by* $\mathcal{A}$ and is denoted by $\langle\mathcal{A}\rangle_+$. When $\mathcal{A}$ is invertible, its production functions are permutations on words of the same length and thus we may consider the group of mappings from Σ^* to Σ^* generated by $\{\sigma_x, x \in Q\}$. This group is called the *group generated by* $\mathcal{A}$ and is denoted by $\langle\mathcal{A}\rangle$. In both cases, the term *selfsimilar* is used as a synonym.

4 Quadratic normalisations and automatic structures

This section gathers the definitions of some classical notions like normal form or automatic structure, together with the little more specific notions of a quadratic normalisation (see [19]) and a Garside family (see [17, 18]).

For any set $\mathcal{Q}$, we denote by $\mathcal{Q}^+$ the free semigroup over $\mathcal{Q}$ (*resp.* by $\mathcal{Q}^*$ the free monoid and by $\mathtt{1}$ its unit element) and call its elements $\mathcal{Q}$-*words*. We write $\|\mathbf{w}\|$ for the length of a $\mathcal{Q}$-word $\mathbf{w}$, and $\mathbf{w}\mathbf{w}'$ for the product of two $\mathcal{Q}$-words $\mathbf{w}$ and $\mathbf{w}'$.

▶ **Definition 3.** Let S be a semigroup with a generating subset $\mathcal{Q}$. A *normal form* for $(S, \mathcal{Q})$ is a (set-theoretic) section of the canonical projection EV from the language of $\mathcal{Q}$-words onto S, that is, a map NF that assigns to each element of S a distinguished representative $\mathcal{Q}$-word:

$$\mathrm{EV} : \mathcal{Q}^+ \longrightarrow S \qquad \mathrm{NF}$$

Whenever $\mathrm{NF}(S)$ is regular, it provides a *right-automatic structure* for S if the language $\mathcal{L}_q = \{\ (\mathrm{NF}(a), \mathrm{NF}(aq)) : a \in S\ \}$ is regular for each $q \in \mathcal{Q}$. The semigroup S then can be called a *(right-)automatic semigroup*.

We mention here the thorough and precious study in [28] of the different notions (right- or left-reading *vs* right- or left-multiplication) of automaticity for semigroups.

▶ Remark. In his seminal work [21, Chapter 9], Thurston shows how the set of these different automata recognizing the multiplication—that is, recognizing the languages $\mathcal{L}_q$ of those pairs of normal forms of elements differing by a right factor $q \in \mathcal{Q}$—in Definition 3 can be replaced with advantage by a single letter-to-letter transducer (see Definition 15) that computes the normal forms via iterated runs, each run both providing one brick of the final normal from and outputting a word still to be normalised.

One will often consider the associated normalisation $\mathrm{N} = \mathrm{NF} \circ \mathrm{EV}$.

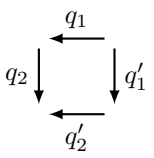
▶ **Definition 4.** A *normalisation* is a pair $(\mathcal{Q}, \mathrm{N})$, where $\mathcal{Q}$ is a set and N is a map from $\mathcal{Q}^+$ to itself satisfying, for all $\mathcal{Q}$-words $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

- $\|\mathrm{N}(\mathbf{w})\| = \|\mathbf{w}\|$,
- $\|\mathbf{w}\| = 1 \Rightarrow \mathrm{N}(\mathbf{w}) = \mathbf{w}$,
- $\mathrm{N}(\mathbf{u}\,\mathrm{N}(\mathbf{w})\,\mathbf{v}) = \mathrm{N}(\mathbf{uwv})$.

A $\mathcal{Q}$-word $\mathbf{w}$ satisfying $\mathrm{N}(\mathbf{w}) = \mathbf{w}$ is called N-*normal*. If S is a semigroup, we say that $(\mathcal{Q}, \mathrm{N})$ is a *normalisation for* S if S admits the presentation

$$\langle\ \mathcal{Q} : \{\ \mathbf{w} = \mathrm{N}(\mathbf{w}) \mid \mathbf{w} \in \mathcal{Q}^+\ \}\ \rangle_+.$$

Following [18], we associate with every element $q \in \mathcal{Q}$ a q-labeled edge and with a product the concatenation of the corresponding edges and represent equalities in the ambient semigroup using commutative diagrams, what we called here *square-diagram*: for instance, the following square illustrates an equality $q_1 q_2 = q_1' q_2'$.



For a normalisation $(\mathcal{Q}, \mathrm{N})$, we denote by $\overline{\mathrm{N}}$ the restriction of N to $\mathcal{Q}^2$ and, for $i \geq 1$, by $\overline{\mathrm{N}}_i$ the (partial) map from $\mathcal{Q}^+$ to itself that consists in applying $\overline{\mathrm{N}}$ to the entries in position i and $i+1$. For any finite sequence $\mathbf{i} = i_1 \cdots i_n$ of positive integers, we write $\overline{\mathrm{N}}_{\mathbf{i}}$ for the composite map $\overline{\mathrm{N}}_{i_n} \circ \cdots \circ \overline{\mathrm{N}}_{i_1}$ (so $\overline{\mathrm{N}}_{i_1}$ is applied first).

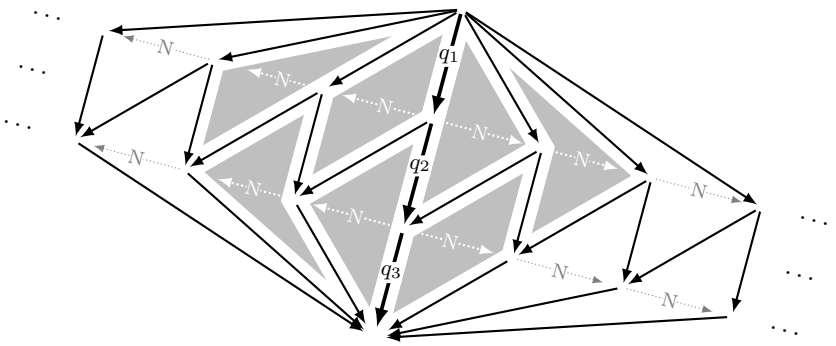


Figure 3 From an initial $\mathcal{Q}$-word $q_1q_2q_3$, one applies normalisations on the first and the second 2-factors alternatively up to stabilisation, beginning either on the first 2-factor q_1q_2 (right-hand side here) or on the second q_2q_3. The gray zone corresponds to Condition ( ) as defined in Definition 6.

▶ **Definition 5.** A normalisation $(\mathcal{Q}, \mathrm{N})$ is *quadratic* if the two conditions hold:

- a $\mathcal{Q}$-word $\mathbf{w}$ is N-normal if, and only if, every length-two factor of $\mathbf{w}$ is;
- for every $\mathcal{Q}$-word $\mathbf{w}$, there exists a finite sequence $\mathbf{i}$ of positions, depending on $\mathbf{w}$, such that $\mathrm{N}(\mathbf{w})$ is equal to $\overline{\mathrm{N}}_{\mathbf{i}}(\mathbf{w})$.

▶ **Definition 6.** As illustrated on Figure 3, with any quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ is associated its *breadth* (d, p) (called minimal left and right classes in [17, 19]) defined as:

$$d = \max_{(q_1,q_2,q_3)\in\mathcal{Q}^3} \min\{\,\ell : \mathrm{N}(q_1q_2q_3) = \overline{\mathrm{N}}\underbrace{_{212\cdots}}_{\text{length }\ell}(q_1q_2q_3)\},$$

and

$$p = \max_{(q_1,q_2,q_3)\in\mathcal{Q}^3} \min\{\,\ell : \mathrm{N}(q_1q_2q_3) = \overline{\mathrm{N}}\underbrace{_{121\cdots}}_{\text{length }\ell}(q_1q_2q_3)\}.$$

Such a breadth is ensured to be finite provided that $\mathcal{Q}$ is finite, and then satisfies $|d-p| \leq 1$. For $d \leq 4$ and $p \leq 3$, $(\mathcal{Q}, \mathrm{N})$ is said to satisfy Condition ( ), corresponding to the so-called *domino rule* in [17, 18, 19] but with a different reading direction.

The first main result of [19] is an axiomatisation of these quadratic normalisations satisfying Condition ( ) in terms of their restrictions to length-two words: any idempotent map $\overline{\mathrm{N}}$ on $\mathcal{Q}^2$ that satisfies $\overline{\mathrm{N}}_{2121} = \overline{\mathrm{N}}_{121} = \overline{\mathrm{N}}_{1212}$ extends into a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ satisfying Condition ( ). For larger breadths, a map on length-two words normalising length-three words needs not normalise words of greater length.
The second main result of [19] involves termination. Every quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ gives rise to a quadratic rewriting system, namely the one with rules $\mathbf{w} \longrightarrow \overline{\mathrm{N}}(\mathbf{w})$ for $\mathbf{w} \in \mathcal{Q}^2$. By Definition 5, such a rewriting system is confluent and normalising, meaning that, for every initial word, there exists a finite sequence of rewriting steps leading to a unique N-normal word, but its convergence, meaning that *any* sequence of rewriting steps is finite, is a quite different problem.

▶ **Theorem 7.** [19] *If $(\mathcal{Q}, \mathrm{N})$ is a quadratic normalisation satisfying Condition ( ), then the associated rewriting system is convergent.*

More precisely, every rewriting sequence starting from a word of $\mathcal{Q}^p$ has length at most $\frac{p(p-1)}{2}$ (*resp.* $2^p - p - 1$) in the case of a breadth $(3, 3)$ (*resp.* either $(3, 4)$ or $(4, 3)$). Theorem 7 is essentially optimal since there exist nonconvergent rewriting systems with breadth $(4, 4)$.

The rest of the current section describes a tiny fragment of Garside theory (see [18] for its foundations). Garside families were recently introduced as a general framework guaranteeing the existence of normal forms. Even if this notion is not necessary for the understanding of the main result, its proof, and the whole of Section 5, several examples of Section 6 could rely on it.

Let S be a monoid. For $a, b, c \in S$, a is a *left-divisor* of b or, equivalently, b is a *right-multiple* of a if $b = ab'$ holds for some b' in S; moreover, b is a minimal common right-multiple, or *right-mcm*, of a and c if b is a right-multiple of a and c, and no proper left-divisor of b is a right-multiple of a and c.
Furthermore, S is said to be *right-cancellative* whenever, for all $a, b, c \in S$, $ab = cb$ implies $a = c$, and S admits no nontrivial *invertible element* whenever $ab = 1$ implies $a = b = 1$. *Right-divisor*, *left-mcm*, and *left-cancellativity* are defined symmetrically.

▶ **Definition 8.** If S is a right-cancellative monoid with no nontrivial invertible element, a *(right-)Garside family* for S is a generating set closed under left-divisor and under left-mcm.

Various practical characterisations of Garside families are known, depending in particular on the specific properties of the considered monoid. The following is especially relevant here.

▶ **Theorem 9.** [19] *Assume that S is a right-cancellative monoid with no nontrivial invertible element and with a finite (right-)Garside family $\mathcal{Q}$. Then the normalisation $(\mathcal{Q}, \mathrm{N})$ defined by $\mathrm{N}(ab) = cd$ for $a, b, c, d \in \mathcal{Q}$ with d maximal, satisfies Condition ( ).*

The results of Section 5 rely on the special Condition ( ) outlined by Dehornoy and Guiraud (see [19]). However, none of their results (in particular Theorems 7 and 9 mentioned here for completeness) is neither applied nor needed to establish ours. The current exposition is thus self-contained and our constructions never require any of their stronger hypotheses (neither cancellativity nor absence of nontrivial invertible elements). We here emphasize that Condition ( ) appears as a common denominator for our different approaches.

5 From an automatic structure to a selfsimilar structure

All the ingredients are now in place to effectively and naturally interpret as an automaton monoid any automatic monoid admitting a special language of normal forms—namely, a quadratic normalisation satisfying Condition ( ). The point is to construct a Mealy automaton encoding the behaviour of its language of normal forms under one-sided multiplication.

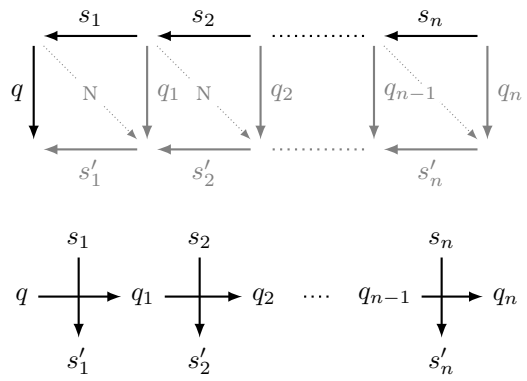
▶ **Definition 10.** Assume that S is a semigroup with a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$. We define the Mealy automaton $\mathcal{M}_{S,\mathcal{Q},\mathrm{N}} = (\mathcal{Q}, \mathcal{Q}, \tau, \sigma)$ such that, for every $(a, b) \in \mathcal{Q}^2$, $\sigma_b(a)$ is the rightmost element of $\mathcal{Q}$ in the normal form $\mathrm{N}(ab)$ of ab and $\tau_a(b)$ is the left one:

$$\mathrm{N}(ab) = \tau_a(b)\sigma_b(a).$$

The latter correspondence can be simply interpreted via square-diagram *vs* cross-diagram:



Then, for $\mathrm{N}(\mathbf{s}) = s_n \cdots s_1$ and $\mathrm{N}(\mathbf{s}q) = q_n s'_n \cdots s'_1$, we obtain diagrammatically:



We choose on purpose to always draw a normalisation square-diagram backward, such that it coincides with the associated cross-diagram. The function σ_q induced by the state q maps any normal form (read backward) to the normal form of the right-product by q (read backward).

We now aim to strike reasonable (most often optimal) hypotheses for a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ associated with an original semigroup S to generate a semigroup $\langle \mathcal{M}_{S,\mathcal{Q},\mathrm{N}} \rangle_+$ that approximates S as sharply as possible. Since the generating sets coincide by Definition 10, we shall first focus on the case where S should be a quotient of $\langle \mathcal{M}_{S,\mathcal{Q},\mathrm{N}} \rangle_+$ (top-approximation), and next, on the case where $\langle \mathcal{M}_{S,\mathcal{Q},\mathrm{N}} \rangle_+$ should be a quotient of S (bottom-approximation).

Before establishing our top-approximation statement, we just recall that semigroups could appear much more difficult to handle, especially when it comes to automaticity (see [28]) or selfsimilarity (see [9, 10]). To any (not monoid) semigroup S with a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$, one obtains a monoid S^1 with a quadratic normalisation $(\mathcal{Q}^1, \mathrm{N}^1)$ by adjoining a unit $\mathtt{1}$ and then by setting $\mathcal{Q}^1 = \mathcal{Q} \sqcup \{\mathtt{1}\}$ and defining N^1 by $\mathrm{N}^1(\mathbf{w}) = \mathrm{N}(\mathbf{w})$ and

$$\mathrm{N}^1(\mathtt{1}\mathbf{w}) = \mathrm{N}^1(\mathbf{w}\mathtt{1}) = \mathtt{1}\mathrm{N}(\mathbf{w}) \qquad (\boxed{\mathrm{N}^1})$$

for $\mathbf{w} \in \mathcal{Q}^+$. The choice made for Condition ($\boxed{\mathrm{N}^1}$) becomes natural whenever we think of the (adjoined or not) unit $\mathtt{1}$ as some *dummy* element that simply ensures the length-preserving property for N^1 (see Definition 4 and also [19, Section 2.2]).

▶ **Lemma 11.** *Assume that S^1 is a monoid with a quadratic normalisation $(\mathcal{Q}^1, \mathrm{N}^1)$ satisfying Condition ($\boxed{\mathrm{N}^1}$). Then the Mealy automaton $\mathcal{M}_{S^1,\mathcal{Q}^1,\mathrm{N}^1}$ generates a monoid of which S^1 is a quotient.*

Proof. Let $S^1 = \mathcal{Q}^*/\equiv_{\mathrm{N}^1}$ and $\mathcal{M}_{S^1,\mathcal{Q}^1,\mathrm{N}^1} = (\mathcal{Q}^1, \mathcal{Q}^1, \tau, \sigma)$ as in Definition 10. We have to prove that any relation in $\langle \mathcal{M}_{S^1,\mathcal{Q}^1,\mathrm{N}^1} \rangle_+^1$ is a relation in S^1, thereby implying for all $\mathbf{u}, \mathbf{v} \in \mathcal{Q}^*$:

$$\sigma_{\mathbf{u}} = \sigma_{\mathbf{v}} \implies \mathbf{u} \equiv_{\mathrm{N}^1} \mathbf{v}.$$

Let $\sigma_{p_1} \cdots \sigma_{p_k} = \sigma_{q_1} \cdots \sigma_{q_\ell}$ be some relation in $\langle \mathcal{M}_{S,\mathcal{Q},\mathrm{N}} \rangle_+^1$ with $p_i \in \mathcal{Q}$ for $0 \le i \le k$ and $q_j \in \mathcal{Q}$ for $0 \le j \le \ell$. Any word $\mathbf{w}$ over $\mathcal{Q}^1$ admits hence the same image under $\sigma_{p_1} \cdots \sigma_{p_k}$ and under $\sigma_{q_1} \cdots \sigma_{q_\ell}$. By taking $\mathbf{w} = \mathtt{1}^\omega$ (or any sufficiently long power of $\mathtt{1}$, precisely any word from $\mathtt{1}^{\max(k,\ell)}\mathtt{1}^*$), such a common image corresponds to their normal forms by very definition of $\mathcal{M}_{S^1,\mathcal{Q}^1,\mathrm{N}^1}$ (see Figure 4). Therefore the resulting letterwise

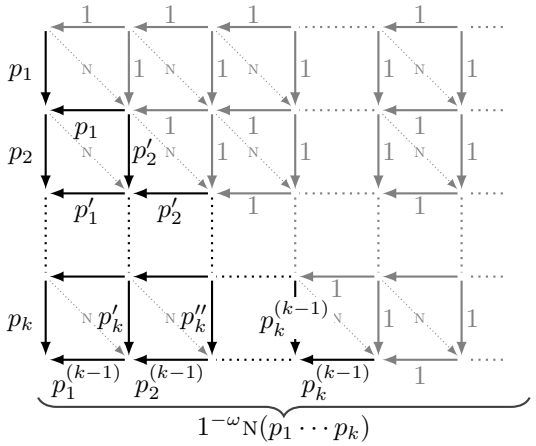


Figure 4 Proof of Lemma 11: any $\mathcal{Q}$-words inducing a same action have normal forms that coincide.

equality $1^{-\omega}\mathrm{N}(p_1 \cdots p_k) = 1^{-\omega}\mathrm{N}(q_1 \cdots q_\ell)$ (where $1^{-\omega}$ denotes the left-infinite word $\cdots 111$) implies that the two corresponding $\mathcal{Q}$-words $p_1 \cdots p_k$ and $q_1 \cdots q_\ell$ represent a same element in S^1 by definition of N^1. ◀

Although specific to a monoidal framework and then requiring the innocuous Condition (N¹), the previous straightforward proof relies only on the definition of a quadratic normalisation and on the well-fitted associated Mealy automaton (Definition 10). For the bottom-approximation statement, we consider an extra assumption, which happens to be necessary and sufficient.

▶ **Proposition 12.** *Assume that S is a semigroup with a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$. If Condition (◆) is satisfied, then the Mealy automaton $\mathcal{M}_{S,\mathcal{Q},\mathrm{N}}$ generates a semigroup quotient of S. The converse holds provided that Condition (N¹) is satisfied.*

Proof. Let $S = \mathcal{Q}^+/\equiv_{\mathrm{N}}$ and $\mathcal{M}_{S,\mathcal{Q},\mathrm{N}} = (\mathcal{Q}, \mathcal{Q}, \tau, \sigma)$ as in Definition 10.

($\Leftarrow$) Assume that Condition (◆) is satisfied and that there exists $(a, b, c, d) \in \mathcal{Q}^4$ with $ab \equiv_{\mathrm{N}} cd$. We have to prove $\sigma_{ab} = \sigma_{cd}$. Without lost of generality, the word ab can be supposed to be N-normal, that is, $\mathrm{N}(ab) = \mathrm{N}(cd) = ab$ holds.
Let $\mathbf{u} = q\mathbf{v} \in \mathcal{Q}^n$ for some $n > 0$ and $q \in \mathcal{Q}$. We shall prove both $\sigma_{ab}(\mathbf{u}) = \sigma_{cd}(\mathbf{u})$ (coordinatewise) and $\tau_{\mathbf{u}}(ab) \equiv_{\mathrm{N}} \tau_{\mathbf{u}}(cd)$ by induction on $n > 0$. For $n = 1$, we obtain the two square-diagrams on Figure 5 (left). With these notations, we have to prove $q''_0 = q''_1$ and $a'b' \equiv_{\mathrm{N}} c'd'$, the latter meaning $\mathrm{N}(a'b') = \mathrm{N}(c'd')$, that is, with the notations from Figure 5, the conjunction of $a'' = c''$ and $b'' = d''$. Now these three equalities hold whenever $(\mathcal{Q}, \mathrm{N})$ satisfies Condition (◆), as shown on Figure 5 (right).
This allows to proceed the induction and to prove the implication ($\Leftarrow$).

($\Rightarrow$) Consider an arbitrary length 3 word over $\mathcal{Q}$, say qcd. Let a, b denote the elements in $\mathcal{Q}$ satisfying $\mathrm{N}(cd) = ab$. By definition, we deduce $ab \equiv_{\mathrm{N}} cd$. This implies $\sigma_{ab} = \sigma_{cd}$ by hypothesis. In particular, the images of any word $q\mathbf{v}$ under σ_{ab} and σ_{cd} coincide: $\sigma_{ab}(q\mathbf{v}) = \sigma_{cd}(q\mathbf{v})$, hence

$$\sigma_{ab}(q) = q''_0 = q''_1 = \sigma_{cd}(q)$$

and

$$\sigma_{\tau_q(ab)}(\mathbf{v}) = \sigma_{a'b'}(\mathbf{v}) = \sigma_{c'd'}(\mathbf{v}) = \sigma_{\tau_q(cd)}(\mathbf{v})$$

(with notations of Figure 5). The last equality holds for any original word $\mathbf{v} \in \mathcal{Q}^*$ and implies $\sigma_{a'b'} = \sigma_{c'd'}$. Whenever, Condition ( ) is satisfied, we deduce $\mathrm{N}(a'b') = \mathrm{N}(c'd')$ according to Lemma 11. We obtain

$$\overline{\mathrm{N}}_{121}(qcd) = \overline{\mathrm{N}}_{2121}(qcd).$$

Therefore $(\mathcal{Q}, \mathrm{N})$ satisfies Condition ( ). ◀

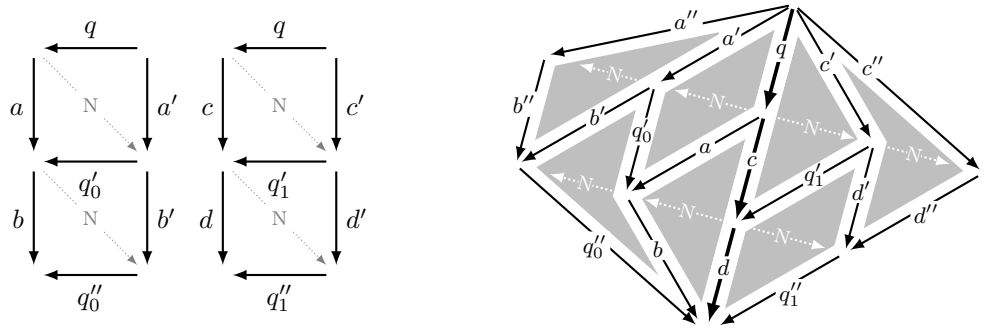


Figure 5 Proof of Proposition 12: initial data (left) can be pasted into Condition ( ) (right).

Gathering Lemma 11 and Proposition 12, we obtain the following main result.

▶ **Theorem 13.** *Assume that S is a monoid with a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ satisfying Conditions ( ) and ( ). Then the Mealy automaton $\mathcal{M}_{S,\mathcal{Q},\mathrm{N}}$ generates a monoid isomorphic to S.*

Proof. By construction, S and $\langle \mathcal{M}_{S,\mathcal{Q},\mathrm{N}} \rangle_+^1$ share a same generating subset $\mathcal{Q}$. Now, any defining relation for S maps to a defining relation for $\langle \mathcal{M}_{S,\mathcal{Q},\mathrm{N}} \rangle_+^1$ by Proposition 12, and conversely by Lemma 11. ◀

▶ **Corollary 14.** *Any monoid with a quadratic normalisation satisfying Conditions ( ) and ( ) is residually finite.*

To conclude this main section, we come back to that remark (following Definition 3) about the original transducer approach by Thurston.

▶ **Definition 15.** With any quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ is associated its *Thurston transducer* defined as the Mealy automaton $\mathcal{T}_{\mathcal{Q},\mathrm{N}}$ with stateset $\mathcal{Q}$, alphabet $\mathcal{Q}$, and transitions as follows:



▶ **Corollary 16.** *Assume that S is a monoid with a quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ satisfying Conditions ( ) and ( ). The Thurston transducer $\mathcal{T}_{\mathcal{Q},\mathrm{N}}$ and the Mealy automaton $\mathcal{M}_{S,\mathcal{Q},\mathrm{N}}$ being dual automaton, S possesses both the explicitly dual properties of automaticity and selfsimilarity.*

6 Examples and counterexamples

Our very first example is straightforward, but enlightening.

▶ **Example 17.** Every finite monoid $\mathcal{J}$ (in particular every finite group) is an automaticon monoid. Consider its quadratic normalisation $(\mathcal{J}, \mathrm{N})$ with $\mathrm{N}(ab) = 1(ab)$ for every $(a, b) \in \mathcal{J}^2$. Figure 6 shows how to compute its breadth $(3, 2)$, witness of Condition ( ) for applying Theorem 13.

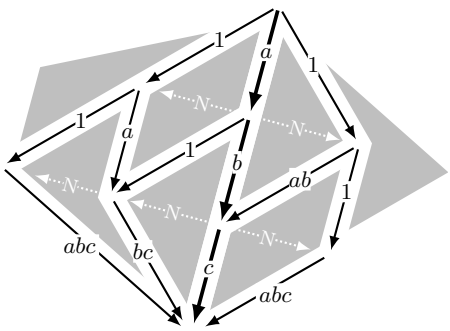


Figure 6 Computing the breadth $(3, 2)$ for any finite monoid $\mathcal{J}$ as in Example 17.

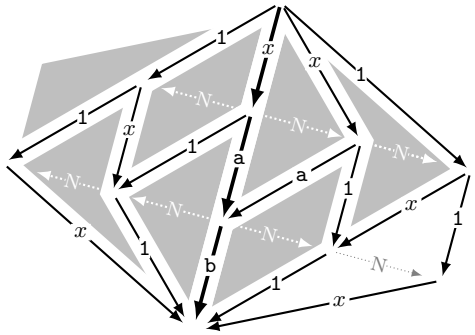


Figure 7 Computing the breadth $(3, 4)$ for the bicylic monoid $\mathbf{B}$ from Example 18.

As mentioned in Section 1 and appearing on Figure 1, there exist automatic semigroups that cannot be automaton semigroups.

▶ **Example 18.** The bicyclic monoid $\mathbf{B} = \langle\, \mathtt{a}, \mathtt{b} : \mathtt{ab} = 1 \,\rangle^1_+$ is known to be automatic and not residually finite, hence cannot be an automaton monoid. Choose for $\mathbf{B}$ the quadratic normalisation $(\{\mathtt{a}, \mathtt{b}, 1\}, \mathrm{N})$ with $\mathrm{N}(\mathtt{ab}) = 1$, $\mathrm{N}(x1) = 1x$ for $x \in \{\mathtt{a}, \mathtt{b}\}$, and $\mathrm{N}(xy) = xy$ otherwise. Figure 7 illustrates the computation (on the witness word $x\mathtt{ab}$ with $x \in \{\mathtt{a}, \mathtt{b}\}$) of its breadth $(3, 4)$. The Condition ( ) is hence not satisfied and Theorem 13 cannot apply. Precisely, according to the proof of Proposition 12 and Figure 5, we have $\sigma_{\mathtt{ab}}(x) = 1 \neq x = \sigma_{11}(x)$ for $x \in \{\mathtt{a}, \mathtt{b}\}$, hence $\sigma_{\mathtt{ab}} \neq \sigma_1 = \sigma_{11}$.

By contrast, one of the simplest nontrivial examples could be the following.

▶ **Example 19.** The automatic (right-cancellative) monoid $\langle\, \mathtt{a}, \mathtt{b} : \mathtt{ab} = \mathtt{a} \,\rangle^1_+$ admits $\mathcal{Q} = \{1, \mathtt{a}, \mathtt{b}\}$ as a (right-)Garside family. According to Proposition 9 and Theorem 13, it is therefore an automaton monoid. The corresponding Mealy automaton is displayed on Figure 8. The latter happens to be the common smallest nontrivial member of the family of Baumslag-Solitar monoids (see [28] for instance), namely $\mathbf{BS}^1_+(1, 0)$, and of a wide family of right-cancellative semigroups, that we readily call *Artin-Krammer* monoids and that have been introduced and studied in [35] (see also [26, 27, 44]), namely $\mathbf{AK}^1_+(\Gamma)$ associated with the Coxeter-like matrix $\Gamma = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

Examples 20 and 21 describe important members from both these families.

▶ **Example 20.** The following Artin-Krammer monoid is emblematic:

$$\mathbf{AK}^1_+\left(\begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}\right) = \left\langle \mathtt{a}, \mathtt{b}, \mathtt{c} : \begin{matrix} \mathtt{abab} = \mathtt{aba} \\ \mathtt{ac} = \mathtt{ca} \\ \mathtt{bcbc} = \mathtt{bcb} \end{matrix} \right\rangle^1_+ .$$

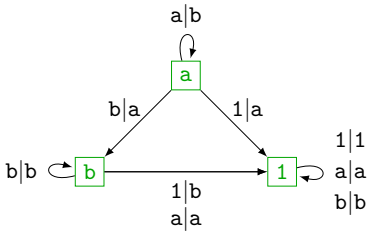


Figure 8 The Mealy automaton associated with the monoid $\langle\, \mathtt{a}, \mathtt{b} : \mathtt{ab} = \mathtt{a} \,\rangle^1_+$ from Example 19.

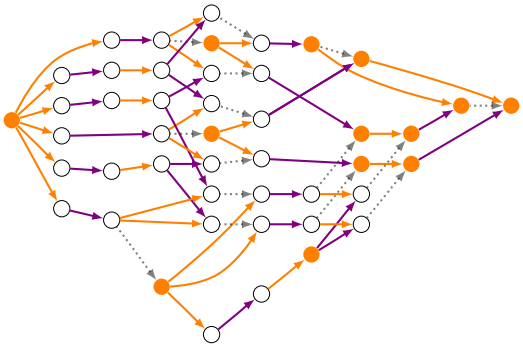


Figure 9 The minimal Garside family of the monoid $\mathbf{AK}^1_+\left(\begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 2 & 4 & 1 \end{bmatrix}\right)$ from Example 20.

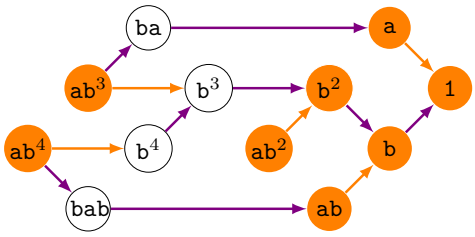


Figure 10 The minimal Garside family of the monoid $\mathbf{BS}^1_+(3,2)$ from Example 21.

As displayed on Figure 9, its minimal Garside family forms like a flint which encodes its whole combinatorics and, according to Theorem 13, makes it an automaticon monoid.

▶ **Example 21.** Consider the Baumslag-Solitar monoid $\mathbf{BS}^1_+(3,2) = \langle\, \mathtt{a}, \mathtt{b} : \mathtt{ab}^3 = \mathtt{b}^2\mathtt{a} \,\rangle^1_+$. Displayed on Figure 10, its minimal Garside family contains eight elements (orange vertices) and makes it an automaticon monoid. This is an example of a group-embeddable automaton monoid whose enveloping group is not an automaton group. Indeed, the Baumslag-Solitar group $\mathbf{BS}(3,2)$ is precisely known as an example of non-residually finite group, hence cannot be an automaton group. The question remains open for those automaton semigroups whose enveloping group is a group of fractions.

▶ **Question 22.** *Is the group of fractions of an automaton monoid an automaton group?*

Concerning again group-embeddability, the following gives now an example of a cancellative automaton semigroup which is not group-embeddable.

▶ **Example 23.** The monoid $\mathbf{T} = \langle\, \mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{a}', \mathtt{b}', \mathtt{c}', \mathtt{d}' : \mathtt{ab} = \mathtt{cd}, \mathtt{a}'\mathtt{b}' = \mathtt{c}'\mathtt{d}', \mathtt{a}'\mathtt{d} = \mathtt{c}'\mathtt{b} \,\rangle^1_+$ is known (by Malcev work [37, 38, 39]) to be cancellative but not group-embeddable: from these three relations, we cannot deduce the fourth $\mathtt{ad}' = \mathtt{cb}'$. The quadratic normalisation $(\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{a}', \mathtt{b}', \mathtt{c}', \mathtt{d}'\}, \mathrm{N})$ defined by $\mathrm{N}(\mathtt{ab}) = \mathtt{cd}$, $\mathrm{N}(\mathtt{a}'\mathtt{b}') = \mathtt{c}'\mathtt{d}'$, and $\mathrm{N}(\mathtt{a}'\mathtt{d}) = \mathtt{c}'\mathtt{b}$ for instance has breadth $(3,3)$, hence satisfies Condition ( ) and Theorem 13 applies. This answers in particular a question by Cain [13].

Some classes of neither left- nor right-cancellative monoids have been studied and shown to admit nice normal forms yielding biautomatic structures (see [14]):

▶ **Example 24.** According to Schützenberger [48], plactic monoids are among the most fundamental monoids. The rank 2 plactic monoid is $\mathbf{P}_2 = \langle\, \mathtt{a}, \mathtt{b} : \mathtt{aba} = \mathtt{baa}, \mathtt{bab} = \mathtt{bba} \,\rangle^1_+$. As noted in [17, 19], $\mathbf{P}_2$ admits the quadratic normalisation $(\mathcal{Q}, \mathrm{N})$ with $\mathcal{Q} = \{1, \mathtt{a}, \mathtt{b}, \mathtt{ba}\}$,

$\mathrm{N}(\mathtt{ba}) = \mathtt{1}(\mathtt{ba})$, $\mathrm{N}((\mathtt{ba})\mathtt{a}) = \mathtt{a}(\mathtt{ba})$, $\mathrm{N}((\mathtt{ba})\mathtt{b}) = \mathtt{b}(\mathtt{ba})$, $\mathrm{N}(\mathtt{1}x) = x\mathtt{1}$ for $x \in \mathcal{Q}$, and $\mathrm{N}(xy) = xy$ otherwise. The latter has a breadth $(3, 3)$, hence satisfies Condition ( ) and Theorem 13 ensures that $\mathbf{P}_2$ is an automaton monoid. Note that, for a higher rank plastic monoid $\mathbf{P}_X$, it suffices similarly to take for $\mathcal{Q}$ the set of *columns*, that is, the strictly decreasing products of elements of X. Chinese monoids [14] admit also quadratic normalisations with breadth $(4, 3)$, hence satisfy Condition ( ), and Theorem 13 ensures that they are selfsimilar as well.

To conclude, we would like to completely illustrate the duality between "being an automatic semigroup" and "being an automaton semigroup" by highlighting a paradigmatic example.

▶ **Example 25.** The braid monoids were used by Thurston [21, Chapter 9] to describe his idea to build a single transducer that computes the now-called Adjan-Garside-Thurston normal form via iterated runs. The 3-strand braid monoid is

$\mathbf{B}^1_{3+} = \langle$  $:$  $\rangle^1_+$.

According to Corollary 16, its Thurston transducer and its Mealy automaton are displayed on Figure 11: these dual automata make $\mathbf{B}^1_{3+}$ both an automatic and an automaton monoid. Such an approach may hopefully shed some light on the question of whether or not the braid groups are selfsimilar.

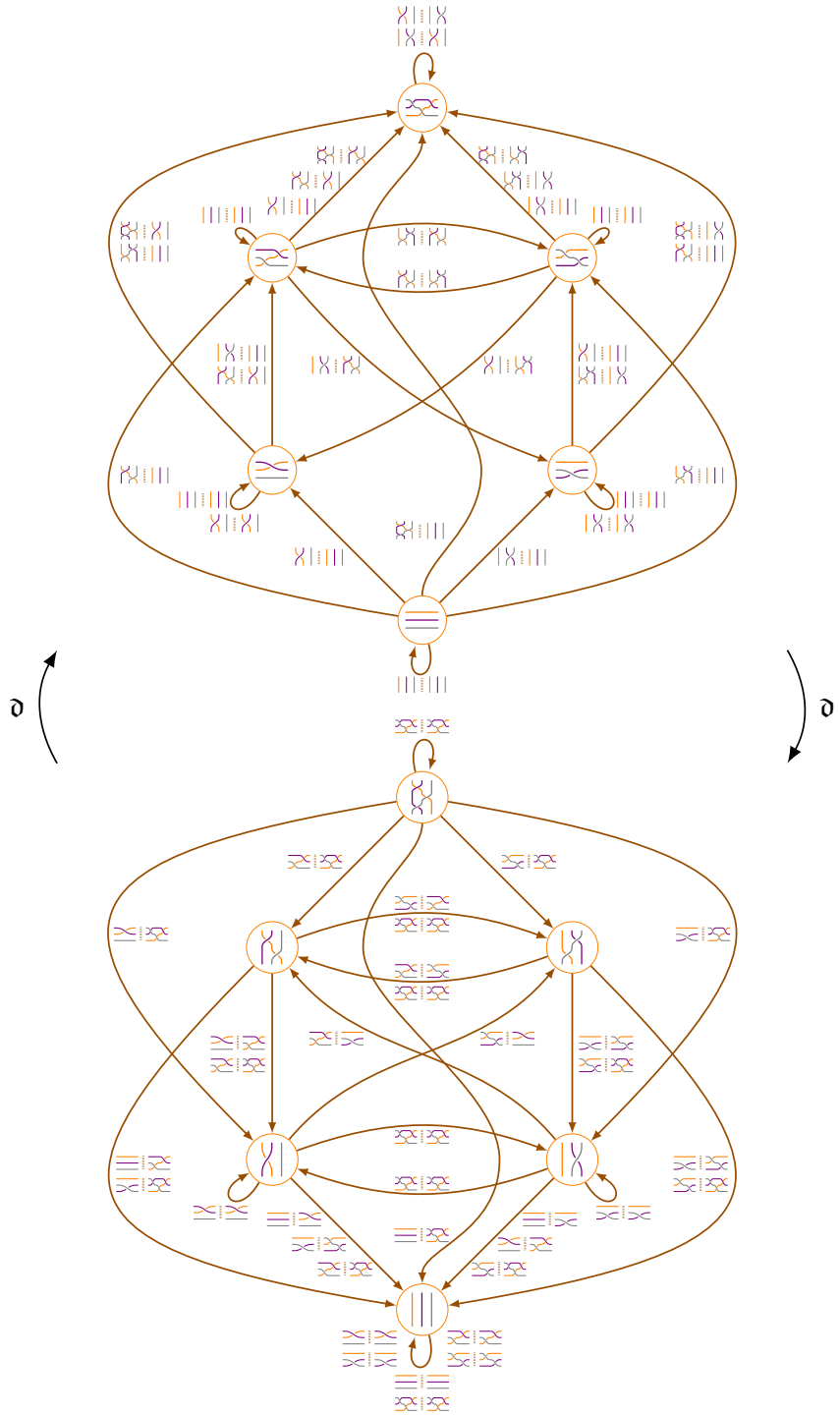


Figure 11 The Thurston transducer (top) *vs* the Mealy automaton (bottom) for the 3-strand braid monoid $\mathbf{B}^1_{3+}$ from Example 25.

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