

Supplemental Material: Field-Embedded Particles Driven by Active Flips

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I. NORMALIZATION

In order to reduce the number of free parameters, we normalize lengths by the lattice spacing a , energies by T , times by $a^2/(\Gamma c)$ and we absorb c in a redefinition of the field ϕ . We thus replace $\mathbf{x}/a \rightarrow \mathbf{x}$, $\Gamma ct/a^2 \rightarrow t$, $c\phi^2/T \rightarrow \phi^2$, $a^2 r/c \rightarrow r$, $B/c \rightarrow B$, $T\mu/(\Gamma c) \rightarrow \mu$, $a^2\alpha/(\Gamma c) \rightarrow \alpha$ and $a^2\gamma/(\Gamma c) \rightarrow \gamma$.

II. SIMULATION DETAILS

We simulate our system on a two dimensional square lattice of size $L_x \times L_y$ with periodic boundary conditions. We use the dimensionless formulation given in the article. The Gaussian field is defined on each site (i, j) and takes continuous real values ϕ_{ij} . The particles move on the lattice sites and we denote $\mathbf{r}_k = (i_k, j_k)$ their position. The discretized Hamiltonian is given by

$$H = \sum_{i,j} \left[\frac{r}{2} \phi_{ij}^2 + \frac{1}{2} (\nabla \phi_{ij})^2 \right] + \sum_{k=1}^N \frac{B}{2} (\phi_k - \phi_0 S_k)^2, \quad (1)$$

where $\phi_k \equiv \phi_{i_k, j_k}$ and $\nabla \phi_{ij} \equiv (\phi_{i+1, j} - \phi_{i, j}, \phi_{i, j+1} - \phi_{i, j})^T$. At each step Δt we first choose randomly if we begin by updating the field and then the particles, or the opposite.

- For the field update, each lattice site is updated according to

$$\phi_{ij}(t+\Delta t) = \phi_{ij}(t) + \Delta t \left(\nabla^2 \phi_{ij}(t) - r \phi_{ij}(t) - B \sum_{k=1}^N [\phi_{ij}(t) - \phi_0 S_k(t)] \delta_{i, i_k} \delta_{j, j_k} \right) + G_{ij}(0, 2\Delta t), \quad (2)$$

where $\nabla^2 \phi_{ij} \equiv \phi_{i+1, j} + \phi_{i-1, j} + \phi_{i, j+1} + \phi_{i, j-1} - 4\phi_{i, j}$ is the discrete Laplacian and the $G_{ij}(0, 2\Delta t)$ are independent random Gaussian variables with mean 0 and variance $2\Delta t$.

- For the particles' update, we choose N times at random a particle among the N particles and decide if it hops to a neighbouring site, flips its spin or does not move. We compute the probability of each event and we apply a tower sampling algorithm [1]. We define $P_u, P_r, P_d, P_\ell, P_f$ the probabilities for a particle to move up, right, down, left, or to flip, respectively. The total energy variation when the particle k moves from site (i_k, j_k) to (i'_k, j'_k) is given by

$$\Delta H_{\mathbf{r}_k \rightarrow \mathbf{r}'_k} = \frac{B}{2} (\phi'_{i'_k, j'_k} - \phi_{i_k, j_k}) (\phi'_{i'_k, j'_k} + \phi_{i_k, j_k} - 2\phi_0 S_k). \quad (3)$$

Similarly, the energy variation of the system when the particle k flips spin at site (i_k, j_k) is given by

$$\Delta H_{S_k \rightarrow -S_k} = 2B\phi_0 \phi_k S_k. \quad (4)$$

Hence, according to the Langevin dynamics, we take

$$P_u(k) = \mu \Delta t \exp \left[-\frac{B}{4} (\phi_{i_k, j_k+1} - \phi_{i_k, j_k}) (\phi_{i_k, j_k+1} + \phi_{i_k, j_k} - 2\phi_0 S_k) \right], \quad (5)$$

$$P_r(k) = \mu \Delta t \exp \left[-\frac{B}{4} (\phi_{i_k+1, j_k} - \phi_{i_k, j_k}) (\phi_{i_k+1, j_k} + \phi_{i_k, j_k} - 2\phi_0 S_k) \right], \quad (6)$$

$$P_d(k) = \mu \Delta t \exp \left[-\frac{B}{4} (\phi_{i_k, j_k-1} - \phi_{i_k, j_k}) (\phi_{i_k, j_k-1} + \phi_{i_k, j_k} - 2\phi_0 S_k) \right], \quad (7)$$

$$P_\ell(k) = \mu \Delta t \exp \left[-\frac{B}{4} (\phi_{i_k-1, j_k} - \phi_{i_k, j_k}) (\phi_{i_k-1, j_k} + \phi_{i_k, j_k} - 2\phi_0 S_k) \right]. \quad (8)$$

The dynamics of the flips depends on the type of particle considered. For SFIPs, the flipping probability is

$$P_f(k) = \epsilon \Delta t \exp[-B\phi_0\phi_k S_k], \quad (9)$$

and satisfies detailed balance, whereas for ASFIPs the flipping probability is independent of the field and is given by

$$P_f(k) = \begin{cases} \alpha\Delta t & \text{if } S_k = -1, \\ \gamma\Delta t & \text{if } S_k = +1. \end{cases} \quad (10)$$

The flipping rate ϵ plays no role on the phase diagram in equilibrium. We take Δt small enough to ensure that the probabilities verify

$$P_u(k) + P_r(k) + P_d(k) + P_\ell(k) + P_f(k) < 1, \quad (11)$$

then the probability $P_n(k)$ that particle k neither jumps nor flips is given by $P_n(k) = 1 - [P_u(k) + P_r(k) + P_d(k) + P_\ell(k) + P_f(k)]$.

III. FORCE BETWEEN TWO PARTICLES

In order to measure the mediated force between two particles, we perform a simulation with only two up-spin particles in the Gaussian field. The first particle is fixed at $\mathbf{r}_1 = 0$, and the second particle is trapped in a quadratic potential centered on site \mathbf{R}_0 . Hence, the Langevin Eq. (5) for the second particle, with $\mathbf{r}_2 = \mathbf{R}$, writes in dimensionless form:

$$\frac{d\mathbf{R}}{dt} = -\mu \frac{\partial H}{\partial \mathbf{R}} - \mu \frac{\partial H_{\text{quad}}}{\partial \mathbf{R}} + \sqrt{2\mu}\boldsymbol{\eta}_2(t), \quad (12)$$

with $H_{\text{quad}} = \lambda(\mathbf{R}(t) - \mathbf{R}_0)^2/2$ the quadratic potential. In equilibrium, when the system reaches a stationary regime, taking the average of Eq. (12) yields the average force

$$\langle \mathbf{f} \rangle = \left\langle -\frac{\partial H}{\partial \mathbf{R}} \right\rangle = \lambda(\langle \mathbf{R} \rangle - \mathbf{R}_0), \quad (13)$$

which corresponds to the field-mediated interaction when λ is large enough to ensure small fluctuations of the position of particle 2.

The field-mediated force can be analytically calculated for two fixed particles using a Hubbard-Stratonovich transformation on the partition function:

$$\mathcal{Z} = \int \mathcal{D}\phi \exp \left(-\frac{1}{2} \int d^2x [r\phi^2 + (\nabla\phi)^2] - \frac{B}{2} \int d^2x (\phi(\mathbf{x}) - \phi_0)^2 [\delta(\mathbf{x}) + \delta(\mathbf{x} - \mathbf{R})] \right) \quad (14)$$

$$\propto \int dk_1 dk_2 \exp \left(-\frac{1}{2} (k_1, k_2) A (k_1, k_2)^T + i\phi_0 (k_1, k_2) (1, 1)^T \right), \quad (15)$$

with

$$A = \begin{pmatrix} B^{-1} + G(0) & G(R) \\ G(R) & B^{-1} + G(0) \end{pmatrix}, \quad (16)$$

in which

$$G(R) = \int \frac{d^2q}{(2\pi)^2} \frac{e^{i\mathbf{q}\cdot\mathbf{R}}}{r + q^2} = \frac{1}{2\pi} K_0(R\sqrt{r}). \quad (17)$$

Since $G(0)$ exhibits a logarithmic UV divergence, we regularize it by introducing a cutoff in Fourier space that takes into account the finite size of the lattice spacing (unity here). Hence $G(0)$ reads

$$G(0) = \frac{1}{2\pi} \int_0^\pi dq \frac{q}{r + q^2} \simeq \frac{1}{2\pi} \ln(\pi r^{-1/2}), \quad (18)$$

for $r \ll 1$. The total free energy $U(R) = -\ln \mathcal{Z}$ is then given by $U = U_C + U_{\text{el}}$ with

$$U_C = \frac{1}{2} \ln \left(1 - \frac{K_0(R\sqrt{r})^2}{Q^2} \right) \quad (19)$$

$$U_{\text{el}} = \frac{2\pi\phi_0^2}{K_0(R\sqrt{r}) + Q}, \quad (20)$$

with $Q = \ln(\pi r^{-1/2}) + 2\pi/B$. In dimensionful form, the energy U_C is proportional to temperature, whereas U_{el} does not depend on temperature. The interpretation is that the total interaction energy splits into a Casimir-like contribution U_C , and an elastic one U_{el} .

IV. ROLE OF MULTIBODY INTERACTIONS

We want to probe the effect of N -body interactions. To do so, we replace the quadratic coupling of the particles to the field with a linear one:

$$H_{\text{int}} = \sum_{k=1}^N \kappa S_k \phi(\mathbf{r}_k), \quad (21)$$

where κ sets the strength of the coupling. In equilibrium, we can integrate out the field which results in an effective Hamiltonian for the particles featuring only pairwise interactions. Indeed, for spin-up particles with local density ρ , the partition function reads

$$\mathcal{Z} = \int \mathcal{D}\phi \exp \left(-\frac{1}{2} \int d^2x [r\phi^2 + (\nabla\phi)^2] - \kappa \int d^2x \phi(\mathbf{x})\rho(\mathbf{x}) \right) \quad (22)$$

$$\propto \exp \left(\frac{\kappa^2}{2} \int \rho(\mathbf{x})G(\mathbf{x}-\mathbf{y})\rho(\mathbf{y}) \right), \quad (23)$$

where G is the same correlator as Eq. (17). From there we read off the direct pairwise potential:

$$U_{\text{el}}^{\text{lin}}(\mathbf{x}, \mathbf{y}) = -\kappa^2 G(\mathbf{x}-\mathbf{y}) = -\frac{\kappa^2}{2\pi} K_0(|\mathbf{x}-\mathbf{y}|\sqrt{r}) \quad (24)$$

which remains independent of temperature in dimensionful form.

In order to carry out a quantitative comparison between the quadratic and the linear coupling, we adjust the parameter κ in order to match the force obtained with the quadratic coupling. This can be done almost perfectly: as shown in Fig. 1, the pairwise force F_{lin} deriving from the linear coupling $U_{\text{el}}^{\text{lin}}$ is equivalent to the force $F_{\text{quad}} = -\nabla U_{\text{el}}$, with U_{el} given by Eq. (20), when κ is correctly tuned (depending on B and ϕ_0). Then, we simulate SFIPs and ASFIPs for both couplings at matching two-body forces (fig. 2). In the linear case, SFIPs undergo strong unphysical condensation (see Fig. 2b) allowed by the absence of direct hard-core repulsion. Furthermore, patterns disappear in the ASFIP system (see Fig. 2e). This demonstrates the importance of multibody interactions.

V. MOVIES

Movie 1: ASFIP at the following parameters : $r = 0.01$, $B = 0.15$, $\phi_0 = 8$, $\mu = 5$, $\rho_0 = 0.4$, and $\alpha = \gamma = 0.02$.

Movie 2: ASFIP at the following parameters : $r = 0.01$, $B = 0.15$, $\phi_0 = 8$, $\mu = 5$, $\rho_0 = 0.4$, and $\alpha = 0.02$, $\gamma = 0.0066$.

[1] W. Krauth, *Statistical mechanics: algorithms and computations*, vol. 13 (OUP Oxford, 2006).

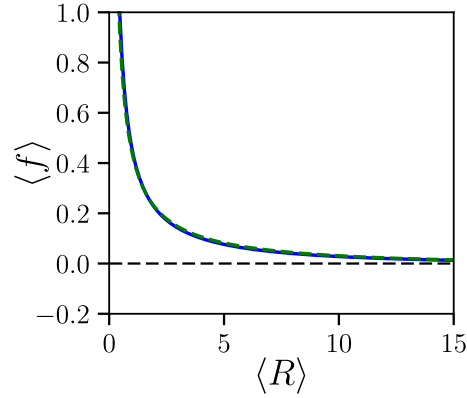


FIG. 1: (color online) Equilibrium average force as a function of the average particle separation. Solid blue line: analytical force deriving from the pairwise interaction when the coupling is linear $U_{\text{el}}^{\text{lin}}$ ($r = 0.01$, $\kappa = 1.7$). Dashed green line (matches exactly the blue line): analytical force deriving from the two particle interaction U_{el} when the coupling is quadratic ($r = 0.01$, $B = 0.26$, $\phi_0 = 8$).

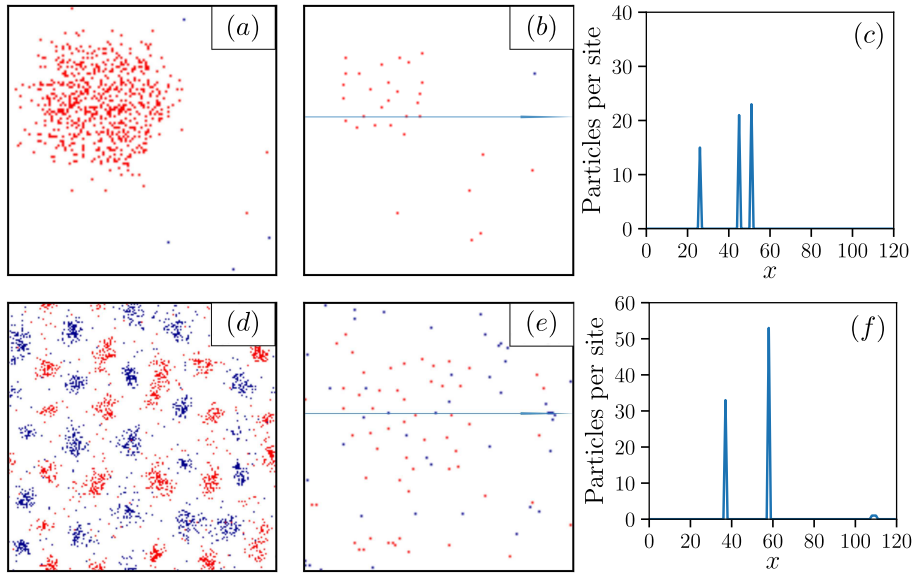


FIG. 2: (a) SFIPs quadratically coupled to the Gaussian field, with $r = 0.01$, $B = 0.26$, $\phi_0 = 8$, $\rho_0 = 0.05$. (b) SFIPs linearly coupled to the Gaussian field with $\kappa = 1.7$ tuned to yield the same interaction as the quadratic case of (a), and the same $\rho_0 = 0.05$, $r = 0.01$. (c) Number of particles per site in the case of a linear coupling: 1D snapshot of (b) along the blue arrow. The system undergoes a strong and unphysical condensation. (d) ASFIPs quadratically coupled to the Gaussian field, with $r = 0.01$, $\rho_0 = 0.1$, $\mu = 2.5$, $\alpha = \gamma = 0.005$, $B = 0.26$ and $\phi_0 = 8$. (e) ASFIPs linearly coupled to the Gaussian field with $\kappa = 1.7$ tuned as before (same parameters as in (d)). (f) Number of particles per site in the case of a linear coupling: 1D snapshot of (e) along the blue arrow. The system undergoes a strong and unphysical condensation and patterns disappear.