

American Options in an Imperfect Complete Market with Default

Roxana Dumitrescu, Marie-Claire Quenez, Agnès Sulem

▶ **To cite this version:**

Roxana Dumitrescu, Marie-Claire Quenez, Agnès Sulem. American Options in an Imperfect Complete Market with Default. ESAIM: Proceedings and Surveys, 2018, pp.93–110. 10.1051/proc/201864093 . hal-01614741v3

HAL Id: hal-01614741
https://inria.hal.science/hal-01614741v3

Submitted on 24 May 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

ESAIM: PROCEEDINGS AND SURVEYS, Vol. ?, 2018, 1-10
Editors: Will be set by the publisher

AMERICAN OPTIONS IN AN IMPERFECT COMPLETE MARKET WITH DEFAULT

Roxana Dumitrescu[1], Marie-Claire Quenez[2] and Agnès Sulem[3]

Abstract. We study pricing and hedging for American options in an imperfect market model with default, where the imperfections are taken into account via the nonlinearity of the wealth dynamics. The payoff is given by an RCLL adapted process (ξ_t). We define the *seller's price* of the American option as the minimum of the initial capitals which allow the seller to build up a superhedging portfolio. We prove that this price coincides with the value function of an optimal stopping problem with a nonlinear expectation $\mathcal{E}^g$ (induced by a BSDE), which corresponds to the solution of a nonlinear reflected BSDE with obstacle (ξ_t). Moreover, we show the existence of a superhedging portfolio strategy. We then consider the *buyer's price* of the American option, which is defined as the supremum of the initial prices which allow the buyer to select an exercise time τ and a portfolio strategy φ so that he/she is superhedged. We show that the *buyer's price* is equal to the value function of an optimal stopping problem with a nonlinear expectation, and that it can be characterized via the solution of a reflected BSDE with obstacle (ξ_t). Under the additional assumption of left upper semicontinuity along stopping times of (ξ_t), we show the existence of a super-hedge (τ, φ) for the buyer.

Key-words: American options, imperfect markets, nonlinear expectation, superhedging, default, reflected backward stochastic differential equations

1. Introduction

We consider an American option associated with a terminal time T and a payoff given by an RCLL adapted process (ξ_t). The case of a classical perfect market has been largely studied in the literature (see e.g. [21, 25]). The *seller's price* (often called superhedging price or fair price in the literature), denoted by u_0, is classically defined as the minimal initial capital which enables the seller to invest in a portfolio which covers his liability to pay to the buyer up to T no matter what the exercise time chosen by the buyer. Moreover, this price is equal to the value function of the following optimal stopping time problem

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}_Q(\tilde{\xi}_\tau), \tag{1}$$

[1] Department of Mathematics, King's College London, Strand, London, WC2R 2LS, United Kingdom,
email: roxana.dumitrescu@kcl.ac.uk
[2] LPMA, Université Paris 7 Denis Diderot, Boite courrier 7012, 75251 Paris Cedex 05, France,
email: quenez@math.univ-paris-diderot.fr
[3] INRIA Paris, 2 rue Simone Iff, CS 42112, 75589 Paris Cedex 12, France, and Université Paris-Est,
email: agnes.sulem@inria.fr

© EDP Sciences, SMAI 2018

where $\mathcal{T}$ is the set of stopping times valued in $[0,T]$. Here, $\tilde{\xi}_t$ denotes the discounted value of ξ_t, equal to $e^{-\int_0^t r_s ds}\xi_t$, where (r_t) is the instantaneous interest process, and $\mathbb{E}_Q$ denotes the expectation under the unique equivalent martingale measure Q. In [12], the seller's price is characterized via a reflected BSDE with lower obstacle (ξ_t).

The aim of the present paper is to study pricing and hedging issues for American options in the case of imperfections in the market model taken into account via the nonlinearity of the dynamics of the wealth (or equivalently, of the portfolio's values), which are modeled via a nonlinear driver g. We moreover include the possibility of a default. The market model we consider here is *complete*, in the sense that for each European option, there exists a unique portfolio such that its value at the exercise time is equal to the payoff. A large class of imperfect market models can fit in our framework: e.g. the case of different borrowing and lending interest rates (see e.g. [4, 5, 22]), or the case of market prices impacted by the hedging strategy of a large seller (see e.g. [14, 23]). In the framework of a market with default, the case of a large seller affecting the default probability is considered in [11].

We provide a characterization of the seller's price u_0 as the value of a corresponding *optimal stopping problem* with a nonlinear expectation, more precisely

$$u_0 = \sup_{\tau\in\mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau), \tag{2}$$

where $\mathcal{E}^g$ is the g-evaluation (also called g-expectation) induced by a nonlinear BSDE with default jump solved under the primitive probability measure P with driver g. Note that in the particular case of a perfect market, the driver g is linear and (2) reduces to (1). We also show that the seller's price can be characterized via the solution of the reflected BSDE with driver g and lower obstacle (ξ_t), and we study the existence of a superhedging portfolio strategy for the seller. The proof of these results is based on the characterization of the value function of an optimal stopping problem with $\mathcal{E}^g$-expectation as the solution of an associated reflected BSDEs with driver g, provided in [14, Section 5.4] in the case of a continuous payoff, and in [27] when the payoff is only RCLL).

We then consider the *buyer's price* of the American option, denoted by $\tilde{u}_0$, defined as the supremum of the initial prices which allow the buyer to select an exercise time τ and a portfolio strategy φ so that he/she is superhedged. We provide a characterization of the buyer's price $\tilde{u}_0$ as the value of a corresponding *optimal stopping problem* with nonlinear expectation. More precisely, we prove that

$$\tilde{u}_0 = \sup_{\tau\in\mathcal{T}} \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau),$$

where $\mathcal{E}^{\tilde{g}}$ is the $\tilde{g}$-expectation, associated with driver $\tilde{g}(t,y,z,k) := -g(t,-y,-z,-k)$. Moreover, we show that the buyer's price can also be characterized via the solution of a nonlinear reflected BSDE with obstacle (ξ_t) and driver $\tilde{g}$. Under the additional assumption of left upper-semicontinuity along stopping times of (ξ_t), we show the existence of a super-hedge $(\tilde{\tau},\tilde{\varphi})$ for the buyer.

When $\tilde{g} = g$, which is the case when the market is perfect, the buyer's price is equal to the seller's price, that is $\tilde{u}_0 = u_0$. When $\tilde{g} \le g$, we obtain $\tilde{u}_0 \le u_0$. The interval $[\tilde{u}_0, u_0]$ can then be seen as a non-arbitrage interval for the price of the American option (see Section 5).

The paper is organized as follows: in Section 2, we introduce our imperfect market model with default and nonlinear wealth dynamics. In Section 3, we study pricing and superhedging of American options from the seller's point of view. In Section 4, we address the pricing and superhedging problem from the buyer's point of view. In Section 5, we discuss some arbitrage issues in our imperfect market model.

2. Imperfect market model with default

2.1. **Market model with default**

Let $(\Omega, \mathcal{G}, P)$ be a complete probability space equipped with two stochastic processes: a unidimensional standard Brownian motion W and a jump process N defined by $N_t = \mathbf{1}_{\vartheta \leq t}$ for all $t \in [0,T]$, where ϑ is a random variable which models a default time. We assume that this default can appear after any fixed time, that is $P(\vartheta \geq t) > 0$ for all $t \geq 0$. We denote by $\mathbb{G} = \{\mathcal{G}_t, t \geq 0\}$ the *augmented filtration* generated by W and N (in the sense of [7, IV-48]). We denote by $\mathcal{P}$ the predictable σ-algebra. We suppose that W is a $\mathbb{G}$-Brownian motion. Let (Λ_t) be the predictable compensator of the nondecreasing process (N_t). Note that $(\Lambda_{t\wedge\vartheta})$ is then the predictable compensator of $(N_{t\wedge\vartheta}) = (N_t)$. By uniqueness of the predictable compensator, $\Lambda_{t\wedge\vartheta} = \Lambda_t$, $t \geq 0$ a.s. We assume that Λ is absolutely continuous w.r.t. Lebesgue's measure, so that there exists a nonnegative process λ, called the intensity process, such that $\Lambda_t = \int_0^t \lambda_s ds$, $t \geq 0$. Since $\Lambda_{t\wedge\vartheta} = \Lambda_t$, λ vanishes after ϑ. The compensated martingale is given by

$$M_t := N_t - \int_0^t \lambda_s ds\,.$$

Let $T > 0$ be the terminal time. We define the following sets:

- S^2 is the set of adapted RCLL processes φ such that $\mathbb{E}[\sup_{0\leq t\leq T} |\varphi_t|^2] < +\infty$.
- $\mathcal{A}^2$ is the set of real-valued non decreasing RCLL predictable processes A with $A_0 = 0$ and $\mathbb{E}(A_T^2) < \infty$.
- $\mathbb{H}^2$ is the set of $\mathbb{G}$-predictable processes Z such that $\|Z\|^2 := \mathbb{E}\Big[\int_0^T |Z_t|^2 dt\Big] < \infty\,.$
- $\mathbb{H}^2_\lambda := L^2(\Omega \times [0,T], \mathcal{P}, \lambda_t\, dP \otimes dt)$, equipped with scalar product $\langle U, V\rangle_\lambda := \mathbb{E}\Big[\int_0^T U_t V_t \lambda_t dt\Big]$, for all U, V in $\mathbb{H}^2_\lambda$. For all $U \in \mathbb{H}^2_\lambda$, we set $\|U\|^2_\lambda := \mathbb{E}\Big[\int_0^T |U_t|^2 \lambda_t dt\Big] < \infty\,.$

For each $U \in \mathbb{H}^2_\lambda$, we have $\|U\|^2_\lambda = \mathbb{E}\Big[\int_0^{T\wedge\vartheta} |U_t|^2 \lambda_t dt\Big]$ because the intensity λ vanishes after ϑ. Note that, without loss of generality, we may assume that U vanishes after ϑ.[1] Moreover, $\mathcal{T}$ denotes the set of stopping times τ such that $\tau \in [0,T]$ a.s. and for each S in $\mathcal{T}$, $\mathcal{T}_S$ is the set of stopping times τ such that $S \leq \tau \leq T$ a.s.

Recall that in this setup, we have a martingale representation theorem with respect to W and M (see [24]).

We consider a complete financial market with default as in [2], which consists of one risk-free asset, with price process S^0 satisfying $dS^0_t = S^0_t r_t dt$, and two risky assets with price processes S^1, S^2 evolving according to the following equations:

$$\begin{cases} dS^1_t = S^1_t[\mu^1_t dt + \sigma^1_t dW_t] \\ dS^2_t = S^2_{t-}[\mu^2_t dt + \sigma^2_t dW_t - dM_t]. \end{cases}$$

The process $S^0 = (S^0_t)_{0\leq t\leq T}$ corresponds to the price of a non risky asset with interest rate process $r = (r_t)_{0\leq t\leq T}$, $S^1 = (S^1_t)_{0\leq t\leq T}$ to a non defaultable risky asset, and $S^2 = (S^2_t)_{0\leq t\leq T}$ to a defaultable asset with total default. The price process S^2 vanishes after ϑ.

All the processes σ^1, σ^2, r, μ^1, μ^2 are predictable (that is $\mathcal{P}$-measurable). We suppose that the coefficients $\sigma^1, \sigma^2 > 0$, and r, σ^1, σ^2, μ^1, μ^2, $(\sigma^1)^{-1}$, $(\sigma^2)^{-1}$ are bounded.
We set $\sigma := (\sigma^1, \sigma^2)'$, where $'$ denotes transposition.

We consider an investor, endowed with an initial wealth equal to x, who can invest his wealth in the three assets of the market. At each time $t < \vartheta$, he chooses the amount φ^1_t (resp. φ^2_t) of wealth invested in the first (resp. second) risky asset. However, after time ϑ, he cannot invest his wealth in the defaultable asset since its price is equal to 0, and he only chooses the amount φ^1_t of wealth invested in the first risky asset. Note that the process φ^2 can be defined on the whole interval $[0,T]$ by setting $\varphi^2_t(\omega) = 0$ for each (ω, t) such that $t > \vartheta(\omega)$.

[1] Indeed, each U in $\mathbb{H}^2_\lambda (= L^2(\Omega \times [0,T], \mathcal{P}, \lambda_t dP \otimes dt))$ can be identified with $U\mathbf{1}_{t\leq\vartheta}$, since $U\mathbf{1}_{t\leq\vartheta}$ is a predictable process satisfying $U_t\mathbf{1}_{t\leq\vartheta} = U_t \;\; \lambda_t dP \otimes dt$-a.s.

A process $\varphi_\cdot = (\varphi_t^1, \varphi_t^2)'_{0\le t\le T}$ is called a *risky assets strategy* if it belongs to $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$. The value of the associated portfolio (also called *wealth*) at time t is denoted by $V_t^{x,\varphi}$ (or simply V_t).

The perfect market model. In the classical case of a perfect market model, the wealth process and the strategy satisfy the self financing condition:

$$dV_t = (r_t V_t + \varphi_t^1(\mu_t^1 - r_t) + \varphi_t^2(\mu_t^2 - r_t))dt + (\varphi_t^1\sigma_t^1 + \varphi_t^2\sigma_t^2)dW_t - \varphi_t^2 dM_t. \tag{3}$$

Setting $K_t := -\varphi_t^2$, and $Z_t := \varphi_t{}'\sigma_t = \varphi_t^1\sigma_t^1 + \varphi_t^2\sigma_t^2$, we get

$$dV_t = (r_t V_t + Z_t\theta_t^1 + K_t\theta_t^2\lambda_t)dt + Z_t dW_t + K_t dM_t,$$

where $\theta_t^1 := \dfrac{\mu_t^1 - r_t}{\sigma_t^1}$ and $\theta_t^2 := \dfrac{\sigma_t^2\theta_t^1 - \mu_t^2 + r_t}{\lambda_t}\,\mathbf{1}_{\{\lambda_t\neq 0\}}$.
Note that (θ_t^2) vanishes after ϑ, as the process (λ_t). We suppose below that the processes θ^1 and $\theta^2\sqrt{\lambda}$ are bounded.

Consider a European contingent claim with maturity $T > 0$ and $\mathcal{G}_T$-measurable payoff ξ in L^2. The problem is to price and hedge this claim by constructing a replicating portfolio. From [11, Proposition 2], there exists a unique process $(X, Z, K) \in \mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ solution of the following BSDE with default jump:

$$-dX_t = -(r_t X_t + Z_t\theta_t^1 + K_t\theta_t^2\lambda_t)dt - Z_t dW_t - K_t dM_t\,; \quad X_T = \xi. \tag{4}$$

The solution (X, Z, K) provides the replicating portfolio. More precisely, the process X corresponds to its value, and the hedging risky assets strategy φ in $\mathbb{H}^2_\lambda$ is given by $\varphi = \Phi(Z, K)$, where Φ is the one-to-one map defined on $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$ by:

Definition 2.1. *Let $\Phi : \mathbb{H}^2 \times \mathbb{H}^2_\lambda \to \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ be the one-to-one map defined for each $(Z, K) \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ by $\Phi(Z, K) := \varphi$, where $\varphi = (\varphi^1, \varphi^2)$ is given by*

$$\varphi_t^2 = -K_t \;\;;\;\; \varphi_t^1 = \frac{Z_t + \sigma_t^2 K_t}{\sigma_t^1},$$

which is equivalent to $K_t = -\varphi_t^2\,;\;\; Z_t = \varphi_t{}'\sigma_t$.

Note that the processes φ^2 and K, which belong to $\mathbb{H}^2_\lambda$, both vanish after time ϑ.

The process X coincides with $V^{X_0,\varphi}$, the value of the portfolio associated with initial wealth $x = X_0$ and portfolio strategy φ. Since $X_T = \xi$ a.s. , we derive that $V_T^{X_0,\varphi} = \xi$ a.s.[2] From the seller's point of view, this portfolio is a hedging portfolio. Indeed, by investing at initial time 0 the initial amount X_0 in the reference assets along the strategy φ, the seller can pay the amount ξ to the buyer at time T (and similarly at each initial time t with initial amount X_t). We derive that X_t is the price at time t of the option, called *hedging price*, and denoted by $X_t(\xi)$. By the representation property of the solution of a λ-linear BSDE with default jump (see [11, Theorem 1]), the solution X of BSDE (4) is given by:

$$X_t(\xi) = \mathbb{E}[e^{-\int_t^T r_s ds}\zeta_{t,T}\xi \,|\, \mathcal{G}_t], \tag{5}$$

where $\zeta_{t,\cdot}$ satisfies $d\zeta_{t,s} = \zeta_{t,s^-}[-\theta_s^1 dW_s - \theta_s^2 dM_s]$ with $\zeta_{t,t} = 1$. This defines a *linear* price system X: $\xi \mapsto X(\xi)$. Suppose now that

$$\theta_t^2 < 1,\; 0 \le t \le T,\; dt \otimes dP - a.s., \tag{6}$$

[2]This property holds for any payoff $\xi \in L^2(\mathcal{F}_T)$, which corresponds to the so-called *completeness* property of the perfect market.

which is a usual assumption in the literature on default risk. Moreover, the price system X is increasing and corresponds to the classical arbitrage-free price system.

The imperfect market model $\mathcal{M}^g$. From now on, we assume that there are imperfections in the market which are taken into account via the *nonlinearity* of the dynamics of the wealth. More precisely, the dynamics of the wealth V associated with strategy $\varphi = (\varphi^1, \varphi^2)'$ can be written via a *nonlinear* driver, defined as follows:

Definition 2.2 (Driver, λ-*admissible* driver)**.** *A function g is said to be a* driver *if* $g : [0,T] \times \Omega \times \mathbb{R}^3 \to \mathbb{R}$*;* $(\omega, t, y, z, k) \mapsto g(\omega, t, y, z, k)$ *is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^3)-$ measurable, and such that $g(.,0,0,0) \in \mathbb{H}^2$. A driver g is called a* λ-admissible driver *if moreover there exists a constant $C \geq 0$ such that $dP \otimes dt$-a.s. , for each (y_1, z_1, k_1), (y_2, z_2, k_2),*

$$|g(t, y_1, z_1, k_1) - g(t, y_2, z_2, k_2)| \leq C(|y_1 - y_2| + |z_1 - z_2| + \sqrt{\lambda_t}|k_1 - k_2|). \tag{7}$$

A nonnegative constant C which satisfies this inequality is called a λ-constant *associated with driver g.*

Note that condition (7) implies that since $\lambda_t = 0$ on the stochastic interval $]\vartheta, T]$, g does not depend on k on $]\vartheta, T]$. More precisely, $dP \otimes dt$-a.s. , for all (y, z, k), we have: $g(t, y, z, k) = g(t, y, z, 0)$, $t \in]\vartheta, T]$.

Let $x \in \mathbb{R}$ be the initial wealth and let $\varphi = (\varphi^1, \varphi^2)$ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$ be a portfolio strategy. We suppose now that the associated *wealth* process $V_t^{x,\varphi}$ (or simply V_t) is defined as the unique solution in $\mathcal{S}^2$ of the forward stochastic differential equation:

$$-dV_t = g(t, V_t, \varphi_t{}'\sigma_t, -\varphi_t^2)dt - \varphi_t{}'\sigma_t dW_t + \varphi_t^2 dM_t, \tag{8}$$

with initial condition $V_0 = x$.Equivalently, setting $Z_t = \varphi_t{}'\sigma_t$ and $K_t = -\varphi_t^2$, the dynamics (8) of the wealth process V_t can be written as follows:

$$-dV_t = g(t, V_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t. \tag{9}$$

In the following, our imperfect market model is denoted by $\mathcal{M}^g$.
Note that in the case of a perfect market (see (3)), we have:

$$g(t, y, z, k) = -r_t y - \theta_t^1 z - \theta_t^2 k\lambda_t, \tag{10}$$

which is a λ-admissible driver since by the assumptions on the coefficients of the model, the processes θ^1 and $\theta^2\sqrt{\lambda}$ are bounded.

2.2. Nonlinear pricing system $\mathcal{E}^g$ (for the seller)

Pricing and hedging European options in the imperfect market $\mathcal{M}^g$ leads to BSDEs with nonlinear driver g and a default jump. By [11, Proposition 2], we have

Proposition 2.3. *Let g be a λ-admissible driver, let $\xi \in L^2(\mathcal{G}_T)$. There exists a unique solution $(X(T,\xi), Z(T,\xi), K(T,\xi))$ (denoted simply by (X, Z, K)) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the following BSDE:*

$$-dX_t = g(t, X_t, Z_t, K_t)dt - Z_t dW_t - K_t dM_t; \quad X_T = \xi. \tag{11}$$

Let us consider a European option with maturity T and terminal payoff ξ in $L^2(\mathcal{G}_T)$ in this market model. Let $x \in \mathbb{R}$ and let $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$. The process φ is called a *hedging risky-assets portfolio strategy for the seller* (associated with initial capital x) if the value of the associated portfolio is equal to ξ at time T, that is $V_T^{x,\varphi} = \xi$ a.s.

Let (X, Z, K) be the solution of BSDE (11). The process X is equal to the value of the portfolio associated with initial capital $x = X_0$ and the risky assets strategy $\varphi = \Phi(Z, K)$ (where Φ is defined in Definition 2.1),

that is $X = V^{X_0,\varphi}$. Since $X_T = \xi$, the process φ is thus a *hedging risky-assets strategy for the seller* associated with initial capital $x = X_0$.

Remark 2.4 (*Completeness* property of the market $\mathcal{M}^g$)**.** *By the above observations, we derive that for each $\xi \in L^2(\mathcal{G}_T)$, there exists a unique initial capital $x \in \mathbb{R}$ and a unique risky-assets strategy φ such that the value of the associated portfolio is equal to ξ at time T. By analogy with the classical case of a perfect market, this property is called* completeness property *of the market $\mathcal{M}^g$.*

Using the previous notation, the initial value $X_0 = X_0(T, \xi)$ of the hedging portfolio is thus a sensible price (at time 0) of the claim ξ for the seller since this amount is the unique initial capital which allows him/her to build a *hedging risky-assets strategy*. Similarly, $X_t = X_t(T, \xi)$ satisfies an analogous property at time t, and is called the *hedging price* (for the seller) at time t. This leads to a *nonlinear pricing* system, first introduced in [14, Section 2.3] in a Brownian framework, then called *g-evaluation* and denoted by $\mathcal{E}^g$ in [26]. For each $S \in [0, T]$, for each $\xi \in L^2(\mathcal{G}_S)$ the associated g-evaluation is defined by $\mathcal{E}^g_{t,S}(\xi) := X_t(S, \xi)$ for each $t \in [0, S]$.

We make the following assumption (see [11, Assumption 4 in Section 3.3]).

Assumption 2.5. *Assume that there exists a map*

$$\gamma : [0, T] \times \Omega \times \mathbb{R}^4 \to \mathbb{R}\,;\, (\omega, t, y, z, k_1, k_2) \mapsto \gamma_t^{y,z,k_1,k_2}(\omega)$$

$\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^4)$-measurable and satisfying $dP \otimes dt$-a.s. , for all $(y, z, k_1, k_2) \in \mathbb{R}^4$,

$$g(t, y, z, k_1) - g(t, y, z, k_2) \geq \gamma_t^{y,z,k_1,k_2}(k_1 - k_2)\lambda_t, \tag{12}$$

and

$$|\gamma_t^{y,z,k_1,k_2}\sqrt{\lambda_t}| \leq C, \quad \textit{and} \quad \gamma_t^{y,z,k_1,k_2} > -1 \tag{13}$$

(where C is a positive constant).

Assumption (12) is satisfied e.g. when g is $\mathcal{C}^1$ in k with $\partial_k g(t, \cdot) > -\lambda_t$. In the special case of a perfect market, g is given by (10), which implies that $\partial_k g(t, \cdot) = -\theta_t^2 \lambda_t$. Then Assumption $\gamma_t^{y,z,k_1,k_2} > -1$ is equivalent to $\theta_t^2 < 1$ (which corresponds to the usual assumption (6)).

Assumption 2.5 ensures the strict monotonicity of the nonlinear pricing system $\mathcal{E}^g$, i.e.
if $\xi \geq \xi'$ a.s. then $\mathcal{E}^g_{\cdot,S}(\xi) \geq \mathcal{E}^g_{\cdot,S}(\xi')$ a.s., and if moreover $P(\xi > \xi') > 0$, then $\mathcal{E}^g_{0,S}(\xi) > \mathcal{E}^g_{0,S}(\xi')$.

Remark 2.6. *Assume that $g(t, 0, 0, 0) = 0$ $dP \otimes dt$-a.s. Then for all $S \in [0, T]$, $\mathcal{E}^g_{\cdot,S}(0) = 0$ a.s. Moreover, by the comparison theorem for BSDEs with default jump (see [11, Theorem 3]), the nonlinear pricing system $\mathcal{E}^g$ is nonnegative, that is, for all $S \in [0, T]$, for all $\xi \in L^2(\mathcal{G}_S)$, if $\xi \geq 0$ a.s., then $\mathcal{E}^g_{\cdot,S}(\xi) \geq 0$ a.s.*

Definition 2.7. *Let $Y \in S^2$. The process (Y_t) is said to be a strong $\mathcal{E}$-martingale if $\mathcal{E}_{\sigma,\tau}(Y_\tau) = Y_\sigma$ a.s. on $\sigma \leq \tau$, for all $\sigma, \tau \in \mathcal{T}_0$.*

Note that, by the flow property of BSDEs, for each $S \in [0, T]$ and for each $\xi \in L^2(\mathcal{G}_S)$, the g-evaluation $\mathcal{E}^g_{\cdot,S}(\xi)$ is an $\mathcal{E}^g$-martingale. Moreover, since $V_t^{x,\varphi} = \mathcal{E}^g_{t,T}(V_T^{x,\varphi})$, we have:

Proposition 2.8. *For each $x \in \mathbb{R}$ and each portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$, the associated wealth process $V^{x,\varphi}$ is an $\mathcal{E}^g$-martingale.*

For examples of imperfect market models of type $\mathcal{M}^g$, the reader is referred to [10, 11, 13].

Note that when the market is perfect, the prices S^0, S^1 and S^2 are $\mathcal{E}^g$-martingales.[3] This property does not necessarily holds for all λ-admissible drivers g (as for example in a market model with taxes on risky investment profits, see [10] and [13]).

[3]This corresponds to the well-kown property that the discounted prices of the reference assets are martingales under the martingale probability measure Q.

3. American option pricing from the seller's point of view

Let us consider an American option associated with horizon $T > 0$ and payoff given by an RCLL adapted process $(\xi_t, 0 \leq t \leq T)$ in S^2. At time 0, it consists in the selection of a stopping time $\tau \in \mathcal{T}$ and the payment of the payoff ξ_τ from the seller to the buyer.

The *seller's price* of the American option at time 0, denoted by u_0, is classically defined as the minimal initial capital which enables the seller to invest in a portfolio which covers his liability to pay to the buyer up to T no matter what the exercise time chosen by the buyer. More precisely, for each initial wealth x, we denote by $\mathcal{A}(x)$ the set of all portfolio strategies $\varphi \in \mathbb{H}^2$ such that $V_t^{x,\varphi} \geq \xi_t$, $0 \leq t \leq T$ a.s. The *seller's price* of the American option is thus defined by

$$u_0 := \inf\{x \in \mathbb{R},\ \exists \varphi \in \mathcal{A}(x)\}. \tag{14}$$

Note that $u_0 \in \overline{\mathbb{R}}$. We shall see below that u_0 is finite.

Remark 3.1. *Suppose that $g(t,0,0,0) = 0$. From Remark 2.6, we derive that if $\xi_\cdot \geq 0$, the infimum in the definition of u_0 can be taken only over nonnegative initial wealths, that is, $u_0 := \inf\{x \geq 0,\ \exists \varphi \in \mathcal{A}(x)\}$.*

We define the *g-value* associated with the American option as the value function (at time 0) of the *$\mathcal{E}^g$-optimal stopping problem* associated with payoff (ξ_t), that is

$$\sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau). \tag{15}$$

Proposition 3.2 (Characterization of the *g-value*)**.** *There exists a unique process (Y, Z, K, A) solution of the reflected BSDE (RBSDE) associated with driver g and obstacle ξ in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda \times \mathcal{A}^2$, that is*

$$-dY_t = g(t, Y_t, Z_t, K_t)dt + dA_t - Z_t dW_t - K_t dM_t;\ \ Y_T = \xi_T,\ \textit{with} \tag{16}$$
$$Y \geq \xi\,,$$
$$\int_0^T (Y_t - \xi_t) dA^c_t = 0 \ \textit{a.s. and}\ \Delta A^d_t = \Delta A^d_t \mathbf{1}_{\{Y_{t^-} = \xi_{t^-}\}}, \tag{17}$$

where A^c denotes the continuous part of A and A^d its discontinuous part. Moreover, for each $S \in \mathcal{T}$, we have

$$Y_S = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^g_{S,\tau}(\xi_\tau) \quad \text{a.s.} \tag{18}$$

*In particular, Y_0 is equal to the g-*value $\sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau)$.

Note that this result was shown in [14, Theorem 5.9] in a Brownian framework with a continuous obstacle, and generalized in [27, Theorem 3.3] to the case of a filtration associated with a Brownian motion and a Poisson random measure, with an RCLL obstacle.

Proof. Let us first show that there exists a unique solution of RBSDE (16). As usual, we first consider the case when the driver $g(t)$ does not depend on the solution. By using the representation property of $\mathbb{G}$-martingales (see [24]) and some results of optimal stopping theory, one can show, proceeding as in [12] (see also [17] and [27]), that there exists a unique solution of the associated RBSDE (16). The proof in the case of a general λ-admissible driver is based on a usual fixed point argument and a priori estimates for RBSDEs with default given in Appendix (see Lemma 8.1). Equality (18) can then be obtained by proceeding as in the proof of [27, Theorem 3.3]. □

Lemma 3.3. *If ξ is left-u.s.c. along stopping time, then A is continuous.*

Proof. Let τ be a predictable stopping time. By (16), we have $\Delta A_\tau = (\Delta Y_\tau)^-$. Using the Skorokhod conditions (17), we get $\Delta A_\tau = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(Y_\tau - Y_{\tau^-})^- = \mathbf{1}_{\{Y_{\tau^-} = \xi_{\tau^-}\}}(Y_\tau - \xi_{\tau^-})^-$ a.s. Now, since by assumption, $\xi_{\tau^-} \leq \xi_\tau$ a.s. , we have $Y_\tau - \xi_{\tau^-} \geq Y_\tau - \xi_\tau \geq 0$ a.s. We derive that $\Delta A_\tau = 0$ a.s. It follows that A is continuous. □

We now provide two characterizations of the seller's price, which generalize those provided in the literature in the case of a perfect market (see [12]) to the case of an imperfect market.

Theorem 3.4 (*Seller's price* of the American option)**.** *The* seller's price u_0 *of the American option is equal to the* g-value*, that is*

$$u_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau). \tag{19}$$

Moreover, we have

$$u_0 = Y_0,$$

where (Y, Z, K, A) *is the solution of the* nonlinear *reflected BSDE* (16) *and the portfolio strategy* $\varphi^* := \Phi(Z, K)$ *(where* Φ *is defined in Definition 2.1) is a superhedging strategy for the seller.*

Note that in the case of a perfect market, equality (19) reduces to the well-known characterization of the price of the American option as the value function of a classical optimal stopping problem, and the equality $u_0 = Y_0$ corresponds to the well-known characterization of this price as the solution of the *linear* reflected BSDE associated with the *linear* driver (10) (see [12]).

Proof. The proof is based on the characterization of the *g-value* as the solution of the reflected BSDE (16) (see Proposition 3.2). It is sufficient to show that $u_0 = Y_0$ and $\varphi^* \in \mathcal{A}(u_0)$.
Let $\mathcal{H}$ be the set of initial capitals which allow the seller to be "super-hedged", that is $\mathcal{H} = \{x \in \mathbb{R} : \exists \varphi \in \mathcal{A}(x)\}$. Note that $u_0 = \inf \mathcal{H}$.

Let us first show that

$$\varphi^* \in \mathcal{A}(Y_0). \tag{20}$$

By (8)-(9), for almost every ω, the trajectory of the value of this portfolio $t \mapsto V_t^{Y_0,\varphi^*}(\omega)$ satisfies the following forward differential equation:

$$V_t^{Y_0,\varphi^*}(\omega) = Y_0 - \int_0^t g(s, \omega, V_s^{Y_0,\varphi^*}(\omega), Z_s(\omega), K_s(\omega))ds + f_t^1(\omega),\ 0 \le t \le T, \tag{21}$$

where $f_t^1 := \int_0^t Z_s dW_s + \int_0^t K_s dM_s$. Moreover, since Y is the solution of the reflected BSDE (16), for almost every ω, the function $t \mapsto Y_t(\omega)$ satisfies:

$$Y_t(\omega) = Y_0 - \int_0^t g(s, \omega, Y_s(\omega), Z_s(\omega), K_s(\omega))ds + f_t^2(\omega),\ 0 \le t \le T, \tag{22}$$

where $f_t^2 := f_t^1 - A_t$. Since $f_t^1 \ge f_t^2$, by a comparison result for forward differential equations (see e.g. [10] in the Appendix), we derive that $V_t^{Y_0,\varphi^*}(\omega) \ge Y_t(\omega)$, $0 \le t \le T$ for almost every ω. Since $Y_t \ge \xi_t$, $0 \le t \le T$ a.s. , we thus have $V_t^{Y_0,\varphi^*} \ge \xi_t$, $0 \le t \le T$ a.s. , which implies the desired property (20). It follows that $Y_0 \ge u_0$.

Let us show the converse inequality. Let $x \in \mathcal{H}$. There exists $\varphi \in \mathcal{A}(x)$ such that $V_t^{x,\varphi} \ge \xi_t$, $0 \le t \le T$ a.s. For each $\tau \in \mathcal{T}$ we thus have $V_\tau^{x,\varphi} \ge \xi_\tau$ a.s. By taking the $\mathcal{E}^g$-evaluation in this inequality, using the monotonicity of $\mathcal{E}^g$ and the $\mathcal{E}^g$-martingale property of the wealth process $V^{x,\varphi}$, we obtain $x = \mathcal{E}^g_{0,\tau}[V_\tau^{x,\varphi}] \ge \mathcal{E}^g_{0,\tau}[\xi_\tau]$. By arbitrariness of $\tau \in \mathcal{T}$, we get $x \ge \sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}[\xi_\tau]$, which holds for any $x \in \mathcal{H}$. By taking the infimum over $x \in \mathcal{H}$, we obtain $u_0 \ge Y_0$. We derive that $u_0 = Y_0$. By (20), we thus have $\varphi^* \in \mathcal{A}(u_0)$, which ends the proof. □

Remark 3.5. *In general, except when* g *does not depend on* y*, by* (22)*, we have*

$$Y_\cdot = Y_0 - \int_0^\cdot g(s, Y_s, Z_s, K_s)ds + \int_0^\cdot Z_s dW_s + \int_0^\cdot K_s dM_s - A_\cdot \not\equiv V_\cdot^{Y_0,\varphi^*} - A_\cdot.$$

Remark 3.6. *In [20], it is proved that the* seller's price *of the American option is equal to the g-*value *in the case of a higher interest rate for borrowing, by using a dual approach. This approach relies on the convexity properties of the driver and cannot be adapted to our case, except when g is convex with respect to* (y, z, k).

We define now the seller's price of the American option at each time/stopping time $S \in \mathcal{T}$. We first define, for each initial wealth $X \in L^2(\mathcal{G}_S)$, a *super-hedge* against the American option as a portfolio strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that $V_t^{S,X,\varphi} \geq \xi_t$, $S \leq t \leq T$ a.s. , where $V^{S,X,\varphi}$ denotes the wealth process associated with initial time S and initial condition X. The *seller's price at time* S is defined by the random variable

$$u(S) := \operatorname{ess\,inf}\{X \in L^2(\mathcal{G}_S),\ \exists \varphi \in \mathcal{A}_S(X)\},$$

where $\mathcal{A}_S(X)$ is the set of all super-hedges associated with initial time S and initial wealth X. Using equality (18) and similar arguments to those used in the proof of Theorem 3.4 above, one can show the following result, which generalizes the results of Theorem 3.4 to any time $S \in \mathcal{T}$.

Proposition 3.7 (*Seller's price process* and characterization)**.** *For each time* $S \in \mathcal{T}$*, the* seller's price $u(S)$ at time S *of the American option satisfies the equalities*

$$u(S) = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_S} \mathcal{E}^g_{S,\tau}(\xi_\tau) = Y_S \quad \text{a.s.},$$

where (Y, Z, K, A) *is the solution of reflected BSDE* (16)*.*
In other terms, the seller's price family of random variables $(u(S), S \in \mathcal{T})$ *can be aggregated by an RCLL adapted process, which we call the* seller's price process *of the American option. Moreover, this price process is both characterized as the value function process of the* $\mathcal{E}^g$*-optimal stopping problem with payoff* (ξ_t)*, and also as the solution* (Y_t) *of the reflected BSDE* (16)*.*

Suppose now that the buyer has bought the American option at the selling price $u_0 = Y_0$. We address the problem of the choice of her/his exercise time. We introduce the following definition.

Definition 3.8. *A stopping time* $\hat{\tau} \in \mathcal{T}$ *is called a* rational exercise time *for the buyer of the American option if it is optimal for Problem* (19)*, that is, if it satisfies* $\sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau) = \mathcal{E}^g_{0,\hat{\tau}}(\xi_{\hat{\tau}})$.

By the optimality criterium provided in [27] (see Proposition 3.5), we have:

Proposition 3.9. *(Characterization of* rational exercise times*) Let* $\tau \in \mathcal{T}$*. Then,* τ *is a a* rational exercise time *for the buyer if and only if* $Y_\tau = \xi_\tau$ *a.s. and* $A_\tau = 0$ *a.s. , where* (Y, Z, K, A) *is the solution of the reflected BSDE* (16)*.*

Suppose now that the price of the American option is equal to the seller's price not only at time 0 but at all time $S \in \mathcal{T}$, that is, the price at time S is equal to $u(S) = Y_S$ (see Proposition 3.7). Suppose that the buyer buys the American option at time 0 (at price $u_0 = Y_0$). Let us show that it is not profitable for him to exercise his option at a stopping time τ which is not a rational exercise time. First, it is not in his interest to exercise at a time t such that $Y_t > \xi_t$ since he would loose a financial asset (the option) with value Y_t and only receive the lower amount ξ_t. Second, it is not in his interest to exercise at a stopping time τ greater than $\bar{\tau}$, defined by $\bar{\tau} := \inf\{s \geq 0,\ A_s \neq 0\}$. Let us show that $Y_{\bar{\tau}} = V_{\bar{\tau}}^{Y_0,\varphi^*}$ a.s. Note that by definition of $\bar{\tau}$, $A_{\bar{\tau}} = 0$ a.s. Hence, for a.e. ω, the trajectories $t \mapsto Y_t(\omega)$ and $t \mapsto V_t^{Y_0,\varphi^*}(\omega)$ are solutions on $[0, \bar{\tau}(\omega)]$ of the same differential equation (with initial value Y_0), which implies that they are equal, by uniqueness of the solution. Hence, $Y_{\bar{\tau}} = V_{\bar{\tau}}^{Y_0,\varphi^*}$ a.s. Without loss of generality, we can suppose that for each ω, we have $Y_{\bar{\tau}}(\omega) = V_{\bar{\tau}}^{Y_0,\varphi^*}(\omega)$. Let $\tau \geq \bar{\tau}$. Let $B := \{\tau > \bar{\tau}\}$. Suppose that $P(B) > 0$. Hence, $A_\tau > 0$ a.s. on B. Then, by exercising the option at time $\bar{\tau}$, the option holder receives the amount $Y_{\bar{\tau}}$ which he can invest in the market along the strategy φ^*. Since $Y_{\bar{\tau}} = V_{\bar{\tau}}^{Y_0,\varphi^*}$, by the flow property of the forward differential equation (8) with $\varphi = \varphi^*$ and $x = Y_0$, the value of the associated portfolio is equal to $V_\tau^{Y_{\bar{\tau}},\varphi^*} = V_\tau^{V_{\bar{\tau}}^{Y_0,\varphi^*},\varphi^*} = V_\tau^{Y_0,\varphi^*}$ at time τ. Since $A_\tau > 0$ on B, by the strict

comparison result for forward differential equations applied to (21) and (22) (see [10] in the Appendix), we get $V_\tau^{Y_0,\varphi^*} > Y_\tau$ a.s. on B, which implies $V_\tau^{Y_{\bar\tau},\varphi^*} = V_\tau^{Y_0,\varphi^*} > \xi_\tau$ a.s. on B. We thus have $V_\tau^{Y_{\bar\tau},\varphi^*} \geq \xi_\tau$ a.s. with $P(V_\tau^{Y_{\bar\tau},\varphi^*} > \xi_\tau) > 0$. Hence, at time $\bar\tau$, it is more interesting for the buyer to exercise immediately than later.

Proposition 3.10. *(Existence of* rational exercise times*) Suppose that the payoff ξ is left u.s.c. along stopping times. Let (Y, Z, K, A) be the solution of the reflected BSDE* (16)*.*
Let $\tau^* := \inf\{s \geq 0,\ Y_s = \xi_s\}$ and $\bar\tau := \inf\{s \geq 0,\ A_s \neq 0\}$.[4]
The stopping time τ^* (resp. $\bar\tau$) is the minimal (resp. maximal) rational exercise time.

Proof. The right continuity of (Y_t) and (ξ_t) ensures that $Y_{\tau^*} = \xi_{\tau^*}$ a.s. By definition of τ^*, we have $Y_t > \xi_t$ a.s. on $[0, \tau^*[$. Hence the process A is constant on $[0, \tau^*[$ and even on $[0, \tau^*]$ because A is continuous (see Lemma 3.3).

By Proposition 3.9, τ^* is thus a rational exercise time and is the minimal one. From the definition of $\bar\tau$, and the continuity of A, we have $A_{\bar\tau} = 0$ a.s. Also, we have a.s. for all $t > \bar\tau,\ A_t > A_{\bar\tau} = 0$. Since A increases only on the set $\{Y_\cdot = \xi_\cdot\}$, it follows that $Y_{\bar\tau} = \xi_{\bar\tau}$. By Proposition 3.9, $\bar\tau$ is a rational exercise time and is the maximal one. □

When ξ is only RCLL, there does not exist necessarily a *rational* exercise time for the buyer. However, we have the following result.

Proposition 3.11. *Suppose ξ is RCLL. For each $\varepsilon > 0$, the stopping time $\tau_\varepsilon := \inf\{t \geq 0 :\ Y_t \leq \xi_t + \varepsilon\}$ satisfies*

$$\sup_{\tau \in \mathcal{T}} \mathcal{E}^g_{0,\tau}(\xi_\tau) \ \leq\ \mathcal{E}^g_{0,\tau_\varepsilon}(\xi_{\tau_\varepsilon}) + K\varepsilon \quad \text{a.s.}, \tag{23}$$

where K is a constant which only depends on T and the λ-constant C of g.

The proof, which relies on similar arguments as those used in the proof of Theorem 3.2 in [27], is left to the reader.

Remark 3.12. *Following the terminology of classical optimal stopping theory, τ^ε is a $K\varepsilon$-optimal stopping time for the $\mathcal{E}^g$-optimal stopping problem $\sup_\tau \mathcal{E}^g_{0,\tau}(\xi_\tau)$.*

4. The buyer's point of view

Let us consider the pricing and hedging problem of a European option with maturity T and payoff $\xi \in L^2(\mathcal{G}_T)$ from the buyer's point of view. Supposing the initial price of the option is z, he starts with the amount $-z$ at time $t = 0$, and wants to find a risky-assets strategy $\tilde\varphi$ such that the payoff that he receives at time T allows him to recover the debt he incurred at time $t = 0$ by buying the option, that is such that $V_T^{-z,\tilde\varphi} + \xi = 0$ a.s. or equivalently, $V_T^{-z,\tilde\varphi} = -\xi$ a.s.

The buyer's price of the option is thus equal to the opposite of the seller's price of the option with payoff $-\xi$, that is $-\mathcal{E}^g_{0,T}(-\xi) = -\tilde X_0$, where $(\tilde X, \tilde Z, \tilde K)$ is the solution of the BSDE associated with driver g and terminal condition $-\xi$. Let us specify the hedging strategy for the buyer. Suppose that the initial price of the option is $z := -\tilde X_0$. The process $\tilde X$ is equal to the value of the portfolio associated with initial value $-z = \tilde X_0$ and strategy $\tilde\varphi := \Phi(\tilde Z, \tilde K)$ (where Φ is defined in Definition 2.1) that is $\tilde X = V^{\tilde X_0,\tilde\varphi} = V^{-z,\tilde\varphi}$. Hence, $V_T^{-z,\tilde\varphi} = \tilde X_T = -\xi$ a.s. , which yields that $\tilde\varphi$ is the hedging risky-assets strategy for the buyer. Similarly, $-\mathcal{E}^g_{t,T}(-\xi) = -\tilde X_t$ satisfies an analogous property at time t, and is called the *hedging price for the buyer* at time t.

Let us now introduce the λ-*admissible* driver $\tilde g$ defined by

$$\tilde g(t, y, z, k) := -g(t, -y, -z, -k). \tag{24}$$

Note that for all $\xi \in L^2(\mathcal{G}_T)$, we have $-\mathcal{E}^g_{\cdot,T}(-\xi) = \mathcal{E}^{\tilde g}_{\cdot,T}(\xi)$.

[4] Note that by Proposition 3.7, the process (Y_t) is equal to the *seller's price process* of the American option.

This leads to the *nonlinear pricing system* $\mathcal{E}^{\tilde{g}}$ *relative to the buyer* (corresponding to the $\tilde{g}$-*evaluation*), defined for each $(S,\xi) \in [0,T] \times L^2(\mathcal{G}_S)$ by $\mathcal{E}^{\tilde{g}}_{\cdot,S}(\xi)$, which is equal to the solution of the BSDE with driver $\tilde{g}$, terminal time S and terminal condition ξ.

Remark 4.1. *If we suppose that* $-g(t,-y,-z,-k) \leq g(t,y,z,k)$ *(which is satisfied when for example* g *is convex with respect to* (y,z,k) *and* $g(t,0,0,0) = 0$*), then, by the comparison theorem for BSDEs, we have* $\mathcal{E}^{\tilde{g}}_{\cdot,S}(\xi) \leq \mathcal{E}^{g}_{\cdot,S}(\xi)$ *for all* $(S,\xi) \in [0,T] \times L^2(\mathcal{G}_S)$*. In other terms, the hedging price of a European option for the buyer is always smaller than the the seller's one.*

Moreover, when $-g(t,-y,-z,-k) = g(t,y,z,k)$ *(which is satisfied when for example* g *is linear with respect to* (y,z,k)*, as in the perfect market case), then the pricing system for the buyer is equal the the seller's one, that is,* $\mathcal{E}^{\tilde{g}} = \mathcal{E}^{g}$*.*

Note that by the flow property of BSDEs, the first coordinate of the solution of a BSDE with driver $\tilde{g}$ is an $\mathcal{E}^{\tilde{g}}$-martingale. Moreover, the opposite of a wealth process is an $\mathcal{E}^{\tilde{g}}$-martingale, that is

Proposition 4.2. *For each* $x \in \mathbb{R}$ *and each portfolio strategy* $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$*, the process* $(-V^{x,\varphi}_t)$ *is an* $\mathcal{E}^{\tilde{g}}$*-martingale.*

Proof. Since $g(t,y,z,k) := -\tilde{g}(t,-y,-z,-k)$, we get that the wealth process $(V^{x,\varphi}_t)$ satisfies:

$$dV^{x,\varphi}_t = \tilde{g}(t,-V^{x,\varphi}_t,-Z_t,-K_t)dt + Z_t dW_t + K_t dM_t, \tag{25}$$

where $(Z,K) = \phi^{-1}(\varphi)$. Hence, $(-V^{x,\varphi},-Z,-K)$ is the solution of the BSDE associated with driver $\tilde{g}$, terminal time T and terminal conditon $-V^{x,\varphi}_T$. The result follows. □

Let us consider the case of the American option from the point of view of the buyer. Supposing the initial price of the American option is z, he starts with the amount $-z$ at time $t = 0$, and wants to find a *super-hedge*, that is an exercise time τ and a risky-assets strategy φ, such that the payoff that he receives allows him to recover the debt he incurred at time $t = 0$ by buying the American option. This notion of *super-hedge for the buyer* can be defined more precisely as follows.

Definition 4.3. *A* super-hedge for the buyer *against the American option with initial price* $z \in \mathbb{R}$ *is a pair* (τ,φ) *of a stopping time* $\tau \in \mathcal{T}$ *and a risky-assets strategy* $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ *such that*

$$V^{-z,\varphi}_\tau + \xi_\tau \geq 0 \ \textit{a.s.}$$

We denote by $\mathcal{B}(z)$ the set of all *super-hedges for the buyer* associated with initial price $z \in \mathbb{R}$.

We now define the *buyer's price* $\tilde{u}_0$ of the American option as the supremum of the initial prices which allow the buyer to be super-hedged, that is

$$\tilde{u}_0 = \sup\{z \in \mathbb{R},\ \ \exists(\tau,\varphi) \in \mathcal{B}(z)\}. \tag{26}$$

Note that $\tilde{u}_0 \in \overline{\mathbb{R}}$. We shall see below that $\tilde{u}_0$ is finite.

Remark 4.4. *We have* $(0,0) \in \mathcal{B}(\xi_0)$*. Hence,* $\tilde{u}_0 \geq \xi_0$*. Moreover, if* $g(t,0,0,0) = 0$ *and* $\xi_0 \geq 0$*, then* $\tilde{u}_0 = \sup\{z \geq 0,\ \ \exists(\tau,\varphi) \in \mathcal{B}(z)\}$.

We first consider the simpler case when ξ is left-u.s.c. along stopping times. In this case, we prove below that the infimum in (26) is attained, which implies that the *buyer's price* $\tilde{u}_0$ allows the buyer to build a super-hedge. Moreover, a super-hedge strategy is provided via the solution of a reflected BSDE associated with driver $\tilde{g}$. More precisely, we have

Theorem 4.5 (*Buyer's price* and *super-hedge*)**.** *Suppose that* (ξ_t) *is left upper-semicontinuous along stopping times. The buyer's price* $\tilde{u}_0$ *of the American option satisfies:*

$$\tilde{u}_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau). \tag{27}$$

Moreover, we have

$$\tilde{u}_0 = \tilde{Y}_0, \tag{28}$$

where $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{A})$ *is the solution of the reflected BSDE associated with driver* $\tilde{g}(t,y,z,k) := -g(t,-y,-z,-k)$ *and lower obstacle* ξ*, that is,*

$$\begin{aligned} &- d\tilde{Y}_t = \tilde{g}(t,\tilde{Y}_t,\tilde{Z}_t,\tilde{K}_t)dt + d\tilde{A}_t - \tilde{Z}_t dW_t - \tilde{K}_t dM_t;\ \ \tilde{Y}_T = \xi_T,\ \textit{with} \\ &\tilde{Y} \geq \xi\,,\ \int_0^T (\tilde{Y}_t - \xi_t) d\tilde{A}_t = 0\ \textit{a.s.}\ \ , \end{aligned} \tag{29}$$

where the non decreasing process $\tilde{A}$ *is continuous.*

Let $\tilde{\tau} := \inf\{t \geq 0 :\ \tilde{Y}_t = \xi_t\}$ *and* $\tilde{\varphi} := \Phi(-\tilde{Z}, -\tilde{K})$ *(where* Φ *is defined in Definition 2.1). The pair* $(\tilde{\tau}, \tilde{\varphi})$ *is a super-hedge for the buyer, that is* $(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{u}_0)$.

Proof. Note first that, by Remark 3.3 applied to the above reflected BSDE (29), since the obstacle (ξ_t) is supposed to be left upper semicontinuous along stopping times, the process $\tilde{A}$ is continuous.

Now, by Proposition 3.2 (applied to the driver $\tilde{g}$ instead of g), we have $\sup_\tau \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau) = \tilde{Y}_0$. In order to show equality (27), it is thus sufficient to show that $\tilde{u}_0 = \tilde{Y}_0$.

Set $\mathcal{S} := \{z \in \mathbb{R} : \exists (\tau,\varphi) \in \mathcal{B}(z)\}$.

Let us first show that $\tilde{Y}_0 \leq \tilde{u}_0$. Since $\tilde{u}_0 = \sup \mathcal{S}$, it is sufficient to show that $\tilde{Y}_0 \in \mathcal{S}$. To this aim, we prove that

$$(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{Y}_0). \tag{30}$$

By definition of $\tilde{\tau}$, we have that $\tilde{Y}_t > \xi_t$ on $[0, \tilde{\tau}[$, which implies that the process $\tilde{A}$ is constant on $[0, \tilde{\tau}[$. Since $\tilde{A}$ is continuous, we derive that $\tilde{A}$ is equal to 0 on $[0, \tilde{\tau}]$. By equation (29), we thus get

$$\tilde{Y}_t = \tilde{Y}_0 - \int_0^t \tilde{g}(s, \tilde{Y}_s, \tilde{Z}_s, \tilde{K}_s)ds + \int_0^t \tilde{Z}_s dW_s + \int_0^t \tilde{K}_s dM_s,\ 0 \leq t \leq \tilde{\tau}\ \text{ a.s.} \tag{31}$$

We now consider the portfolio associated with the initial capital $-\tilde{Y}_0$ and the strategy $\tilde{\varphi} = \Phi(-\tilde{Z}, -\tilde{K})$. Using the equality $g(t,y,z,k) = -\tilde{g}(t,-y,-z,-k)$, we derive that its value $V^{-\tilde{Y}_0,\tilde{\varphi}}$ satisfies

$$-V_t^{-\tilde{Y}_0,\tilde{\varphi}} = \tilde{Y}_0 - \int_0^t \tilde{g}(s, -V_s^{-\tilde{Y}_0,\tilde{\varphi}}, \tilde{Z}_s, \tilde{K}_s)ds + \int_0^t \tilde{Z}_s dW_s + \int_0^t \tilde{K}_s dM_s,\ 0 \leq t \leq T\ \text{ a.s.} \tag{32}$$

Hence, $-V^{-\tilde{Y}_0,\tilde{\varphi}}$ and $\tilde{Y}$ satisfy the same forward differential equation on $[0, \tilde{\tau}]$ with the same initial condition. By uniqueness of the solution, they coincide. We thus get the equality $V_{\tilde{\tau}}^{-\tilde{Y}_0,\tilde{\varphi}} = -\tilde{Y}_{\tilde{\tau}}$ a.s. Moreover, the definition of the stopping time $\tilde{\tau}$ together with the right-continuity of the processes $(\tilde{Y}_t)$ and (ξ_t) yield the equality $\tilde{Y}_{\tilde{\tau}} = \xi_{\tilde{\tau}}$ a.s. We thus have the equality $V_{\tilde{\tau}}^{-\tilde{Y}_0,\tilde{\varphi}} = -\xi_{\tilde{\tau}}$ a.s. The desired property (30) follows. Hence, we have $\tilde{Y}_0 \in \mathcal{S}$, and thus $\tilde{Y}_0 \leq \tilde{u}_0$.

It remains to prove that $\tilde{u}_0 \leq \tilde{Y}_0$. Let $z \in \mathcal{S}$. By definition of $\mathcal{S}$, there exists $(\tau,\varphi) \in \mathcal{B}(z)$ such that $-V_\tau^{-z,\varphi} \leq \xi_\tau$ a.s. By taking the $\mathcal{E}^{\tilde{g}}$-evaluation, using and the $\mathcal{E}^{\tilde{g}}$-martingale property of the process $(-V_t^{-z,\varphi})$ (see Proposition 4.2) and the monotonicity of the $\tilde{g}$-expectation $\mathcal{E}^{\tilde{g}}$, we derive that $z = \mathcal{E}^{\tilde{g}}_{0,\tau}(-V_\tau^{-z,\varphi}) \leq \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau)$, which implies the inequality

$$z \leq \sup_{\tau \in \mathcal{T}} \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau) = \tilde{Y}_0.$$

Since this inequality holds for any $z \in \mathcal{S}$, by taking the supremum over $z \in \mathcal{S}$, we get $\tilde{u}_0 \leq \tilde{Y}_0$. It follows that $\tilde{u}_0 = \tilde{Y}_0$. By (30), we get $(\tilde{\tau}, \tilde{\varphi}) \in \mathcal{B}(\tilde{u}_0)$, which completes the proof. □

We now consider the general case when ξ is only RCLL. In this case, the characterizations (27) and (28) of the buyer's price $\tilde{u}_0$ still hold. However, the supremum in (26) is not necessarily attained; in other words, the price $\tilde{u}_0$ does not necessarily allow the buyer to build a *super-hedge* against the American option.

We introduce the definition of an *ε-super-hedge for the buyer*:

Definition 4.6. *For each initial price z and for each $\varepsilon > 0$, an* ε-super-hedge for the buyer *against the American option is a pair (τ, φ) of a stopping time $\tau \in \mathcal{T}$ and a risky-assets strategy $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that*

$$V_\tau^{-z,\varphi} + \xi_\tau \geq -\varepsilon \quad \text{a.s.}$$

Theorem 4.7 (*Buyer's price* and *ε-super-hedge*)**.** *Suppose that ξ is only RCLL.*
The buyer's price $\tilde{u}_0$ *of the American option satisfies*

$$\tilde{u}_0 = \sup_{\tau \in \mathcal{T}} \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau) = \tilde{Y}_0,$$

where $(\tilde{Y}, \tilde{Z}, \tilde{K}, \tilde{A})$ is the solution of the reflected BSDE associated with driver $\tilde{g}(t,y,z,k) := -g(t,-y,-z,-k)$ and lower obstacle ξ, that is,

$$\begin{aligned} &- d\tilde{Y}_t = \tilde{g}(t, \tilde{Y}_t, \tilde{Z}_t, \tilde{K}_t)dt + d\tilde{A}_t - \tilde{Z}_t dW_t - \tilde{K}_t dM_t; \;\; \tilde{Y}_T = \xi_T, \text{ with} \\ &\tilde{Y} \geq \xi\,, \;\; \int_0^T (\tilde{Y}_t - \xi_t) d\tilde{A}^c_t = 0 \text{ a.s. and } \Delta\tilde{A}^d_t = \Delta\tilde{A}^d_t \mathbf{1}_{\{\tilde{Y}_{t^-} = \xi_{t^-}\}}. \end{aligned} \tag{33}$$

Let $\tilde{\varphi} := \Phi(-\tilde{Z}, -\tilde{K})$ and for each $\varepsilon > 0$, let

$$\tau_\varepsilon := \inf\{t \geq 0 : \;\; \tilde{Y}_t \leq \xi_t + \varepsilon\}. \tag{34}$$

The pair $(\tau_\varepsilon, \tilde{\varphi})$ is an ε-super-hedge for the buyer *(associated with the initial price $\tilde{u}_0$).*

Proof. Let $\varepsilon > 0$. We have $\tilde{Y}_\cdot \geq \xi_\cdot + \varepsilon$ on $[0, \tau_\varepsilon[$. Since $\tilde{A}$ satisfies the Skorohod condition, it follows that almost surely, $\tilde{A}$ is constant on $[0, \tau_\varepsilon[$. Also, $\tilde{Y}_{(\tau^\varepsilon)^-} \geq \xi_{(\tau^\varepsilon)^-} + \varepsilon$ a.s. , which implies that $\Delta\tilde{A}_{\tau_\varepsilon} = 0$ a.s. Hence, $\tilde{A}_{\tau^\varepsilon} = 0$ a.s. It follows that $\tilde{Y}_\cdot(\omega)$ satisfies the *forward* differential equation (31) on $[0, \tau_\varepsilon]$. Now, the wealth $-V_\cdot^{-\tilde{Y}_0,\tilde{\varphi}}$ is the solution of the *forward* differential equation (32). Hence, $\tilde{Y}$ satisfies the same *forward* differential equation on $[0, \tau_\varepsilon]$ as $-V^{-\tilde{Y}_0,\tilde{\varphi}}$, with the same initial condition $\tilde{Y}_0$. By uniqueness of the solution, they coincide, which implies in particular that $V_{\tau_\varepsilon}^{-\tilde{Y}_0,\tilde{\varphi}} = -\tilde{Y}_{\tau_\varepsilon}$ a.s. Moreover, by definition of the stopping time τ_ε and the right-continuity of $(\tilde{Y}_t)$ and (ξ_t), we have

$$\tilde{Y}_{\tau_\varepsilon} \leq \xi_{\tau_\varepsilon} + \varepsilon \quad \text{a.s.} \tag{35}$$

We thus get the inequality $V_{\tau_\varepsilon}^{-\tilde{Y}_0,\tilde{\varphi}} + \xi_{\tau_\varepsilon} \geq -\varepsilon$ a.s. Hence, the pair $(\tau_\varepsilon, \tilde{\varphi})$ is an *ε-super-hedge for the buyer* associated with the initial price $\tilde{Y}_0$.

Let us now show that $\tilde{Y}_0 = \tilde{u}_0$. First, by Proposition 3.2, we have $\sup_\tau \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau) = \tilde{Y}_0$. Using this property together with the same arguments as those used in the second part of the proof of Theorem 4.5 (which do not require the continuity of $\tilde{A}$), we obtain that $\tilde{u}_0 \leq \tilde{Y}_0$.

It remains to prove that $\tilde{u}_0 \geq \tilde{Y}_0$. Let $\varepsilon > 0$. Let (Y', Z', K') be the solution of the BSDE associated with driver $\tilde{g}$, terminal time τ_ε and terminal condition $\xi_{\tau_\varepsilon} \wedge \tilde{Y}_{\tau_\varepsilon}$. Now, since $\tilde{A}_{\tau^\varepsilon} = 0$, the process $(\tilde{Y}, \tilde{Z}, \tilde{K})$ is the solution of the BSDE associated with driver $\tilde{g}$, terminal time τ_ε and terminal condition $\tilde{Y}_{\tau_\varepsilon}$. By an a priori estimate on BSDEs with default jump (see [11, Proposition 1]), since by (35) we have $\tilde{Y}_{\tau_\varepsilon} \leq \xi_{\tau_\varepsilon} \wedge \tilde{Y}_{\tau_\varepsilon} + \varepsilon$, we derive that $\tilde{Y}_0 \leq Y'_0 + K\varepsilon$, where K is a constant which only depends on T and where C is a λ-constant associated with driver $\tilde{g}$ (or equivalently with driver g). Moreover, since by assumption $Y'_{\tau_\varepsilon} = \xi_{\tau_\varepsilon} \wedge \tilde{Y}_{\tau_\varepsilon}$, we have

$Y'_{\tau_\varepsilon} \leq \xi_{\tau_\varepsilon}$. Now, one can show that $Y' = -V^{-Y'_0,\varphi'}$, where $\varphi' := \Phi(-Z', -K')$. We thus get $-V_{\tau_\varepsilon}^{-Y'_0,\varphi'} \leq \xi_{\tau_\varepsilon}$ We derive that $(\tau_\varepsilon, \varphi')$ is a super-hedge (for the buyer) associated with initial price Y'_0. The initial price Y'_0 $(\geq \tilde{Y}_0 - K\varepsilon)$ thus allows the buyer to be super-hedged. By definition of $\tilde{u}_0$, we derive that $\tilde{u}_0 \geq Y'_0 \geq \tilde{Y}_0 - K\varepsilon$. We thus get $\tilde{u}_0 \geq \tilde{Y}_0 - K\varepsilon$ for each $\varepsilon > 0$. Hence, $\tilde{u}_0 \geq \tilde{Y}_0$, which completes the proof. □

5. Links with arbitrage issues

We address now arbitrage issues associated to the problem of pricing European and American options in our imperfect market model $\mathcal{M}^g$. Related issues are studied in [19] and [20] in markets with convex portfolio constraints.

5.1. **European option case**

Consider the market $\mathcal{M}^g$ with a European option with maturity T and payoff $\xi \in L^2(\mathcal{G}_T)$. Recall that $\mathcal{E}^g_{0,T}(\xi)$ is the *hedging price* for the seller (see Section 2.2).

Definition 5.1. *Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$ and φ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$. We say that (y, φ) is an* arbitrage opportunity for the seller of the European option with initial price x *if*

$$y < x \quad \textit{and} \quad V_T^{y,\varphi} - \xi \geq 0 \quad \textit{a.s.} \tag{36}$$

This definition means that the seller sells the European option at the price x strictly greater than the amount y which is enough to be hedged (by using the strategy φ). He thus makes the profit $x - y > 0$ at time 0.

Proposition 5.2. *Let $x \in \mathbb{R}$. There exists an* arbitrage opportunity for the seller *of the European option with price x if and only if $x > \mathcal{E}^g_{0,T}(\xi)$.*

Proof. Let $\phi^\star := \Phi(Z, K)$, where (X, Z, K) is the solution of BSDE associated with driver g and terminal condition ξ. Note that $X_0 = \mathcal{E}^g_{0,T}(\xi)$. Suppose that $X_0 < x$. Since $V_T^{X_0,\phi^\star} = \xi$ a.s., $(X_0, \phi^\star)$ is an arbitrage opportunity for the seller.

Suppose now that there exists an arbitrage opportunity (y, φ) for the seller. Then $y < x$ and $V_T^{y,\varphi} - \xi \geq 0$ a .s. Since $\mathcal{E}^g$ is monotonous and $V^{y,\varphi}$ is an $\mathcal{E}^g$-martingale, we get $y = \mathcal{E}^g_{0,T}(V_T^{y,\varphi}) \geq \mathcal{E}^g_{0,T}(\xi)$. Hence $x > \mathcal{E}^g_{0,T}(\xi)$. □

Consider now the buyer. Recall that $\mathcal{E}^{\tilde{g}}_{0,T}(\xi)$ is the *buyer's hedging price* (see Section 4).

Definition 5.3. *Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$ and φ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$. We say that (y, φ) is an* arbitrage opportunity for the buyer of the European option with initial price x *if*

$$y > x \quad \textit{and} \quad V_T^{-y,\varphi} + \xi \geq 0 \quad \textit{a .s.}$$

This definition means that the buyer buys the European option at the price x, stricly smaller than the amount y , which, borrowed at time 0, allows him to recover his debt at time T (by using the strategy φ). He thus makes the profit $y - x > 0$ at time 0.

Using the same arguments as above, we get

Proposition 5.4. *There exists an arbitrage opportunity for the buyer of the European option with price x if and only if $x < \mathcal{E}^{\tilde{g}}_{0,T}(\xi)$.*

Definition 5.5. *A real number x is called an* arbitrage-free price for the European option *if there exists no arbitrage opportunity, neither for the seller nor for the buyer of the European option.*

By Propositions 5.2 and 5.4, we get

Proposition 5.6. *If $\mathcal{E}^g_{0,T}(\xi) < \mathcal{E}^{\tilde{g}}_{0,T}(\xi)$, there does not exist arbitrage-free prices of the European option.*

If $\mathcal{E}^g_{0,T}(\xi) \geq \mathcal{E}^{\tilde{g}}_{0,T}(\xi)$, the interval $[\mathcal{E}^{\tilde{g}}_{0,T}(\xi), \mathcal{E}^g_{0,T}(\xi)]$ is the set of all arbitrage-free prices. We call it the arbitrage-free *interval for the European option.*

Remark 5.7. *Note that by similar arguments as above, one can easily show that $\mathcal{E}^g_{0,T}(\xi) = \inf\{x \in \mathbb{R}, \exists \varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ s.t. $V_T^{x,\varphi} \geq \xi$ a.s.$\}$ and $\mathcal{E}^{\tilde{g}}_{0,T}(\xi) = \sup\{z \in \mathbb{R}, \exists \varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ s.t. $V_T^{-z,\varphi} + \xi \geq 0$ a.s.$\}$.*

Remark 5.8. *Suppose that $\xi \geq 0$ a.s. and that $g(t,0,0,0) = 0$. Then, by Remark 2.6, we can restrict ourselves to nonnegative prices x. In this case, we take $x, y \in \mathbb{R}^+$ in the definitions above, and the propositions remain true (with $x \in \mathbb{R}^+$).*

Remark 5.9. *Note that it is possible that $\mathcal{E}^g_{0,T}(\xi) < \mathcal{E}^{\tilde{g}}_{0,T}(\xi)$, in which case there is no arbitrage-free price. A simple example is given by taking $g(t,y,z,k) = -|y|$ and $\xi = 1$. In this case, $\mathcal{E}^g_{0,T}(1) = e^{-T}$ and $\mathcal{E}^{\tilde{g}}_{0,T}(1) = e^T$.*

5.2. **American option case**

Consider now the market $\mathcal{M}^g$ with an American option with maturity T and payoff (ξ_t) in S^2.
Let u_0 be the *superhedging price* for the seller defined by (14).

Definition 5.10. *Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$, and let φ in $\mathbb{H}^2 \times \mathbb{H}^2_\lambda$. We say that (y, φ) is an* arbitrage opportunity for the seller *of the American option with initial price x if*

$$y < x \quad \text{and} \quad V_\tau^{y,\varphi} - \xi_\tau \geq 0 \quad \text{a.s.} \quad \text{for all } \tau \in \mathcal{T}.$$

Proposition 5.11. *Let $x \in \mathbb{R}$. There exists an arbitrage opportunity for the seller of the American option with price x if and only if $x > u_0$.*

Proof. Suppose that there exists an arbitrage opportunity (y, φ) for the seller of the American option with price x. Then $y < x$ and $V_\tau^{y,\varphi} - \xi_\tau \geq 0$ a.s. for all $\tau \in \mathcal{T}$. Hence, φ is a superhedging strategy for the seller associated with initial wealth y. By definition of u_0 as an infimum (see (14)), we get $y \geq u_0$. Since $x > y$, it follows that $x > u_0$.
Suppose now that $x > u_0$. Then, by definition of u_0 as an infimum, we derive that there exists x' such that $x > x' \geq u_0$ and $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that φ is a superhedging strategy for the seller associated with initial wealth x', that is, such that $V_\tau^{x',\varphi} \geq \xi_\tau$ a.s. for all $\tau \in \mathcal{T}$. Since $x' > x$, it follows that (x', φ) is an arbitrage opportunity for the seller of the American option with price x. □

Consider now the buyer's point of view. Let $\tilde{u}_0$ be the *superhedging price for the buyer* defined in (26).

Definition 5.12. *Let $x \in \mathbb{R}$. Let $y \in \mathbb{R}$, let $\tau \in \mathcal{T}$ and let $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$. We say that (y, τ, φ) is an arbitrage opportunity for the buyer of the American option with initial price x, if*

$$y > x \quad \text{and} \quad V_\tau^{-y,\varphi} + \xi_\tau \geq 0 \quad \text{a.s.}$$

This definition means that the buyer buys the American option at the price x strictly smaller than the amount y that, borrowed at time 0, allows him to recover his debt at the exercise time τ (by using the strategy φ). He thus makes the profit $y - x > 0$ at time 0.

Proposition 5.13. *Let $x \in \mathbb{R}$. There exists an arbitrage opportunity for the buyer of the American option with price x if and only if $x < \tilde{u}_0$.*

Proof. Suppose that there exists an arbitrage opportunity (y, τ, φ) for the buyer of the American option with price x. We thus have $y > x$ and $V_\tau^{-y,\varphi} + \xi_\tau \geq 0$ a.s. Hence, (τ, φ) is a super-hedge for the buyer associated with initial wealth y. By definition of $\tilde{u}_0$ as a supremum (see (26)), we get $y \leq \tilde{u}_0$. Since $x < y$, we get $x < \tilde{u}_0$.
Suppose now that $x < \tilde{u}_0$. Then, by definition of $\tilde{u}_0$ as a supremum, we derive that there exists z such that $x < z \leq \tilde{u}_0$, $\tau \in \mathcal{T}$ and $\varphi \in \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ such that (τ, φ) is a super-hedge for the buyer associated with z, that is,

such that $V_\tau^{-z,\varphi} + \xi_\tau \geq 0$ a.s. Since $z > x$, it follows that (z, τ, φ) is an arbitrage opportunity for the buyer of the American option with price x. □

Definition 5.14. *A real number x is called an* arbitrage-free price for the American option *if there exists no arbitrage opportunity, neither for the seller nor for the buyer of the American option.*

By Propositions 5.11 and 5.13, we derive the following result.

Proposition 5.15. *If $u_0 < \tilde{u}_0$, there does not exist arbitrage-free price for the American option.*

If $u_0 \geq \tilde{u}_0$, the interval $[\tilde{u}_0, u_0]$ is the set of all arbitrage-free prices. We call it the arbitrage-free *interval for the American option.*

Remark 5.16. *When $\tilde{g} = g$, that is $g(t, y, z, k) = -g(t, -y, -z, -k)$ (which is satisfied for example when g is linear with respect to (y, z, k)), then $u_0 = \tilde{u}_0$ is the unique* arbitrage-free price for the American option.

When $g \geq \tilde{g}$, that is $g(t, y, z, k) \geq -g(t, -y, -z, -k)$ (which is satisfied for example when g is convex with respect to (y, z, k) and $g(t, 0, 0, 0) = 0$), then, for all $\tau \in \mathcal{T}$, we have $\mathcal{E}^g_{0,\tau}(\xi_\tau) \geq \mathcal{E}^{\tilde{g}}_{0,\tau}(\xi_\tau)$. By taking the supremum over $\tau \in \mathcal{T}$, using Theorems 3.4 and 4.5, we get $u_0 \geq \tilde{u}_0$.

Similarly, when $g \leq \tilde{g}$, that is $g(t, y, z, k) \leq -g(t, -y, -z, -k)$ (which is satisfied for example when g is concave with respect to (y, z, k) and $g(t, 0, 0, 0) = 0$), then $u_0 \leq \tilde{u}_0$.

Note that it is possible that $u_0 < \tilde{u}_0$, and hence, that there does not exist an arbitrage-free price for the American option. A simple example is given by $g(t, y, z, k) = -|y|$ and $\xi_t = 1$ for all t. In this case, we have $u_0 = \sup_{\tau \in [0,T]} e^{-\tau} = 1$ and $\tilde{u}_0 = \sup_{\tau \in [0,T]} e^{\tau} = e^T$.

Remark 5.17. *Suppose that $\xi_0 \geq 0$ and that $g(t, 0, 0, 0) = 0$. Then, by Remarks 2.6, 3.1, and 4.4, we can restrict ourselves to nonnegative prices x. In this case, we take $x, y \in \mathbb{R}^+$ in the definitions above, and the propositions remain true (with $x \in \mathbb{R}^+$). Then, Proposition 5.15 corresponds to Theorem 4.3 of [20].*

6. American options with intermediate cashflows

Suppose that a European option pays a terminal payoff ξ at terminal time S and intermediate cashflows. The cumulative intermediate cashflows generated by the option is modeled by a finite variational RCLL adapted process (D_t) with square integrable total variation (with $D_0 = 0$). By [11, Proposition 2], there exists a unique solution (X, Z, K) in $\mathcal{S}^2 \times \mathbb{H}^2 \times \mathbb{H}^2_\lambda$ of the following BSDE:

$$-dX_t = g(t, X_t, Z_t, K_t)dt + dD_t - Z_t dW_t - K_t dM_t; \quad X_S = \xi. \tag{37}$$

The process X, denoted also by $X_{\cdot,S}(\xi, D)$, is the wealth process associated with initial value $x = X_0$ and strategy $\varphi = \Phi(Z, K)$. Here, D_t represents the cumulative cash amount at time t withdrawn from the portfolio in order to pay the intermediate cashflows to the buyer. Hence, the amount X_0 allows the seller to be perfectly hedged against the option, in the sense that it allows him/her to pay the intermediate cashflows and the terminal payoff to the buyer, by investing the amount X_0 along the strategy φ in the market. The price for the seller (at time 0) of this option is thus given by $X_0 = X_{0,S}(\xi, D)$ and the associated hedging strategy is equal to φ. Note that the driver of BSDE (37) is given by the λ-*admissible* "generalized" driver $g(t, X_t, Z_t, K_t)dt + dD_t$. This leads to the following *nonlinear pricing* system for the seller (see [11, Section 3.3]):
For each $S \in [0, T]$, for each payoff $\xi \in L^2(\mathcal{G}_S)$ and for each cumulative intermediate cashflows (D_t), the associated price is defined by $\mathcal{E}^{g,D}_{t,S}(\xi) := X_{t,S}(\xi, D)$ for each $t \in [0, S]$. Some properties of this nonlinear pricing system are provided in [11, Section 3.3].
Note that $\mathcal{E}^{g,D}_{t,S}(\xi)$ can be defined on the whole interval $[0, T]$ by setting $\mathcal{E}^{g,D}_{t,S}(\xi) := \mathcal{E}^{g^S,D^S}_{t,T}(\xi)$ for $t \geq S$, where $g^S(t, .) := g(t, .)\mathbf{1}_{t \leq S}$ and $D^S_t := D_{t \wedge S}$. The operator $\mathcal{E}^{g,D}$ is called the (g, D)-conditional expectation (cf. [11, Section 3.4]). If $D = 0$, $\mathcal{E}^{g,D}$ corresponds to the g-conditional expectation $\mathcal{E}^g$. Concerning the pricing of the American option, the results of Section 3 can be shown by using a similar approach to the one used in Section 3, replacing the driver g by the "generalized" driver $g(\cdot)dt + dD_t$, and $\mathcal{E}^g$ by $\mathcal{E}^{g,D}$.

7. Conclusion and perspectives

This paper deals with the pricing and hedging problem of American options in a complete imperfect market model with default. First, we have shown that the seller's price, defined as the minimal initial wealth which allows the seller to be superhedged, admits a dual representation as the supremum $\sup_\tau \mathcal{E}^g_{0,\tau}(\xi_\tau)$. Moreover, we have obtained an infinitesimal characterization of the price process as the first coordinate of the solution of a *nonlinear* reflected BSDE, and also provided a superhedging portfolio strategy, defined in terms of the solution of this reflected BSDE. Our proof relies on the characterization of the value function of an $\mathcal{E}^g$-optimal stopping problem as the solution of a *nonlinear* reflected BSDE (see [14] and [27]).

We have also considered the buyer's point of view : we have shown a dual representation of the buyer's price, and an infinitesimal characterization of the price process as the first coordinate of the solution of a reflected BSDE. Moreover, under an additional regularity assumption on the payoff, we have obtained a super-hedge pair, consisting of a suitable portfolio strategy and an appropriate exercise time, defined in terms of the solution of this reflected BSDE.

We study the problem of *game* options pricing in an imperfect market with default in [10]. We provide dual representations of the seller's and the buyer's price as well as infinitesimal characterizations of these prices in terms of solutions of *nonlinear* doubly reflected BSDEs. Our proof relies on a result we provided in [9, Section 4], which allows us to characterize the *common value* of an $\mathcal{E}^g$-optimal stopping *game* problem as the solution of a *nonlinear* doubly reflected BSDE.

Note that the characterizations of the buyer's price of the American option cannot be directly derived from those of the seller's price, contrary to the case of *game* options. Indeed, there is in a way a symmetry between the buyer and the seller of game options (see [10, Section 5.2]), which is not the case for American options.

In a future work, we address the pricing and hedging problem of American options when only one of the two risky assets is tradable. In this case, the market model is no longer complete.

8. Appendix

We give here some a priori estimates for reflected BSDEs with default jump.

Lemma 8.1 (A priori estimate for RBSDEs)**.** *Let f^1 and f^2 be two* λ-admissible *drivers. Let C be a λ-constant associated with f^1. Let ξ be an adapted RCLL processes. For $i = 1, 2$, let (Y^i, Z^i, K^i, A^i) be a solution of the RBSDE associated with terminal time T, driver f^i and obstacle ξ. Let $\eta, \beta > 0$ be such that $\beta \geq \frac{3}{\eta} + 2C$ and $\eta \leq \frac{1}{C^2}$.*
Let $\bar{f}(s) := f^1(s, Y^2_s, Z^2_s, K^2_s) - f^2(s, Y^2_s, Z^2_s, K^2_s)$. For each $t \in [0, T]$, we then have

$$e^{\beta t}(Y^1_s - Y^2_s)^2 \leq \eta\, \mathbb{E}[\int_t^T e^{\beta s} \bar{f}(s)^2 ds \mid \mathcal{G}_t] \quad \text{a .s.} \tag{38}$$

Moreover, $\|\bar{Y}\|^2_\beta \leq T\eta \|\bar{f}\|^2_\beta$, and if $\eta < \frac{1}{C^2}$, we then have $\|\bar{Z}\|^2_\beta + \|\bar{K}\|^2_{\lambda,\beta} \leq \frac{\eta}{1-\eta C^2} \|\bar{f}\|^2_\beta$.

The proof is similar to the one given for doubly reflected SBDEs in the same framework with default (see the proof of Proposition 6.1 in the Appendix in [10]), and left to the reader.

References

[1] Bank, P. and Kramkov D. (2015): A model for a large investor trading at market indifference prices. II: Continuous-time case, *The Annals of Applied Probability* No. 5, 2708-2742

[2] Bielecki, T., Jeanblanc, M., and Rutkowski, M., PDE approach to valuation and hedging of credit derivatives, *Quantitative finance* 5, 257–270.

[3] Bielecki, T., Crepey, S., Jeanblanc, M., and Rutkowski, M., *Defaultable game options in a hazard process model*, International Journal of Stochastic Analysis, 2009.

[4] Brigo, D., Franciscello, M., and Pallavicini: Analysis of nonlinear valuation equations under credit and funding effects. In *Innovations in Derivative Markets, Springer Proceedings in Mathematics and Statistics*, Vol. 165, Glau, K., Grbac, Z., Scherer, M., and Zagst, R. (Eds.), Springer, Heidelberg, 2016, 37-52.
[5] Crépey, S, Bilateral Counterparty Risk under Funding Constraints Part I: CVA. Pricing. *Mathematical Finance* 25 (1), 1-22, 2015. 113, 2015.
[6] Crépey, S. and Matoussi, A., Reflected and Doubly Reflected BSDEs with jumps, *Annals of App. Prob.* 18(5), 2041-2069 (2008).
[7] Dellacherie, C. and Meyer, P.-A. (1975). *Probabilités et Potentiel, Chap. I-IV*. Nouvelle édition. Hermann. **MR**0488194
[8] Dumitrescu, R., Quenez M.C., Sulem A., Optimal stopping for dynamic risk measures with jumps and obstacle problems, *Journal of Optimization Theory and Applications*, Volume 167, Issue 1 (2015), 219–242. DOI:10.1007/s10957-014-0635-2.
[9] Dumitrescu R., M.-C. Quenez, and A. Sulem (2016), Generalized Dynkin Games and Doubly reflected BSDEs with jumps, *Electronic Journal of Probability*, 21(64), 32p.
[10] Dumitrescu, R., Quenez M.C., and A. Sulem, Game options in an imperfect market with default, *SIAM Journal on Financial Mathematics,* Vol 8, pp 532–559, 2017.
[11] Dumitrescu, R. Grigorova, Quenez M.C., M., Sulem A., , BSDEs with default jump, in *Computation and Combinatorics in Dynamics, Stochastics and Control*, Eds. Elena Celledoni, Giulia Di Nunno, Kurusch Ebrahimi-Fard, Hans Munthe-Kaas, The Abel Symposium 2016, The Abel Symposia book series, volume 13, Springer. (Expected 2018)
[12] El Karoui N., Pardoux E. and M.C. Quenez (1997), Reflected Backward SDEs and American Options, *Numerical methods in finance*, edited by L. C. G. Rogers and D. Talay
[13] El Karoui N., Peng S. and M.C. Quenez (2001), "A dynamic maximum principle for the optimization of recursive utilities under constraints", Vol.11, n.3, 664-693 *Annals of Applied Probability.*
[14] El Karoui N. and M.C. Quenez (1996), "Non-linear Pricing Theory and Backward Stochastic Differential Equations", Lectures Notes in Mathematics 1656, Bressanone, 1996, Editor:W.J.Runggaldier, collection Springer,1997.
[15] Hamadène, S., Mixed zero-sum stochastic differential game and American game options, *SIAM J. Control Optim.*, 45(2), (2006), 496-518.
[16] Hamadène S. (2002) Reflected BSDE's with discontinuous barrier and application, *Stochastics and Stochastic Reports* 74(3-4), 571-596.
[17] Hamadène, S. and Ouknine, Y., Reflected Backward SDEs with general jumps, *Teor. Veroyatnost. i Primenen.* (2015), **60**(2), 357-376.
[18] Jeanblanc, M., Yor M., and Chesney M. (2009): Mathematical methods for financial markets, Springer Finance.
[19] Karatzas I. and Kou S., On the pricing of contingent claims under constrained portfolios, *The Annals of Applied Probability* 6(2) 321-369, 1996.
[20] Karatzas I. and Kou S., Hedging American contingent claims with constrained portfolios, *Finance and Stochastics* 2(3) 215-258, 1998.
[21] Karatzas I. and Shreve S.E., Methods of Mathematical Finance. Applications of Mathematics (New York) 39. Springer, New York, 1998.
[22] Korn R., Contingent claim valuation in a market with different interest rates *Mathematical Methods of Operation Research* (1995) 42: 255-274.
[23] Kühn C. and Kraft H. , Large traders and illiquid options: hedging vs. manipulation, *Stochastics*, 2010.
[24] Kusuoka, S., A remark on default risk models, *Adv. Math. Econ.*, 1, 69–82, 1999.
[25] Musiela M., Rutkowski M. American Options. In: Martingale Methods in Financial Modelling. Applications of Mathematics (Stochastic Modelling and Applied Probability), vol 36. Springer, Berlin, Heidelberg, 1997.
[26] Peng, S. (2004), Nonlinear expectations, nonlinear evaluations and risk measures, 165-253, *Lecture Notes in Math.*, Springer, Berlin.
[27] Quenez M.-C. and Sulem A., Reflected BSDEs and robust optimal stopping for dynamic risk measures with jumps, *Stochastic Processes and Applications* 124, (2014) 3031–3054.