MODULI OF STOKES TORSORS AND SINGULARITIES OF DIFFERENTIAL EQUATIONS

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Abstract. — Let $\mathcal{M}$ be a meromorphic connection with poles along a smooth divisor D in a smooth algebraic variety. Let $\operatorname{Sol}\mathcal{M}$ be the solution complex of $\mathcal{M}$. We prove that the good formal decomposition locus of $\mathcal{M}$ coincides with the locus where the restrictions to D of $\operatorname{Sol}\mathcal{M}$ and $\operatorname{Sol}\operatorname{End}\mathcal{M}$ are local systems. By contrast to the very different natures of these loci (the first one is defined via algebra, the second one is defined via analysis), the proof of their coincidence is geometric. It relies on moduli of Stokes torsors.

The problematic of this paper is to understand how the geometry of the Stokes phenomenon in any dimension sheds light on the interplay between the singularities of a differential equation and the singularities of its solutions.

Consider an algebraic linear system $\mathcal{M}$ of differential equations with n variables

$$\frac{\partial X}{\partial x_i} = \Omega_i X \qquad i = 1, \dots, n$$

where Ω_i is a square matrix of size r with coefficients into the ring $\mathbb{C}[x_1, \dots, x_n][x_n^{-1}]$ of Laurent polynomials with poles along the hyperplane D in $\mathbb{C}^n$ given by $x_n = 0$. At a point away from D, the holomorphic solutions of the system $\mathcal{M}$ are fully understood by means of Cauchy's theorem. At a point of D, the situation is much more complicated. It is still the source of challenging unsolved problems. We call D *the singular locus of* $\mathcal{M}$. Two distinguished open subsets of D where the singularities of $\mathcal{M}$ are mild can be defined.

First, the set $\operatorname{Good}(\mathcal{M})$ of *good formal decomposition points of* $\mathcal{M}$ is the subset of D consisting of points P at the formal neighbourhood of which $\mathcal{M}$ admits a *good decomposition*. For P being the origin, and modulo ramification issues that will be neglected in this introduction, this means roughly that there exists a base change with coefficients in $\mathbb{C}[\![x_1, \dots, x_n]\!][x_n^{-1}]$ splitting $\mathcal{M}$ as a direct sum of well-understood systems easier to work with.

Good formal decomposition can always be achieved in the one variable case [**Sv00**]. It is desirable in general because it provides a concrete description of the system, at least formally at a point. In the higher variable case however, it was observed in

[**Sab00**] that $\mathcal{M}$ may not have good formal decomposition at every point of D. Thus, the set $\mathrm{Good}(\mathcal{M})$ is a non trivial invariant of $\mathcal{M}$. As proved by André [**And07**], the set $\mathrm{Good}(\mathcal{M})$ is the complement in D of a Zariski closed subset F of D either purely of codimension 1 in D or empty. Traditionally, F is called the *Turning point locus of* $\mathcal{M}$, by reference to the way the Stokes directions of $\mathcal{M}$ move along a small circle in D going around a turning point. In a sense, **the good formal decomposition locus of $\mathcal{M}$ is the open subset of D where the singularities of the system $\mathcal{M}$ are as simple as possible**.

To define the second distinguished subset of D associated to $\mathcal{M}$, let us view $\mathcal{M}$ as a $\mathcal{D}$-module, that is a module over the Weyl algebra of differential operators. Let us denote by $\mathrm{Sol}\,\mathcal{M}$ the solution complex of the analytification of $\mathcal{M}$. Concretely, $\mathcal{H}^0\,\mathrm{Sol}\,\mathcal{M}$ encodes the holomorphic solutions of our differential system while the higher cohomologies of $\mathrm{Sol}\,\mathcal{M}$ keep track of higher Ext groups in the category of $\mathcal{D}$-modules. As proved by Kashiwara [**Kas75**], the complex $\mathrm{Sol}\,\mathcal{M}$ is perverse. From a theorem of Mebkhout [**Meb90**], the restriction of $\mathrm{Sol}\,\mathcal{M}$ to D, that is, the *irregularity complex of $\mathcal{M}$ along D*, denoted by $\mathrm{Irr}^*_D\,\mathcal{M}$ in this paper, is also perverse. In particular, $(\mathrm{Sol}\,\mathcal{M})_{|D}$ is a local system on D away from a closed analytic subset of D. The *smooth locus of* $(\mathrm{Sol}\,\mathcal{M})_{|D}$ denotes the biggest open in D on which $(\mathrm{Sol}\,\mathcal{M})_{|D}$ is a local system. In a sense, **the smooth locus of $(\mathrm{Sol}\,\mathcal{M})_{|D}$ is the open subset of D where the singularities of the (derived) solutions of $\mathcal{M}$ are as simple as possible**.

As observed in [**Tey13**], the open set $\mathrm{Good}(\mathcal{M})$ is included in the smooth locus of $(\mathrm{Sol}\,\mathcal{M})_{|D}$ and $(\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_{|D}$, and the reverse inclusion was conjectured in [**Tey13**, 15.0.5]. Coincidence of $\mathrm{Good}(\mathcal{M})$ with the smooth locus of $(\mathrm{Sol}\,\mathcal{M})_{|D}$ and $(\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_{|D}$ seems surprising at first sight, since goodness is an *algebraic* notion whereas $\mathrm{Sol}\,\mathcal{M}$ is *transcendental*. The main goal of this paper is to prove via *geometric* means the following

Theorem 1. — *The good formal decomposition locus of an algebraic meromorphic connection $\mathcal{M}$ with poles along a smooth divisor D in a smooth algebraic variety is exactly the locus of D where $(\mathrm{Sol}\,\mathcal{M})_{|D}$ and $(\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_{|D}$ are local systems.*

Other criteria detecting good points of meromorphic connections are available in the literature. Let us mention André's criterion [**And07**, 3.4.1] in terms of specialisations of Newton polygons. Let us also mention Kedlaya's criterion [**Ked10**, 4.4.2] in terms of the variation of spectral norms under varying Gauss norms on rings of formal power series. This criterion is numerical in nature. By contrast, the new criterion given by Theorem 1 is *transcendental*.

The main tool at stake in the proof of Theorem 1 is geometric, via *moduli of Stokes torsors* [**Tey17**]. For a detailed explanation of the line of thoughts that brought them into the picture, let us refer to 2.1. In this introduction, we explain how these moduli are used by giving the main ingredients of the proof of Theorem 1 in dimension 2. In that case, we have to show the goodness of a point $0 \in D$ provided we know that $(\mathrm{Sol}\,\mathcal{M})_{|D}$ and $(\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_{|D}$ are local systems in a neighbourhood of 0. The

main problem is to extend the good formal decomposition of $\mathcal{M}$ across 0. This decomposition can be seen as a system of linear differential equations $\mathcal{N}$ defined in a neighbourhood of a small disc Δ^* of D punctured at 0.

To show that $\mathcal{N}$ extends across 0, we first construct via Stokes torsors a moduli space $\mathcal{X}$ parametrizing very roughly systems defined in a neighbourhood of Δ and formally isomorphic to $\mathcal{M}$ along Δ. A distinguished point of $\mathcal{X}$ is given by $\mathcal{M}$ itself. Similarly, we construct a moduli space $\mathcal{Y}$ parametrizing roughly systems defined in a neighbourhood of Δ^* and formally isomorphic to $\mathcal{M}_{|\Delta^*}$ along Δ^*. Two distinguished points of $\mathcal{Y}$ are $\mathcal{M}_{|\Delta^*}$ and $\mathcal{N}$. Restriction from Δ to Δ^* provides a morphism of algebraic varieties $\mathbf{res} : \mathcal{X} \longrightarrow \mathcal{Y}$. The problem of extending $\mathcal{N}$ is then the problem of proving that **res** hits $\mathcal{N}$. The moduli $\mathcal{X}$ and $\mathcal{Y}$ have the wonderful property that the tangent map $T_{\mathcal{M}}\,\mathbf{res}$ of **res** at $\mathcal{M}$ is exactly the map

$$\Gamma(\Delta, \mathcal{H}^1\,\mathrm{Sol}\,\mathrm{End}\,\mathcal{M}) \longrightarrow \Gamma(\Delta^*, \mathcal{H}^1\,\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})$$

associating to $s \in \Gamma(\Delta, \mathcal{H}^1\,\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})$ the restriction of s to Δ^*. In this geometric picture, the smoothness of $(\mathcal{H}^1\,\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_{|D}$ around 0 thus translates into the fact that $T_{\mathcal{M}}\,\mathbf{res}$ *is an isomorphism of vector spaces.* Since $\mathcal{X}$ and $\mathcal{Y}$ are smooth, we deduce that **res** is *étale* at the point $\mathcal{M}$. Thus, the image of **res** in $\mathcal{Y}$ contains a non empty open set. We prove furthermore that **res** is proper, so its image is closed in $\mathcal{Y}$. Since $\mathcal{Y}$ is irreducible, we conclude that **res** is surjective, which proves the existence of the sought-after extension of $\mathcal{N}$.

As a by-product of the tools developed to prove Theorem 1, we show furthermore the following rigidity result refining [**Tey17**, Th 3]. In a sense, it says that at a singular point of a divisor, the existence of a non trivial Stokes structure is an exceptional phenomenon

Theorem 2. — *Let $\mathcal{N}$ be a good unramified split meromorphic flat bundle in a neighbourhood of the origin in $\mathbb{C}^n$. If the pole locus of $\mathcal{N}$ has at least two components, and if $\mathcal{N}$ is very general, then $\mathcal{N}$ itself is the only germ of good meromorphic flat bundle formally isomorphic to $\mathcal{N}$ at 0.*

In this statement, very general means roughly that the residues of each regular constituent contributing to $\mathcal{N}$ lie away from a countable union of strict Zariski closed subsets in an affine space. The main idea to prove Theorem 2 is to show that under the genericity assumption, the moduli of Stokes torsors of $\mathcal{N}$ has dimension 0 and is connected. It is thus reduced to a point.

A last application of the tools developed to prove Theorem 1 deals with degenerations of irregular singularities. Let X be a smooth algebraic variety and let D be a germ of smooth divisor at $0 \in X$. Let $\mathcal{M}$ be a germ of meromorphic connection defined in a neighbourhood of D in X and with poles along D. Motivated by Dubrovin's conjecture and the study of Frobenius manifolds, Cotti, Dubrovin and Guzzetti [**CDG17**] studied how much information on the Stokes data of $\mathcal{M}$ can be retrieved from the restriction of $\mathcal{M}$ to a smooth curve C transverse to D and passing through 0.

Under the assumption that $\mathcal{M}_{\widehat{D}}$ splits as a direct sum of regular connections twisted by meromorphic functions $a_1, \ldots, a_n \in \mathcal{O}_X(*D)$ with simple poles along D, they

proved that *the Stokes data of the restriction $\mathcal{M}_{|C}$ determine in a bijective way the Stokes data of $\mathcal{M}$ in a small neighbourhood of* 0 *in* D. This is striking, since the numerators of the $a_i - a_j$ may vanish at 0, thus inducing a discontinuity at 0 in the configuration of the Stokes directions. Using different methods, this was reproved by Sabbah in [**Sab17b**, Th 1.4]. If X is a surface, we give a short conceptual proof of a stronger version of Cotti, Dubrovin and Guzzetti's injectivity theorem, which generalizes it in several directions: we don't make any assumption on the shape of $\mathcal{M}_{\widehat{D}}$, nor suppose that D is smooth, nor assume that C is transverse to D. The price to pay for this generality is the use of resolution of turning points in dimension 2, as proved in the fundamental work of Kedlaya [**Ked10**] and Mochizuki [**Moc09**]. The intuition that the techniques developed in this paper could be applied to the questions considered by Cotti, Dubrovin and Guzzetti is due to C. Sabbah.

To state our result, let us recall that a *$\mathcal{M}$-marked connection* is the data of a couple (M, iso) where M is a germ of meromorphic connection with poles along D defined in a neighbourhood of D in X, and where $\mathrm{iso} : M_{\widehat{D}} \longrightarrow \mathcal{M}_{\widehat{D}}$ is an isomorphism of formal connections.

Theorem 3. — *Let X be a smooth algebraic surface, let $0 \in X$ and let D be a divisor defined in a neighbourhood of* 0. *Let $\mathcal{M}$ be a germ of meromorphic connection at* 0 *and with poles along D. Let C be a smooth curve passing through* 0 *and not contained in any of the irreducible components of D. If (M_1, iso_1) and (M_2, iso_2) are $\mathcal{M}$-marked connections such that*

$$(M_1, \mathrm{iso}_1)_{|C} \simeq (M_2, \mathrm{iso}_2)_{|C}$$

then (M_1, iso_1) and (M_2, iso_2) are isomorphic in a neighbourhood of 0.

Let us give an outline of the paper. In section 1, we recall the *Level filtration* for the Stokes sheaf in any dimension. We then apply it to prove Theorem 2. In section 2, we introduce the global variant of the moduli of Stokes torsors constructed in [**Tey17**] suited for the proof of Theorem 1. We then prove Theorem 3. In section 3, we show how to reduce the proof of Theorem 1 to the dimension 2 case. We then show in dimension 2 that Theorem 1 reduces to extending the good formal model of $\mathcal{M}$ across the point 0 under study. In the last section, we show that the sought-after extension exists provided the moduli of Stokes torsors associated to a resolution of the turning point 0 for $\mathcal{M}$ satisfies suitable geometric conditions. Finally, we show that these geometric conditions are always satisfied when the hypothesis of Theorem 1 are satisfied, thus concluding the proof of Theorem 1.

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1. Level filtration and application

We first introduce some notations and recall some definitions. A reference for good meromorphic flat bundles is Part I, Chapter 2 from [**Moc11b**]. For basics concerning Stokes torsors in any dimension, we refer to [**Tey17**].

1.1. Irregular values and truncation. — Let D be the germ of normal crossing divisor at $0 \in \mathbb{C}^n$ given by $z_1 \cdots z_m = 0$. We endow $\mathbb{Z}^m$ with the order given by $\mathbf{m} \leqslant \mathbf{m}'$ if and only if $m_i \leqslant m_i'$ for every $i = 1, \dots m$. For $a \in \mathcal{O}_{\mathbb{C}^n}(*D)/\mathcal{O}_{\mathbb{C}^n}$, we write $a = \sum_{\mathbf{m}\in\mathbb{Z}^m} a_{\mathbf{m}} z^{\mathbf{m}}$ and denote by $\operatorname{ord} a$ the minimum of

$$\{\mathbf{m} \in \mathbb{Z}^m_{\leqslant 0} \text{ such that } a_{\mathbf{m}} \neq 0\}$$

when it exists.

Let $\mathcal{I}$ be a good set of irregular values with poles contained in D. By definition, $\mathcal{I}$ is a subset of $\mathcal{O}_{\mathbb{C}^n}(*D)/\mathcal{O}_{\mathbb{C}^n}$ such that

– For every non zero $a \in \mathcal{I}$, $\operatorname{ord} a$ exists and $a_{\operatorname{ord} a}$ is invertible in a neighbourhood of 0.

– For every distinct $a, b \in \mathcal{I}$, $\operatorname{ord} a - b$ exists and $(a-b)_{\operatorname{ord} a-b}$ is invertible in a neighbourhood of 0.

– The set $\Phi(\mathcal{I}) := \{\operatorname{ord} a - b, a, b \in \mathcal{I} \text{ distinct}\}$ is totally ordered.

The elements of $\Phi(\mathcal{I})$ are the *levels of* $\mathcal{I}$. In particular, the set $\{\operatorname{ord} a, a \in \mathcal{I}\}$ is totally ordered. Let $\mathbf{m}(0)$ be its minimum. Let $(\mathbf{m}(0), \dots, \mathbf{m}(L), \mathbf{m}(L+1))$ be an auxiliary sequence for $\mathcal{I}$. This means that $\mathbf{m}(i+1) = \mathbf{m}(i) + (0, \dots, 1, \dots, 0)$ with 1 located in position $h_i \leqslant m$, that $\Phi(\mathcal{I}) \subset \{\mathbf{m}(0), \dots, \mathbf{m}(L+1)\}$ and that $\mathbf{m}(L+1) = 0$ by convention. We set for every $a \in \mathcal{I}$ and every $i = 0, \dots, L+1$,

$$\xi_{\mathbf{m}(i)}(a) := \sum_{\mathbf{n} \not\geqslant \mathbf{m}(i)} a_{\mathbf{n}} z^{\mathbf{n}}$$

and $a_{\geqslant \mathbf{m}(i)} := a - \xi_{\mathbf{m}(i)}(a)$.

1.2. Real blow-up. — Let $p : \widetilde{X} \longrightarrow \mathbb{C}^n$ be the fiber product of the real blow-ups of $\mathbb{C}^n$ along the $z_i = 0$, $i = 1, \dots, m$. We have

$$\widetilde{X} \simeq ([0, +\infty[\times S^1)^m \times \mathbb{C}^{n-m}$$

and p reads

$$((r_k, \theta_k)_k, y) \longrightarrow ((r_k e^{i\theta_k})_k, y)$$

In particular, $\mathbb{T} := p^{-1}(0)$ is a torus. Let $\pi : \mathbb{R}^m \longrightarrow \mathbb{T}$ be the canonical projection.

1.3. Good unramified split bundle. — For every $a \in \mathcal{I}$, set $\mathcal{E}^a = (\mathcal{O}_{\mathbb{C}^n,0}(*D), d - da)$. We fix once for all a germ of split unramified good meromorphic flat bundle of rank r with poles along D

$$\mathcal{N} := \bigoplus_{a\in\mathcal{I}} \mathcal{E}^a \otimes \mathcal{R}_a$$

where the $\mathcal{R}_a$ are regular. Let $i_a : \mathcal{E}^a \otimes \mathcal{R}_a \longrightarrow \mathcal{N}$ be the canonical inclusion and $p_a : \mathcal{N} \longrightarrow \mathcal{E}^a \otimes \mathcal{R}_a$ the canonical projection. For $i = 0, \dots, L+1$, we set $\mathcal{I}(i) := \xi_{\mathbf{m}(i)}(\mathcal{I})$ and

$$\mathcal{N}(i) := \bigoplus_{a\in\mathcal{I}} \mathcal{E}^{\xi_{\mathbf{m}(i)}(a)} \otimes \mathcal{R}_a$$

The levels of $\mathcal{N}(i)$ belong to $\{\mathbf{m}(0), \dots, \mathbf{m}(i-1)\}$. For $\alpha \in \mathcal{I}(i)$, we set

$$\mathcal{N}_\alpha := \bigoplus_{a\in\mathcal{I}, \xi_{\mathbf{m}(i)}(a)=\alpha} \mathcal{E}^a \otimes \mathcal{R}_a$$

The levels of $\mathcal{N}_\alpha$ belong to $\{\mathbf{m}(i), \dots, \mathbf{m}(L+1)\}$.

1.4. The Stokes sheaf. — Let $\mathrm{St}_\mathcal{N}$ be the Stokes sheaf of $\mathcal{N}$. This is a sheaf of complex unipotent algebraic groups over $\mathbb{T}$. By definition, the germs of $\mathrm{St}_\mathcal{N}$ at $\theta \in \mathbb{T}$ are the automorphisms of $\mathcal{N}$ defined on small sectors emanating from 0 containing the direction θ and asymptotic to id at 0 along the direction θ. For a formal definition, let us refer to [**Tey17**, 1.4].

1.5. The level filtration. — We recall the definition of the level filtration on the Stokes sheaf $\mathrm{St}_\mathcal{N}$ of a good unramified split bundle $\mathcal{N}$ as in 1.3. It is a straightforward generalization of [**BV89**, II 3.2.1]. We include it for the reader's convenience due to a lack of reference in the higher dimensional case. For $i = 0, \dots, L+1$, let us set

$$\mathrm{St}^i_\mathcal{N} := \{g \in \mathrm{St}_\mathcal{N} \,|\, e^a(g - \mathrm{id}) \text{ has rapid decay for every } a \text{ with } \operatorname{ord} a > \mathbf{m}(i-1)\}$$

The sheaf $\mathrm{St}^i_\mathcal{N}$ is a sheaf of normal algebraic subgroups of $\mathrm{St}_\mathcal{N}$. Let us define two diagonal matrices $M := \mathrm{Diag}(e^a, i \in \mathcal{I})$ and $M_{\geqslant} := \mathrm{Diag}(e^{a_{\geqslant \mathbf{m}(i)}}, i \in \mathcal{I})$. The sheaf $\mathrm{St}^i_\mathcal{N}$ admits the following Stokes theoretic description:

Lemma 1.5.1. — *The map*

$$\begin{array}{rcl} \varphi : \mathrm{St}_{\mathcal{N}(i)} & \longrightarrow & \mathrm{St}_\mathcal{N} \\ s & \longrightarrow & e^{M_{\geqslant}} s e^{-M_{\geqslant}} \end{array}$$

induces an isomorphism between $\mathrm{St}_{\mathcal{N}(i)}$ *and* $\mathrm{St}^i_\mathcal{N}$.

Proof. — The statement is local. Hence, it is enough to work on an open set $\mathcal{S}$ contained in a product of strict open intervals. For such an open, a choice of fundamental matrix F of flat sections for $\bigoplus_{a\in\mathcal{I}} \mathcal{R}_a$ yields a commutative diagram with injective arrows

$$\begin{array}{ccc} \Gamma(\mathcal{S}, \mathrm{St}_{\mathcal{N}(i)}) & \longrightarrow & \Gamma(\mathcal{S}, \mathrm{St}_\mathcal{N}) \\ & \searrow & \downarrow \iota \\ & & \mathrm{GL}_r \end{array} \tag{1.5.2}$$

where ι is given by $s \longrightarrow e^{-M}F^{-1}sFe^{M}$. By definition, $\iota(\Gamma(\mathcal{S},\mathrm{St}_{\mathcal{N}}))$ is the subgroup of elements $g \in \mathrm{GL}_r$ such that for every $a,b \in \mathcal{I}$,

$$\begin{cases} g_{aa} = \mathrm{id} \\ g_{ab} = 0 \qquad \text{if } a \neq b \text{ and } a \not<_{\mathcal{S}} b \end{cases}$$

Throughout the diagonal arrow of (1.5.2), the group $\Gamma(\mathcal{S},\mathrm{St}_{\mathcal{N}(i)})$ identifies with the subgroup of elements of $g \in GL_r$ such that for every $a,b \in \mathcal{I}$,

$$\begin{cases} g_{aa} = \mathrm{id} \\ g_{ab} = 0 \qquad \text{if } \xi_{\mathbf{m}(i)}(a) \neq \xi_{\mathbf{m}(i)}(b) \text{ and } \xi_{\mathbf{m}(i)}(a) \not<_{\mathcal{S}} \xi_{\mathbf{m}(i)}(b) \\ g_{ab} = 0 \qquad \text{if } a \neq b \text{ and } \xi_{\mathbf{m}(i)}(a) = \xi_{\mathbf{m}(i)}(b) \end{cases}$$

Note that if $a,b \in \mathcal{I}$ with $\xi_{\mathbf{m}(i)}(a) \neq \xi_{\mathbf{m}(i)}(b)$, then

$$a \not<_{\mathcal{S}} b \text{ if and only if } \xi_{\mathbf{m}(i)}(a) \not<_{\mathcal{S}} \xi_{\mathbf{m}(i)}(b)$$

Hence, $\Gamma(\mathcal{S},\mathrm{St}_{\mathcal{N}(i)})$ identifies with the subgroup of elements $g \in \iota(\Gamma(\mathcal{S},\mathrm{St}_{\mathcal{N}}))$ such that for every $a,b \in \mathcal{I}$,

$$g_{ab} = 0 \text{ if } \xi_{\mathbf{m}(i)}(a) = \xi_{\mathbf{m}(i)}(b) \text{ and } a \neq b$$

Let $s \in \Gamma(\mathcal{S},\mathrm{St}^i_{\mathcal{N}})$, and let $a,b \in \mathcal{I}$ with $a \neq b$. If $\xi_{\mathbf{m}(i)}(a) = \xi_{\mathbf{m}(i)}(b)$, then

$$\iota(s)_{ab} = e^{b-a}F_a^{-1}s_{ab}F_b = F_a^{-1}(e^{b_{\geqslant \mathbf{m}(i)}-a_{\geqslant \mathbf{m}(i)}}s_{ab})F_b$$

By definition, $e^{b_{\geqslant \mathbf{m}(i)}-a_{\geqslant \mathbf{m}(i)}}s_{ab}$ has rapid decay. Since F_a and F_b have moderate growth at 0, we deduce that the constant matrix $\iota(s)_{ab}$ has rapid decay. Hence, $\iota(s)_{ab} = 0$. Thus $\mathrm{St}^i_{\mathcal{N}} \subset \varphi(\mathrm{St}_{\mathcal{N}(i)})$. On the other hand, let $s \in \varphi(\mathrm{St}_{\mathcal{N}(i)})$ and let c with $\operatorname{ord} c > \mathbf{m}(i-1)$. We have to show that for every $a,b \in \mathcal{I}$ with $a \neq b$,

$$e^{c}s_{ab} = e^{c+a-b}F_a\varphi(s)_{ab}F_b^{-1}$$

has rapid decay. We can suppose $\xi_{\mathbf{m}(i)}(a) <_{\mathcal{S}} \xi_{\mathbf{m}(i)}(b)$. In particular $a <_{\mathcal{S}} b$. Since the leading term of $c+a-b$ is the leading term of $a-b$, the exponential e^{c+a-b} has rapid decay on $\mathcal{S}$. Thus, so does $e^{c}s_{ab}$. Hence, $s \in \mathrm{St}^i_{\mathcal{N}}$ and we deduce $\mathrm{St}^i_{\mathcal{N}} = \varphi(\mathrm{St}_{\mathcal{N}(i)})$. □

1.6. Quotients of the level filtration. —

Lemma 1.6.1. — *There is a split exact sequence of sheaves of algebraic groups*

$$1 \longrightarrow \mathrm{St}_{\mathcal{N}(i)} \xrightarrow{\varphi} \mathrm{St}_{\mathcal{N}} \xrightarrow{\psi} \prod\nolimits_{\alpha\in\mathcal{I}(i)} \mathrm{St}_{\mathcal{N}_\alpha} \longrightarrow 1$$

In particular, $\mathrm{Gr}^i\,\mathrm{St}_{\mathcal{N}} := \mathrm{St}^{i+1}_{\mathcal{N}} / \mathrm{St}^i_{\mathcal{N}} \simeq \prod_{\alpha\in\mathcal{I}(i)} \mathrm{St}_{\mathcal{N}(i+1)_\alpha}$

Proof. — Let us define

$$\begin{array}{rccl} \psi : \mathrm{St}_{\mathcal{N}} & \longrightarrow & \prod_{\alpha\in\mathcal{I}(i)} \mathrm{St}_{\mathcal{N}_\alpha} \\ s & \longrightarrow & \times (s_{ab})_{\substack{\xi_{\mathbf{m}(i)}(a)=\alpha \\ \xi_{\mathbf{m}(i)}(b)=\alpha}} \end{array}$$

From the local description of $\mathrm{St}_{\mathcal{N}(i)}$ given in the proof of 1.5.1, we see that the only a priori non obvious thing to prove is the fact that ψ is a group homomorphism. Let $\mathcal{S}$ be an open of $\mathbb{T}$, let $s,t \in \Gamma(\mathcal{S},\mathrm{St}_{\mathcal{N}})$, let $\alpha \in \mathcal{I}(i)$ and let $a,b \in \mathcal{I}$ such that

$\xi_{\mathbf{m}(i)}(a) = \xi_{\mathbf{m}(i)}(b) = \alpha$. Let us denote by ψ_α the component of ψ associated to α. Then

$$(\psi_\alpha(st))_{ab} = \sum_{c\in\mathcal{I}} s_{ac}t_{cb} = \sum_{\substack{c\in\mathcal{I}\\ a\leqslant_{\mathcal{S}} c\leqslant_{\mathcal{S}} b}} s_{ac}t_{cb}$$

If $c_{\leqslant\mathbf{m}} \neq \alpha$, the leading coefficient of $c-a$ is that of $\xi_{\mathbf{m}(i)}(c)-\xi_{\mathbf{m}(i)}(a) = \xi_{\mathbf{m}(i)}(c)-\alpha$. Hence, $a \leqslant_{\mathcal{S}} c$ if and only if $\alpha <_{\mathcal{S}} \xi_{\mathbf{m}(i)}(c)$. Similarly, $c \leqslant_{\mathcal{S}} b$ if and only if $\xi_{\mathbf{m}(i)}(c) <_{\mathcal{S}} \alpha$. Hence, for $\xi_{\mathbf{m}(i)}(c) \neq \alpha$, the condition $a \leqslant_{\mathcal{S}} c \leqslant_{\mathcal{S}} b$ is empty. Thus

$$(\psi_\sigma(st))_{ab} = \sum_{\substack{c\in\mathcal{I}\\ \xi_{\mathbf{m}(i)}(c)=\alpha}} s_{ac}t_{cb} = (\psi_\alpha(s)\psi_\alpha(t))_{ab}$$

□

1.7. Action of the fundamental group in the one level case. — We consider in this paragraph the case where $\mathcal{N}$ has a unique level $\mathbf{m}$ and we fix a smooth curve C passing through 0 as in 1.2. We denote by $S^1_C \subset \mathbb{T}$ the circle of directions in C emanating from 0. For an hyperplane H of $\mathbb{R}^n$ and for an interval I of S^1_C, set $\mathbb{T}(H, I) := \pi(H + \pi^{-1}(I))$. For $\mathbf{m} \in \mathbb{Z}^m_{\leqslant 0}$, we set

$$\mathbb{T}(\mathbf{m}, I) := \mathbb{T}(\sum_i \mathbf{m}_i x_i = 0, I)$$

For every $x \in \mathbb{T}$, the translation t_x by x provides an isomorphism $\pi_1(\mathbb{T}(\mathbf{m}, 0), 0) \longrightarrow \pi_1(x + \mathbb{T}(\mathbf{m}, 0), x)$. Hence, $\pi_1(\mathbb{T}(\mathbf{m}, 0), 0)$ acts on $\mathrm{St}_{\mathcal{N}_C} \simeq (\mathrm{St}_{\mathcal{M}})_{|S^1_C}$ via the parallel transport. We deduce that $\pi_1(\mathbb{T}(\mathbf{m}, 0), 0)$ acts on $H^1(S^1_C, \mathrm{St}_{\mathcal{N}_C})$. To simplify notations, we will denote by $H^1(S^1_C, \mathrm{St}_{\mathcal{N}_C})^{\pi_1}$ the set of invariants of the action of $\pi_1(\mathbb{T}(\mathbf{m}, 0), 0)$ on $H^1(S^1_C, \mathrm{St}_{\mathcal{N}_C})$.

For a connected open $\mathcal{S} \subset \mathbb{T}$, the path γ acts on $\Gamma(\mathcal{S}, \mathcal{L}(\mathcal{N}))$ via a linear map $\rho(\gamma)$. The induced action on $\Gamma(\mathcal{S}, \mathcal{L}(\mathrm{End}\,\mathcal{N}))$ is the conjugation by $\rho(\gamma)$.

Lemma 1.7.1. — *Let C be a smooth curve passing through the origin. For every cover $\underline{I} = (I_i)_{i\in\mathbb{Z}/N\mathbb{Z}}$ of S^1_C adapted to $\mathcal{N}_C$, the morphisms in the commutative triangle*

$$\begin{array}{ccc} H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}}) & \longrightarrow & H^1(S^1_C, \mathrm{St}_{\mathcal{N}_C})^{\pi_1} \\ & \nwarrow \quad \nearrow & \\ & Z^1(\mathbb{T}(\mathbf{m}, \underline{I}), \mathrm{St}_{\mathcal{N}}) & \end{array}$$

are isomorphisms. In particular, $H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}})$ is an affine space.

For a single level connection in dimension 1, an *adapted cover* is a cover by consecutive intervals with empty triple intersections such that every $I \in \underline{I}$ and every pair of irregular values a, b, the interval I contains exactly one Stokes direction associated to $a - b$.

Proof. — Since $\mathcal{N}$ has only one level, the same holds for $\mathcal{N}_C$. Let $\mathcal{T} \in H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}})$. From **[BV89]**, the restriction $\mathcal{T}_{S^1_C} \in H^1(S^1_C, \mathrm{St}_{\mathcal{N}_C})$ of $\mathcal{T}$ to S^1_C admits a unique trivialisation t_i on each I_i, $i \in \mathbb{Z}/N\mathbb{Z}$ and

$$Z^1(\underline{I}, \mathrm{St}_{\mathcal{N}_C}) \xrightarrow{\sim} H^1(S^1_C, \mathrm{St}_{\mathcal{N}_C}) \tag{1.7.2}$$

Since $\mathrm{St}_{\mathcal{N}}$ has no non trivial global section on $\mathbb{T}(\mathbf{m}, I_i)$, the section t_i extends uniquely into a section of $\mathcal{T}$ on $\mathbb{T}(\mathbf{m}, I_i)$. Hence, the cocycle corresponding to $\mathcal{T}_{S^1_C}$ via (1.7.2) extends uniquely into a cocycle for $\mathcal{T}$ relative to the cover $\mathbb{T}(\mathbf{m}, \underline{I})$. In particular, the upper horizontal arrow of the diagram

$$\begin{array}{ccc} Z^1(\mathbb{T}(\mathbf{m}, \underline{I}), \mathrm{St}_{\mathcal{N}}) & \longrightarrow & H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}}) \\ \downarrow & & \downarrow \mathbf{res}_C \\ Z^1(\underline{I}, \mathrm{St}_{\mathcal{N}_C}) & \xrightarrow{\sim} & H^1(S^1_C, \mathrm{St}_{\mathcal{N}_C}) \end{array} \tag{1.7.3}$$

is surjective. Every section of $\mathrm{St}_{\mathcal{N}}$ on a connected open is determined by its germ at a point. Hence, the left vertical arrow of (1.7.3) is injective. We deduce that the upper horizontal arrow of (1.7.3) is a bijection and that $\mathbf{res}_C$ is injective. Tautologically, the image of the left vertical arrow is exactly formed by those collections of $g \in Z^1(\underline{I}, \mathrm{St}_{\mathcal{N}_C})$ extending to $\mathbb{T}(\mathbf{m}, \underline{I})$. These are exactly the invariants under the action of $\pi_1(\mathbb{T}(\mathbf{m}, 0), 0)$ constructed in 1.7.

To conclude, we observe that $\mathrm{St}_{\mathcal{N}}$ being a sheaf of unipotent algebraic groups, the scheme $\Gamma(U, \mathrm{St}_{\mathcal{N}})$ is an affine space for every open subset $U \subset \mathbb{T}$. Since $Z^1(\mathbb{T}(\mathbf{m}, \underline{I}), \mathrm{St}_{\mathcal{N}})$ is a product of such schemes, it is also an affine space. □

1.8. Proof of Theorem 2. — We are now in position to prove Theorem 2. This is a local statement, so we work in a neighbourhood of 0 in $\mathbb{C}^n$ and use notations from section 1. We argue recursively on the number of levels of $\mathcal{N}$. Suppose that $\mathcal{N}$ has only one level. From 1.7.1, we know that $H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}})$ is an affine space. From **[Tey17**, Th 3], we know that $H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}})$ has dimension 0. Hence, $H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}})$ is a point, so Theorem 2 is proved in the single level case. Suppose that $\mathcal{N}$ has at least two levels. Let $(\mathbf{m}(0), \ldots, \mathbf{m}(L), \mathbf{m}(L+1))$ be an auxiliary sequence for $\mathcal{I}$. Then, there is an index i such that $\mathcal{N}(i)$ has only one level and such that the number of levels of $\mathrm{St}_{\mathcal{N}_\alpha}$ is strictly less than number of levels of $\mathcal{N}$ for every $\alpha \in \mathcal{I}(i)$. Since the $\mathcal{N}_\alpha$ are direct summands of $\mathcal{N}$, they are also very generic. By recursion hypothesis applied to the $\mathcal{N}_\alpha$, we obtain that the right term of the exact sequence of pointed sets

$$H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}(i)}) \longrightarrow H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}}) \longrightarrow \prod\nolimits_{\alpha \in \mathcal{I}(i)} H^1(\mathbb{T}, \mathcal{N}_\alpha)$$

deduced from 1.6.1 is trivial. Hence $H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}}) \simeq H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}(i)})$. Since $\mathcal{N}$ is very generic, so is $\mathcal{N}(i)$. Since $\mathcal{N}(i)$ has only one level, $H^1(\mathbb{T}, \mathrm{St}_{\mathcal{N}(i)})$ is a point. This finishes the proof of Theorem 2.

2. Moduli of Stokes torsors. Global aspects

2.1. Why moduli of Stokes torsors?— Let us explain in this subsection how the moduli of Stokes torsors were found to be relevant to the proof of Theorem 1. We use the notations from the introduction and work in dimension 2. We suppose that $0 \in D$ lies in the smooth locus of $(\operatorname{Sol}\mathcal{M})_{|D}$ and $(\operatorname{Sol}\operatorname{End}\mathcal{M})_{|D}$, and we want to prove that 0 is a good formal decomposition point for $\mathcal{M}$.

From a theorem of Kedlaya [**Ked10**][**Ked11**] and Mochizuki [**Moc09**][**Moc11b**], our connection $\mathcal{M}$ acquires good formal decomposition at any point after pulling-back by a suitable sequence of blow-ups above D. To test the validity of the conjecture [**Tey13**, 15.0.5], a natural case to consider was the case where only one blow-up is needed. Using results of André [**And07**], it was shown in [**Tey14**] that the conjecture reduces in this case to the following

Question. — *Given two good meromorphic connections $\mathcal{M}$ and $\mathcal{N}$ with poles along the coordinate axis in $\mathbb{C}^2$ and formally isomorphic at 0, is it true that*

$$\dim(\mathcal{H}^1 \operatorname{Sol}\operatorname{End}\mathcal{M})_0 = \dim(\mathcal{H}^1 \operatorname{Sol}\operatorname{End}\mathcal{N})_0 \quad ? \tag{2.1.1}$$

It turns out that each side of (2.1.1) appeared as dimensions of moduli spaces of Stokes torsors constructed by Babbitt-Varadarajan in [**BV89**]. These moduli were associated with germs of meromorphic connections in dimension 1. Babbitt and Varadarajan proved that they are affine spaces. This suggested the existence of a moduli $\mathcal{X}$ with two points $P, Q \in \mathcal{X}$ such that the left-hand side of (2.1.1) would be $\dim T_P\mathcal{X}$ and the right-hand side of (2.1.1) would be $\dim T_Q\mathcal{X}$. The equality (2.1.1) would then follow from the smoothness and connectedness of the putative moduli. This is what led to [**Tey17**], but the question of smoothness and connectedness was left open. In the meantime, a positive answer to the above question was given by purely analytic means by C. Sabbah in [**Sab17a**].

2.2. Relation with [Tey17]. — In [**Tey17**], a moduli for local Stokes torsors was constructed in any dimension. This moduli suffers two drawbacks in view of the proof of Theorem 1. First, the Stokes sheaf used in [**Tey17**] only makes sense at a neigbourhood of a point, whereas our situation will be global as soon as we apply Kedlaya-Mochizuki's resolution of turning points. Second, the relation between Irregularity and the tangent spaces of the moduli from [**Tey17**] only holds in particular cases. To convert the hypothesis on Irregularity appearing in Theorem 1 into a geometric statement on a moduli of torsors, we need to replace the Stokes sheaf $\mathrm{St}_{\mathcal{M}}$ of a connection $\mathcal{M}$ by a subsheaf denoted by $\mathrm{St}^{<D}_{\mathcal{M}}$. We will abuse terminology be also calling the torsors under $\mathrm{St}^{<D}_{\mathcal{M}}$ Stokes torsors. The sheaf $\mathrm{St}^{<D}_{\mathcal{M}}$ has the advantage of being globally defined when $\mathcal{M}$ is globally defined. Along the smooth locus of D, the sheaf $\mathrm{St}^{<D}_{\mathcal{M}}$ is the usual Stokes sheaf. The only difference between $\mathrm{St}_{\mathcal{M}}$ and $\mathrm{St}^{<D}_{\mathcal{M}}$ appears at a singular point of D.

Note that the only global moduli of Stokes torsors needed in this paper come from the case where X is a surface. Hence, this case is of independent interest regarding the general theory and thus deserves a special treatment. To keep the level of technicality as low as possible, we will thus stick to the case of surfaces. The general case

will appear in a subsequent work, along with applications of different nature than the one we aim at in the present paper.

2.3. Geometric setup. — In this section, D denotes a normal crossing divisor in a smooth algebraic surface X. Let $D_1, \dots, D_n$ be the irreducible components of D. For every sheaf of $\mathcal{O}_X$-module $\mathcal{F}$, we set

$$\mathcal{F}_{\widehat{D}} = \mathcal{O}_{\widehat{X|D}} \otimes_{\mathcal{O}_X} \mathcal{F}$$

Let D_i° be the complement in D_i of the singular locus $\mathrm{Sing}(D)$ of D. Let $p : \widetilde{X} \longrightarrow X$ be the real blow-up of X along D. For every subset $A \subset X$, we set $\partial A := p^{-1}(A) \cap p^{-1}(D)$ and denote by $\iota_A : \partial A \longrightarrow \partial D$ the canonical inclusion. We denote by $\mathcal{A}$ the sheaf of functions on ∂D admitting an asymptotic development along D [**Sab00**]. We denote by $\mathcal{A}^{<D} \subset \mathcal{A}$ the sheaf of functions on ∂D with rapid decay along D. Concretely, this means the following. Let (x_1, x_2) be local coordinates such that D is defined locally by $x_1 x_i = 0$ with $i \in \{1, 2\}$. Then, the germ of $\mathcal{A}^{<D}$ at $\theta \in \partial 0$ is given by those holomorphic functions u defined over the trace on $X \backslash D$ of a neighbourhood Ω of θ in $\widetilde{X}$, and such that for every compact $K \subset \Omega$, for every $N_1, N_i \in \mathbb{N}$, there exists a constant $C_{N_1,N_i} > 0$ such that

$$|u(x)| \leqslant C_{N_1,N_i} |x_1|^{N_1} |x_i|^{N_i} \text{ for any } x \in K \cap (X \backslash D)$$

2.4. Definition of the moduli. — Let $\mathcal{M}$ be a good meromorphic connection defined in a neighbourhood of D with poles along D. We set

$$\partial\mathcal{M} = \mathcal{A} \otimes_{p^{-1}\mathcal{O}_X} p^{-1}\mathcal{M}$$

We define $\mathrm{St}_{\mathcal{M}}^{<D}$ as the subsheaf of $\mathcal{H}^0 \,\mathrm{DR}\, \partial\, \mathrm{End}\, \mathcal{M}$ of sections asymptotic to the Identity along D, that is of the form $\mathrm{Id} + f$ where f has coefficients in $\mathcal{A}^{<D}$.

The sheaf $\mathrm{St}_{\mathcal{M}}^{<D}$ is a sheaf of complex unipotent algebraic groups. In particular, $\mathrm{St}_{\mathcal{M}}^{<D}(R)$ is defined as a sheaf of groups on ∂D for every $R \in \mathbb{C}$-alg. For every subset $A \subset D$, we denote by $H^1(\partial A, \mathrm{St}_{\mathcal{M}}^{<D})$ the functor

$$\begin{array}{ccc} \mathbb{C}\text{-alg} & \longrightarrow & \mathrm{Set} \\ R & \longrightarrow & H^1(\partial A, \mathrm{St}_{\mathcal{M}}^{<D}(R)) \end{array}$$

2.5. Representability by a scheme. — The purpose of this subsection is to prove that $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ is representable by an affine scheme of finite type over $\mathbb{C}$. To do this, the idea is to analyse separately the contributions coming from each stratum of D. On the smooth locus of D, representability will essentially be a consequence of Babbitt-Varadarajan's works [**BV89**]. At a singular point P of D, representability will be achieved by comparison with the situation on a well-chosen component passing through P.

Let $P \in \mathrm{Sing}(D)$ and let D_i be a component of D containing P. Then, there exists a disc $\Delta_{D_i,P} \subset D_i$ centred at P such that any $\iota_P^{-1}\, \mathrm{St}_{\mathcal{M}}^{<D}$-torsor extends above $\Delta_{D_i,P}$.

Set $\Delta^*_{D_i,P} = \Delta_{D_i,P}\backslash\{P\}$. Let $j_{i,P} : \partial\Delta^*_{D_i,P} \longrightarrow \partial D$ be the canonical inclusion. Hence, there is a canonical morphism of functors

$$H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial\Delta^*_{D_i,P}, \mathrm{St}^{<D}_{\mathcal{M}}) \tag{2.5.1}$$

On the other hand, restriction of torsors provides a morphism of functors

$$H^1(\partial D_i^\circ, \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial\Delta^*_{D_i,P}, \mathrm{St}^{<D}_{\mathcal{M}}) \tag{2.5.2}$$

The collection of morphisms (2.5.1) and (2.5.2) defines a finite diagram of functors. Since Stokes torsors have no non trivial automorphisms [**Tey17**, 1.8.1], the limit of this diagram is $H^1(\partial D, \mathrm{St}^{<D}_{\mathcal{M}})$. In particular, to understand $H^1(\partial D, \mathrm{St}^{<D}_{\mathcal{M}})$ amounts to understand what happens at a singular point of D and what happens on the smooth locus.

Lemma 2.5.3. — *For every $i = 1, \ldots, n$, the functor $H^1(\partial D_i^\circ, \mathrm{St}^{<D}_{\mathcal{M}})$ is a scheme of finite type over $\mathbb{C}$. The restriction morphism* (2.5.2) *is a closed immersion.*

Proof. — Let $R \in \mathbb{C}$-alg. Since Stokes torsors have no non trivial automorphisms, the relative non abelian cohomology functor

$$\begin{array}{rcl} R^1p_*\,\mathrm{St}^{<D}_{\mathcal{M}}(R) : \mathrm{Open}(D) & \longrightarrow & \mathrm{Set} \\ U & \longrightarrow & H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}}(R)) \end{array}$$

is a sheaf of sets on D. From [**Sab02**, II 6.1] (see also [**Mal83a**] for the one level case), the restriction of $R^1p_*\,\mathrm{St}^{<D}_{\mathcal{M}}(R)$ to D_i° is a local system on D_i° whose stack at $P \in D_i^\circ$ is $H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}}(R))$. Hence, for a ball B in D_i°, for every connected open set $U \subset B$ and every $P \in U$, restriction induces an identification

$$\Gamma(U, R^1p_*\,\mathrm{St}^{<D}_{\mathcal{M}}(R)) \xrightarrow{\sim} H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}}(R))$$

functorial in R. From works of Babbit-Varadarajan [**BV89**], the functor $H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}})$ is an affine space. Hence, the restriction of $R^1p_*\,\mathrm{St}^{<D}_{\mathcal{M}}$ to D_i° is a local system of schemes in the sense of [**Sim94**]. Let $P \in D_i^\circ$. Since

$$H^1(\partial D_i^\circ, \mathrm{St}^{<D}_{\mathcal{M}}) = \Gamma(D_i^\circ, R^1p_*\,\mathrm{St}^{<D}_{\mathcal{M}})$$

the functor $H^1(\partial D_i^\circ, \mathrm{St}^{<D}_{\mathcal{M}})$ identifies with the invariants of the action of $\pi_1(D_i^\circ, P)$ on $H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}})$. Since this action is functorial in R, Yoneda lemma implies that $\pi_1(D_i^\circ, P)$ acts on the scheme $H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}})$ via algebraic maps. Hence, its invariants form a closed sub-scheme in $H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}})$. In particular, $H^1(\partial D_i^\circ, \mathrm{St}^{<D}_{\mathcal{M}})$ is a scheme.

The fact that (2.5.2) is a closed immersion is an immediate consequence of the fact that $H^1(\partial D_i^\circ, \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}})$ is a closed immersion.

□

Proposition 2.5.4. — *For every $P \in \mathrm{Sing}(D)$, the functor $H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}})$ is a scheme of finite type over $\mathbb{C}$.*

Proof. — If $\mathcal{M}$ has only one irregular value at P, then $H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D})$ is the trivial scheme so there is nothing to do. Suppose that $\mathcal{M}$ has at least two irregular values at P. Goodness implies that there is a component of D passing through P such that the difference of any two irregular values of $\mathcal{M}$ at P has poles along this component. Let D_i be the other component of D passing through P. We take local coordinates (x, y) such that D_i is given by $x = 0$. For every $\mathcal{T} \in H^1(\partial\Delta^*_{D_i,P}, \mathrm{St}_{\mathcal{M}}^{<D})$, the sheaf $\iota_P^{-1} j_{i,P*}\mathcal{T}$ is a $\iota_P^{-1} j_{i,P*}\,\mathrm{St}_{\mathcal{M}}^{<D}$-torsor on ∂P. So if we prove that the adjunction morphism

$$\iota_P^{-1}\,\mathrm{St}_{\mathcal{M}}^{<D} \longrightarrow \iota_P^{-1} j_{i,P*} j_{i,P}^{-1}\,\mathrm{St}_{\mathcal{M}}^{<D} \tag{2.5.5}$$

is an isomorphism, then $\iota_P^{-1} j_{i,P*}$ will provide us with an inverse for (2.5.1), and 2.5.4 will be a consequence of 2.5.3. We now prove that (2.5.5) is an isomorphism. By a standard Galois argument, we can suppose that $\mathcal{M}$ is unramified. Injectivity of (2.5.5) is obvious so we are left to prove surjectivity. Since this is a local statement on ∂P, Mochizuki's asymptotic development theorem [**Moc11b**, 3.2.10] reduces the question to the case where $\mathcal{M}$ is split unramified. We thus treat that case and borrow the notations from 1.3. We put

$$\mathcal{S} = ([0, r[\times I_1) \times ([0, r[\times I_2)$$

where I_1, I_2 are intervals. Sections of $\mathrm{St}_{\mathcal{M}}^{<D}$ on $\partial S := S \cap \partial D$ are automorphisms of $\mathcal{M}$ on $\mathcal{S} \cap (X\backslash D)$ of the form $\mathrm{Id} + f$ where $p_a f i_b = 0$ unless

$$e^{a-b} \in \Gamma(\partial S, \mathcal{A}^{<D}) \tag{2.5.6}$$

Sections of $\mathrm{St}_{\mathcal{M}}^{<D}$ over

$$\partial\mathcal{S}_i^{\circ} := \partial\mathcal{S} \cap \partial\Delta^*_{D_i,P} = (\{0\} \times I_1) \times (]0, r[\times I_2)$$

are automorphisms of $\mathcal{M}$ on $\mathcal{S} \cap (X\backslash D)$ of the form $\mathrm{Id} + f$ where $p_a f i_b = 0$ unless

$$e^{a-b} \in \Gamma(\partial\mathcal{S}_i^{\circ}, \mathcal{A}^{<D}) \tag{2.5.7}$$

We thus have to show that for every distinct irregular values a, b, the conditions (2.5.7) and (2.5.6) are equivalent for a small enough choice of $\mathcal{S}$. A change of variable reduces the problem to the case where $a - b = 1/x^\alpha y^\beta$ where $(\alpha, \beta) \in \mathbb{N} \times \mathbb{N}^*$. Since $\mathcal{A}^{<D}$ is a sheaf, condition (2.5.6) trivially implies condition (2.5.7). Suppose that $e^{1/x^\alpha y^\beta} \in \Gamma(\partial\mathcal{S}_i^{\circ}, \mathcal{A}^{<D})$. At the cost of shrinking $\mathcal{S}$, this means that for every $N \in \mathbb{N}$, every $\epsilon > 0$, there exists a constant $C > 0$ such that for

$$(x, y) \in (]0, r[\times\overline{I_1}) \times (]\epsilon, r[\times\overline{I_2})$$

we have

$$|e^{1/x^\alpha y^\beta}| \leqslant C|x|^N$$

Writing $x = (r_1, \theta_1)$ and $y = (r_2, \theta_2)$, this means

$$e^{\cos(\alpha\theta_1 + \beta\theta_2)/r_1^\alpha r_2^\beta} \leqslant C r_1^N$$

In particular, $\alpha > 0$ and $\cos(\alpha\theta_1 + \beta\theta_2) < 0$ for every $(\theta_1, \theta_2) \in \overline{I_1} \times \overline{I_2}$. Let $c > 0$ such that $\cos(\alpha\theta_1 + \beta\theta_2) < -c$ on $I_1 \times I_2$. Then, we have

$$|e^{1/x^\alpha y^\beta}| \leqslant e^{-c/|x|^\alpha |y|^\beta}$$

on $\mathcal{S}$. Since $\alpha > 0$ and since $\beta > 0$ from our choice of component D_i, we deduce that (2.5.7) holds, which proves the equivalence between conditions (2.5.7) and (2.5.6). □

Putting 2.5.3 and 2.5.4 together gives the following

Proposition 2.5.8. — *The functor $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ is an affine scheme of finite type over $\mathbb{C}$.*

We have the following

Proposition 2.5.9. — *For very $P \in \mathrm{Sing}(D)$ and every component D_i of D passing through P, the restriction morphism* (2.5.1) *is a closed immersion.*

Proof. — We can suppose that $\mathcal{M}$ is unramified in a neighbourhood of P. If not all the two by two differences of $\mathcal{M}$'s irregular values at P have poles along D_i, then the proof of 2.5.4 shows that (2.5.1) is an isomorphism, so 2.5.9 is true in that case. Let us suppose that the difference of any two distinct irregular values for $\mathcal{M}$ at P has poles along D_i. Let $\mathrm{St}_{\mathcal{M}_P}$ be the Stokes sheaf of $\mathcal{M}$ on ∂P as defined in [**Tey17**]. Then, $\mathrm{St}_{\mathcal{M}}^{<D}$ is distinguished in $\mathrm{St}_{\mathcal{M}_P}$. We thus have an exact sequence of sheaves of algebraic groups on ∂P

$$1 \longrightarrow \mathrm{St}_{\mathcal{M}}^{<D} \xrightarrow{\iota} \mathrm{St}_{\mathcal{M}_P} \longrightarrow \mathcal{Q} \longrightarrow 1$$

At the cost of restricting $\Delta^*_{D_i,P}$, any section of $\mathrm{St}_{\mathcal{M}_P}$ extends to $\partial\Delta^*_{D_i,P}$. We thus have an adjunction morphism

$$\mathrm{St}_{\mathcal{M}_P} \longrightarrow \iota_P^{-1} j_{i,P*} j_{i,P}^{-1} \mathrm{St}_{\mathcal{M}_P} = \iota_P^{-1} j_{i,P*} j_{i,P}^{-1} \mathrm{St}_{\mathcal{M}}^{<D} \tag{2.5.10}$$

Hence, there is a factorization

$$\begin{array}{ccc} H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) & \xrightarrow{\iota*} & H^1(\partial P, \mathrm{St}_{\mathcal{M}_P}) \\ & \searrow & \downarrow \wr \\ & & H^1(\partial\Delta^*_{D_i,P}, \mathrm{St}_{\mathcal{M}}^{<D}) \end{array} \tag{2.5.11}$$

From an argument similar to that in 2.5.4, the map (2.5.10) is an isomorphism of sheaves on ∂P. Hence, the vertical arrow in (2.5.11) is an isomorphism of schemes. To prove 2.5.9, it is enough to prove that

$$\iota_* : H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}_P})$$

is a closed immersion. From [**Fre57**, I.2], there is an exact sequence of pointed functors

$$H^0(\partial P, \mathcal{Q}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \xrightarrow{\iota*} H^1(\partial P, \mathrm{St}_{\mathcal{M}_P}) \longrightarrow H^1(\partial P, \mathcal{Q}) \tag{2.5.12}$$

The complex of sheaves

$$\mathrm{St}_{\mathcal{M}_P} \longrightarrow \partial \operatorname{End} \mathcal{M}_{\widehat{D}} \longrightarrow \partial \operatorname{End} \mathcal{M}_{\widehat{0}}$$

induces a sequence

$$\mathcal{Q} \hookrightarrow \partial \operatorname{End} \mathcal{M}_{\widehat{D}} \longrightarrow \partial \operatorname{End} \mathcal{M}_{\widehat{0}} \tag{2.5.13}$$

Taking global sections, we deduce via [**Sab00**, p 44] the following sequence

$$0 \longrightarrow H^0(\partial P, \mathcal{Q}) \longrightarrow \operatorname{End} \mathcal{M}_{\widehat{D},0} \longrightarrow \operatorname{End} \mathcal{M}_{\widehat{0}} \tag{2.5.14}$$

By flatness, the second map of (2.5.14) is injective. Hence, $H^0(\partial P, \mathcal{Q})$ is trivial. From the exactness of (2.5.12), we deduce that the following diagram of functors

$$\begin{array}{ccc} H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) & \longrightarrow & * \\ {\scriptstyle \iota_*}\downarrow & & \downarrow \\ H^1(\partial P, \mathrm{St}_{\mathcal{M}_P}) & \longrightarrow & H^1(\partial P, \mathcal{Q}) \end{array} \tag{2.5.15}$$

is cartesian, where $*$ denotes the trivial $\mathcal{Q}$-torsor. If we knew that $H^1(\partial P, \mathcal{Q})$ is a scheme, we would directly obtain that ι_* is a closed immersion. This question does not seem to follow from the use of skeletons [**Tey17**]. We will circumvent this problem with a group theoretic argument.

From [**Tey17**, 1.9.1], any cover $\mathcal{U}$ of ∂P by good open subsets induces a morphism of schemes

$$Z^1(\mathcal{U}, \mathrm{St}_{\mathcal{M}_P}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}_P}) \tag{2.5.16}$$

which is surjective at the level of R-points for every $R \in \mathbb{C}$-alg. From [**BV89**, 2.7.3], the morphism (2.5.16) admits a section. Composing this section with

$$Z^1(\mathcal{U}, \mathrm{St}_{\mathcal{M}_P}) \longrightarrow Z^1(\mathcal{U}, \mathcal{Q})$$

gives rise to a commutative triangle of functors

$$\begin{array}{ccc} H^1(\partial P, \mathrm{St}_{\mathcal{M}_P}) & \longrightarrow & H^1(\partial P, \mathcal{Q}) \\ & \searrow & \uparrow \\ & & Z^1(\mathcal{U}, \mathcal{Q}) \end{array}$$

The algebraic group

$$G := \prod\nolimits_{U \in \mathcal{U}} \Gamma(U, \mathcal{Q})$$

acts on $Z^1(\mathcal{U}, \mathcal{Q})$. Let

$$G \longrightarrow Z^1(\mathcal{U}, \mathcal{Q}) \tag{2.5.17}$$

be the morphism of schemes obtained by restricting the action of G to the trivial cocycle. Since $H^0(\partial P, \mathcal{Q}) \simeq 0$, the morphism (2.5.17) is a monomorphism. The

diagram (2.5.15) thus splits into cartesian diagrams of functors

$$\begin{array}{ccccc} H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) & \longrightarrow & G & \longrightarrow & * \\ {\scriptstyle \iota *}\downarrow & & \downarrow & & \downarrow \\ H^1(\partial P, \mathrm{St}_{\mathcal{M}_P}) & \longrightarrow & Z^1(\mathcal{U}, \mathcal{Q}) & \longrightarrow & H^1(\partial P, \mathcal{Q}) \end{array}$$

We are thus left to show that (2.5.17) is a closed immersion. From the general theory of algebraic group actions, (2.5.17) factors into

$$G \xrightarrow{\alpha} \mathcal{O} \xrightarrow{\beta} Z^1(\mathcal{U}, \mathcal{Q})$$

where α is faithfully flat, where $\mathcal{O}$ denotes the orbit of the trivial cocycle under G and where β is an immersion of schemes. Since smoothness is a local property for the fppf topology, smoothness of G implies that $\mathcal{O}$ is smooth. By definition, α is an isomorphism at the level of $\mathbb{C}$-points. Hence, α is an isomorphism of varieties. We are thus left to show that $\mathcal{O}$ is closed in $Z^1(\mathcal{U}, \mathcal{Q})$. It is enough to show that $\mathcal{O}$ is closed in $Z^1(\mathcal{U}, \mathcal{Q})^{\mathrm{red}}$. From Kostant-Rosenlicht theorem [**Bor91**, I 4.10], it is enough to show that G is a unipotent algebraic group.

Let $\mathcal{N}$ be the good split unramified bundle formally isomorphic to $\mathcal{M}$ at 0. Let us choose a formal isomorphism $\mathrm{iso} : \mathcal{M}_{\widehat{0}} \longrightarrow \mathcal{N}_{\widehat{0}}$. At the cost of refining $\mathcal{U}$, we can suppose that there exists an isomorphism $\theta_U : \partial\mathcal{M}_{|U} \longrightarrow \partial\mathcal{N}_{|U}$ asymptotic to iso for every $U \in \mathcal{U}$. Then, conjugation by θ_U provides an isomorphism $\mathrm{St}_{\mathcal{M}_{P|U}} \longrightarrow \mathrm{St}_{\mathcal{N}_{P|U}}$ carrying $\mathrm{St}_{\mathcal{M}|U}^{<D}$ to $\mathrm{St}_{\mathcal{N}|U}^{<D}$. To prove that $\Gamma(U, \mathcal{Q})$ is unipotent, we can thus suppose that $\mathcal{M}$ is good split unramified.

Let $(\mathbf{m}(0), \dots, \mathbf{m}(L), \mathbf{m}(L+1))$ be an auxiliary sequence for the good set of irregular values of $\mathcal{M}$ at P. If i denotes the smallest index for which $\mathbf{m}(i)$ does not have poles along every component of D passing through P, then $\mathrm{St}_{\mathcal{M}|\partial P}^{<D} = \mathrm{St}_{\mathcal{M}_P(i)}$. From the description 1.6.1 of the quotients of the level filtration on $\mathrm{St}_{\mathcal{M}_P}$, we deduce that $\mathcal{Q}$ is a product of Stokes sheaves. Since Stokes sheaves are sheaves of unipotent algebraic groups, we conclude that G is a unipotent algebraic group. This concludes the proof of 2.5.9.

□

Since $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ is a limit of morphisms of the form (2.5.1) and (2.5.2), lemmas 2.5.3 and 2.5.9 give the following

Corollary 2.5.18. — *For every $P \in D$, the restriction morphism*

$$H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D})$$

is a closed immersion.

2.6. Stokes torsors and marked connections. — Let us recall that a $\mathcal{M}$-*marked connection* is the data of a couple (M, iso) where M is a germ of meromorphic connection with poles along D defined in a neighbourhood of D in X, and where $\mathrm{iso} : M_{\widehat{D}} \longrightarrow \mathcal{M}_{\widehat{D}}$ is an isomorphism of formal connections. We denote by

$\mathrm{Isom}_{\mathrm{iso}}(M,\mathcal{M})$ the $\mathrm{St}^{<D}_{\mathcal{M}}(\mathbb{C})$-torsor of isomorphisms between ∂M and $\partial\mathcal{M}$ which are asymptotic to iso along D.

The proof of the following statement was suggested to me by T. Mochizuki. I thank him for kindly sharing it. When D is smooth, It it was known to Malgrange [**Mal83b**]. See also [**Sab02**, II 6.3].

Lemma 2.6.1. — *The map associating to every isomorphism class of $\mathcal{M}$-marked connection (M,iso) the $\mathrm{St}^{<D}_{\mathcal{M}}(\mathbb{C})$-torsor $\mathrm{Isom}_{\mathrm{iso}}(M,\mathcal{M})$ is bijective.*

Proof. — Let us construct an inverse. Take $\mathcal{T}\in\mathrm{St}^{<D}_{\mathcal{M}}(\mathbb{C})$ and let $g=(g_{ij})$ be a cocycle for $\mathcal{T}$ associated to a cover $(U_i)_{i\in I}$ of ∂D. Let $\mathcal{L}$ be the Stokes filtered local system on ∂D associated to $\mathcal{M}$. Set $\mathcal{L}_i:=\mathcal{L}_{|U_i}$. Then, g allows to glue the $\mathcal{L}_i$ into a Stokes filtered local system $\mathcal{L}_{\mathcal{T}}$ on ∂D independent of the choice of g. From the irregular Riemann-Hilbert correspondence [**Moc11a**, 4.11], $\mathcal{L}_{\mathcal{T}}$ is the Stokes filtered local system associated to a unique (up to isomorphism) good meromorphic connection $\mathcal{M}_{\mathcal{T}}$ defined in a neighbourhood of D and with poles along D. By construction, the isomorphism $\mathcal{L}_{\mathcal{T}|U_i}\longrightarrow\mathcal{L}_{|U_i}$ corresponds to an isomorphism $\partial\mathcal{M}_{\mathcal{T}|U_i}\longrightarrow\partial\mathcal{M}_{|U_i}$. We thus obtain a formal isomorphism $\mathrm{iso}_i:\partial\mathcal{M}_{\mathcal{T},\widehat{D}|U_i,}\longrightarrow\partial\mathcal{M}_{\widehat{D}|U_i}$. On U_{ij}, the discrepancy between iso_i and iso_j is measured by the asymptotic of g_{ij} along D. By definition, this asymptotic is Id. Hence, the iso_i glue into a globally defined isomorphism $\partial\mathcal{M}_{\mathcal{T},\widehat{D}}\longrightarrow\partial\mathcal{M}_{\widehat{D}}$. Applying p_* thus yields an isomorphism $\mathrm{iso}:\mathcal{M}_{\mathcal{T},\widehat{D}}\longrightarrow\mathcal{M}_{\widehat{D}}$. It is then standard to check that the map $\mathcal{T}\longrightarrow(\mathcal{M}_{\mathcal{T}},\mathrm{iso})$ is the sought-after inverse. □

2.7. Proof of Theorem 3. — We are now in position to prove Theorem 3, whose notations we use. Let $\pi:Y\longrightarrow X$ be a resolution of the turning point 0 for $\mathcal{M}$. Such a resolution exists by works of Kedlaya [**Ked10**] and Mochizuki [**Moc09**]. Set $E:=\pi^{-1}(D)$. At the cost of blowing up further, we can suppose that the strict transform C' of C is transverse to E at a point P in the smooth locus of E.

From 2.6.1, the $\pi^+\mathcal{M}$-marked connections $(\pi^+M_1,\pi^+\mathrm{iso}_1)$ and $(\pi^+M_2,\pi^+\mathrm{iso}_2)$ define two $\mathbb{C}$-points of $H^1(\partial E,\mathrm{St}^{<E}_{\pi^+\mathcal{M}})$. Since

$$(\pi_+\pi^+M_i,\pi_+\pi^+\mathrm{iso}_i)\simeq(M_i,\mathrm{iso}_i)$$

for $i=1,2$, it is enough to show $(\pi^+M_1,\pi^+\mathrm{iso}_1)\simeq(\pi^+M_2,\pi^+\mathrm{iso}_2)$.

By assumption,

$$\begin{aligned}(\pi^+M_1,\pi^+\mathrm{iso}_1)_{|C'}&\simeq(M_1,\mathrm{iso}_1)_{|C}\\&\simeq(M_2,\mathrm{iso}_2)_{|C}\\&\simeq(\pi^+M_2,\pi^+\mathrm{iso}_2)_{|C'}\end{aligned}$$

In particular, the image of $(\pi^+M_1,\pi^+\mathrm{iso}_1)$ and $(\pi^+M_2,\pi^+\mathrm{iso}_2)$ by the restriction map

$$H^1(\partial E,\mathrm{St}^{<E}_{\pi^+\mathcal{M}})\longrightarrow H^1(\partial P,\mathrm{St}^{<E}_{\pi^+\mathcal{M}}) \tag{2.7.1}$$

are the same. From 2.5.18, the map (2.7.1) is a closed immersion. Hence, $(\pi^+M_1,\pi^+\mathrm{iso}_1)\simeq(\pi^+M_2,\pi^+\mathrm{iso}_2)$, which concludes the proof of Theorem 3.

2.8. Obstruction theory and tangent space. — Let us compute the obstruction theory of $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ at a point $\mathcal{T}_0 \in H^1(\partial D, \mathrm{St}_{\mathcal{N}}^{<D}(\mathbb{C}))$. We fix a morphism of infinitesimal extensions of $\mathbb{C}$-algebras

$$R' \longrightarrow R \longrightarrow \mathbb{C}, \qquad I := \mathrm{Ker}\, R' \longrightarrow R$$

such that I is annihilated by $\mathrm{Ker}\, R' \longrightarrow \mathbb{C}$. In particular, $I^2 = 0$ and I is endowed with a structure of $\mathbb{C}$-vector space, which we suppose to be finite dimensional. Let $\mathcal{T} \in H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D}(R))$ lifting $\mathcal{T}_0$. Choose a cover $\mathcal{U} = (U_i)_{i\in K}$ of ∂D such that $\mathcal{T}$ comes from a cocycle $g = (g_{ij})_{i,j\in K}$. Set $L_i(R) := \mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(R)_{|U_i}$. The identifications

$$\begin{array}{ccc} L_i(R)_{|U_{ij}} & \xrightarrow{\sim} & L_j(R)_{|U_{ij}} \\ M & \longrightarrow & g_{ij}^{-1} M g_{ij} \end{array}$$

allow to glue the $L_i(R)$ into a sheaf of R-Lie algebras over ∂D denoted by $\mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(R)^{\mathcal{T}}$ and depending only on $\mathcal{T}$ and not on g. For $t = (t_{ijk}) \in \check{C}^2(\mathcal{U}, \mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(R)^{\mathcal{T}})$, we denote by s_{ijk} the unique representative of t_{ijk} in $\Gamma(U_{ijk}, L_i(R))$. Then

$$\begin{aligned} (dt)_{ijkl} &= t_{jkl} - t_{ikl} + t_{ijl} - t_{ijk} \\ &= [g_{ij} s_{jkl} g_{ij}^{-1} - s_{ikl} + s_{ijl} - s_{ijk}] \end{aligned}$$

We have the following

Lemma 2.8.1. — *There exists*

$$\mathrm{ob}(\mathcal{T}) \in I \otimes_{\mathbb{C}} \check{H}^2(\partial D, \mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}_0})$$

such that $\mathrm{ob}(\mathcal{T}) = 0$ *if and only if* $\mathcal{T}$ *lifts to* $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D}(R'))$.

Proof. — For every $i, j \in K$, let $h_{ij} \in \Gamma(U_{ij}, \mathrm{St}_{\mathcal{M}}^{<D}(R'))$ be an arbitrary lift of g_{ij} to R'. We can always choose the h_{ij} to satisfy $h_{ii} = \mathrm{Id}$ and $h_{ij}h_{ji} = \mathrm{Id}$. Since $\mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(R')$ is locally free,

$$I \cdot \mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(R') \simeq I \otimes_{R'} \mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(R') \simeq I \otimes_{\mathbb{C}} \mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})$$

We will use both descriptions without mention. We set

$$s_{ijk} := h_{ij}h_{jk}h_{ki} - \mathrm{Id} \in \Gamma(U_{ijk}, I \cdot \mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(R'))$$

We see s_{ijk} as a section of $I \otimes_{\mathbb{C}} L_i(\mathbb{C})$ over U_{ijk} and denote by $[s_{ijk}]$ its class in $I \otimes_{\mathbb{C}} \mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}_0}$. We want to prove that the $[s_{ijk}]$ define a cocycle. As seen above, this amounts to prove the following equality in $\Gamma(U_{ijk}, I \otimes_{\mathbb{C}} \mathrm{Lie}\, \mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C}))$

$$g_{ij}(0) s_{jkl} g_{ij}^{-1}(0) - s_{ikl} + s_{ijl} - s_{ijk} = 0 \tag{2.8.2}$$

Where $g_{ij}(0)$ is the image of g_{ij} by $R \longrightarrow \mathbb{C}$. We have

$$\begin{aligned} g_{ij}(0)s_{jkl}g_{ij}^{-1}(0) &= h_{ij}h_{jk}h_{kl}h_{lj}h_{ji} - \mathrm{Id} \\ &= (h_{ij}h_{jk} - h_{ik} + h_{ik})h_{kl}h_{lj}h_{ji} - \mathrm{Id} \\ &= (h_{ij}h_{jk} - h_{ik})g_{kl}(0)g_{lj}(0)g_{ji}(0) + h_{ik}h_{kl}h_{lj}h_{ji} - \mathrm{Id} \\ &= (h_{ij}h_{jk} - h_{ik})g_{ki}(0) + h_{ik}h_{kl}h_{lj}h_{ji} - \mathrm{Id} \\ &= (h_{ij}h_{jk} - h_{ik})h_{ki} + h_{ik}h_{kl}h_{lj}h_{ji} - \mathrm{Id} \\ &= h_{ij}h_{jk}h_{ki} + h_{ik}h_{kl}h_{lj}h_{ji} - 2\,\mathrm{Id} \end{aligned}$$

We now see how the second term of the last line above interacts with the second term of the left-hand side of (2.8.2).

$$\begin{aligned} h_{ik}h_{kl}h_{lj}h_{ji} - s_{ikl} &= h_{ik}h_{kl}h_{lj}h_{ji} - h_{ik}h_{kl}h_{li} + \mathrm{Id} \\ &= h_{ik}h_{kl}(h_{lj}h_{ji} - h_{li}) + \mathrm{Id} \\ &= g_{ik}(0)g_{kl}(0)(h_{lj}h_{ji} - h_{li}) + \mathrm{Id} \\ &= g_{il}(0)(h_{lj}h_{ji} - h_{li}) + \mathrm{Id} \\ &= h_{il}h_{lj}h_{ji} \end{aligned}$$

Hence,

$$\begin{aligned} g_{ij}(0)s_{jkl}g_{ij}^{-1}(0) - s_{ikl} + s_{ijl} - s_{ijk} &= h_{il}h_{lj}h_{ji} + h_{ij}h_{jl}h_{li} - 2\,\mathrm{Id} \\ &= (h_{ij}h_{jl}h_{li})^{-1} + h_{ij}h_{jl}h_{li} - 2\,\mathrm{Id} \\ &= (h_{ij}h_{jl}h_{li})^{-1}((h_{ij}h_{jl}h_{li})^2 - 2h_{ij}h_{jl}h_{li} + \mathrm{Id}) \\ &= (h_{ij}h_{jl}h_{li})^{-1}s_{ijl}^2 \\ &= 0 \end{aligned}$$

where the last equality comes from $I^2 = 0$. Hence, the $[s_{ijk}]$ define a cocycle of $I \otimes_{\mathbb{C}} \operatorname{Lie}\operatorname{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}_0}$. An other choice of lift gives rise to homologous cocycles. We denote by $\mathrm{ob}(\mathcal{T})$ the class of $([s_{ijk}])_{ijk}$ in $\check{H}^2(\partial D, I \otimes_{\mathbb{C}} \operatorname{Lie}\operatorname{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}_0})$. It is standard to check that $\mathrm{ob}(\mathcal{T})$ has the sought-after property. □

Corollary 2.8.3. — *Let (M, iso) be a $\mathcal{M}$-marked connection. Then, the space $H^2(D, \mathrm{Irr}_D^* \operatorname{End} M))$ is an obstruction theory for $H^1(\partial D, \operatorname{St}_{\mathcal{M}}^{<D})$ at $\mathrm{Isom}_{\mathrm{iso}}(M, \mathcal{M})$.*

Proof. — Set $\mathcal{T} := \mathrm{Isom}_{\mathrm{iso}}(M, \mathcal{M})$. As observed in [**Tey17**, 5.2], the canonical identification

$$\mathcal{H}^0 \operatorname{DR}^{<D} \operatorname{End} \mathcal{M} \xrightarrow{\sim} \operatorname{Lie}\operatorname{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}}$$

induces

$$\begin{aligned} \check{H}^i(\partial D, \operatorname{Lie}\operatorname{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}}) &\simeq H^i(\partial D, \operatorname{Lie}\operatorname{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}}) \\ &\simeq H^i(\partial D, \mathcal{H}^0 \operatorname{DR}^{<D} \operatorname{End} M) \\ &\simeq H^i(\partial D, \operatorname{DR}^{<D} \operatorname{End} M) \\ &\simeq H^i(D, \mathrm{Irr}_D^* \operatorname{End} M)) \end{aligned}$$

The second identification comes from the fact [**Hie09**, Prop. 1] that $\mathrm{DR}^{<D}\,\mathrm{End}\,\mathcal{M}$ is concentrated in degree 0. The third identification comes from [**Sab17a**, 2.2]. Then, corollary 2.8.3 follows from 2.8.1. □

Reasoning exactly as in [**Tey17**, 5.2.1], we prove the following

Lemma 2.8.4. — *For every $\mathcal{M}$-marked connection (M, iso), the tangent space of $H^1(\partial D, \mathrm{St}^{<D}_{\mathcal{M}})$ at (M, iso) identifies canonically to $H^1(D, \mathrm{Irr}^*_D\,\mathrm{End}\,M)$.*

3. Moduli of Stokes torsors in the one level case

3.1. Roadmap. — The goal of this section is to describe the moduli of Stokes torsors in the case where the irregular values have only one level. To do this, we compare a relative version of the absolute Stokes groups from [**MR91**][**Lod94**] with the relative non abelian cohomology of the Stokes sheaf defined in 2.5.3. For the problem raised by this comparison in the multi-level case, we refer to 3.5. Note that over a smooth base (corresponding in this paper to the case where D is smooth), relative Stokes groups appeared in the one level case in [**JMU81**] and in more generality in [**Boa02**]. In particular, over a smooth base, they were already considered in the multi-level case in [**Boa14**].

The reader interested only in the proof of Theorem 1 can skip this part, since it will not be used in the sequel.

3.2. Relative Stokes groups. — We keep the setup and notations from 2.3 and 2.4. We recall that $\mathcal{M}$ stands for a good meromorphic connection defined in a neighbourhood of a normal crossing divisor D in an algebraic surface X and with poles along D. Let $\mathcal{I}$ be the sheaf of irregular values of $\mathcal{M}$. We first suppose that $\mathcal{M}$ is unramified. In that case, $\mathcal{I}$ is a subsheaf of $\mathcal{O}_X(*D)/\mathcal{O}_X$. For $a, b \in \mathcal{I}$, the function $G_{a,b} := (a-b)/|a-b|$ induces a C^∞-function $\partial G_{a,b}$ on ∂D. The *anti-Stokes lines of* (a, b) are the connected components of

$$H^<_{ab} := \{\theta \in \partial D \text{ such that } \partial G_{a,b}(\theta) \in \mathbb{R}^-\}$$

The set $H^<_{ab}$ is a smooth C^∞-hypersurface in ∂D. Let $H^<$ be the union of all $H^<_{ab}$, $a, b \in \mathcal{I}$ distinct. Let $\imath : H^< \longrightarrow \partial D$ be the inclusion. Let $\mathbb{S}to_{\mathcal{M}}$ be the subsheaf of $\imath_*\imath^{-1}\,\mathrm{St}^{<D}_{\mathcal{M}}$ whose germ at $\theta \in \partial D$ is

$$\mathbb{S}to_{\mathcal{M},\theta} = \{g \in \mathrm{St}^{<D}_{\mathcal{M},\theta} \text{ such that for every } a, b \in \mathcal{I} \text{ distinct}, g_{ab} = 0 \text{ unless } \theta \in H^<_{ab}\}$$

We call $p_*\mathbb{S}to_{\mathcal{M}}$ the *relative Stokes group of* $\mathcal{M}$. For a possibly ramified connection $\mathcal{M}$, we define the relative Stokes group of $\mathcal{M}$ via Galois descent from the unramified case.

Suppose that D is smooth. Then, for every $a, b \in \mathcal{I}$, the Stokes lines of (a, b) are parallel to the anti-Stokes lines of (a, b). Hence, $H^<_{ab}$ does not meet any Stokes line of (a, b). Thus, for any $\theta \in \partial D$ and any $g \in \mathbb{S}to_{\mathcal{M},\theta}$, the section g extends uniquely on a small product $\Delta \times I$ containing θ, where Δ is a disc in D centred at $p(\theta)$ and where I is an interval of S^1. This product only depends on θ and not on g. We deduce that when D is smooth, $p_*\mathbb{S}to_{\mathcal{M}}$ is a local system on D.

3.3. One level along the smooth locus of D. —

Lemma 3.3.1. — *We suppose that D is smooth and that $\mathcal{M}$ admits a unique level. Then, the sheaf $p_*\mathbb{S}to_{\mathcal{M}}$ is canonically isomorphic to $R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D}$.*

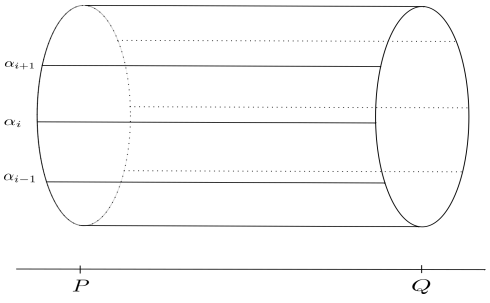
Proof. — Since D is smooth, the sheaves $p_*\mathbb{S}to_{\mathcal{M}}$ and $R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D}$ are local systems on ∂D. For every $P \in D$, we have a diagram with canonical vertical arrows

$$\begin{array}{ccc} (p_*\mathbb{S}to_{\mathcal{M}})_P & \longrightarrow & (R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D})_P \\ \wr\downarrow & & \downarrow\wr \\ \Gamma(\partial P, \mathbb{S}to_{\mathcal{M}}) & \xrightarrow[\mathrm{LR}_P]{} & H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \end{array} \tag{3.3.2}$$

where LR_P is the isomorphism constructed by Loday-Richaud [**Lod94**, II 1.9], and where the upper arrow makes (3.3.2) commutative. To prove 3.3.1, we have to show that the identifications LR_P glue into an isomorphism of local systems. This amounts to show that the LR_P are compatible with the parallel transports of $p_*\mathbb{S}to_{\mathcal{M}}$ and $R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D}$. That is, for every $P, Q \in D$ and every continuous path γ in D joining P to Q, the following diagram commutes

$$\begin{array}{ccc} \Gamma(\partial P, \mathbb{S}to_{\mathcal{M}}) & \longrightarrow & \Gamma(\partial Q, \mathbb{S}to_{\mathcal{M}}) \\ {\scriptstyle \mathrm{LR}_P}\downarrow & & \downarrow{\scriptstyle \mathrm{LR}_Q} \\ H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) & \longrightarrow & H^1(\partial Q, \mathrm{St}_{\mathcal{M}}^{<D}) \end{array} \tag{3.3.3}$$

where the horizontal arrows are the parallel transports along γ. This compatibility question is a local question on D. Let us thus suppose that P and Q belong to a small disc Δ in D. By Galois descent, we can suppose that $\mathcal{M}$ is unramified. Via a local rectification $\partial\Delta \simeq \Delta \times S^1$ as in [**Sab02**, 6.8], the anti-Stokes hyperplanes of $\mathcal{M}$ above Δ can be pictured as follows. Let us order the connected components of $\partial\Delta \cap H^<$



cyclically $\alpha_1, \ldots, \alpha_d$ and denote by $\alpha_i(x)$ the point $\alpha_i \cap \partial x$ for every $x \in \Delta$. For $\epsilon > 0$ small enough and for $i \in \mathbb{Z}/d\mathbb{Z}$, consider the open sector $S_i = \Delta \times]\alpha_i - \epsilon, \alpha_{i+1} + \epsilon[$. Set $\mathcal{S} := (S_i)_{i\in\mathbb{Z}/d\mathbb{Z}}$. Take $g = (g_i \in \mathbb{S}to_{\mathcal{M},\alpha_i(P)})_{i\in\mathbb{Z}/d\mathbb{Z}}$. Since ϵ has been chosen small enough, g_i can be seen as a section of $\mathrm{St}_{\mathcal{M}}^{<D}$ above $]\alpha_i(P)-\epsilon, \alpha_i(P)+\epsilon[$. By definition, $\mathrm{LR}_P(g)$ is the Stokes torsor on ∂P associated to the cocycle $g \in Z^1(\mathcal{S} \cap \partial P, \mathrm{St}_{\mathcal{M}}^{<D})$.

The image of $\mathrm{LR}_P(g)$ by the parallel transport of $R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D}$ is the restriction to ∂Q of the unique $\mathcal{T} \in H^1(\partial\Delta, \mathrm{St}_{\mathcal{M}}^{<D})$ such that $\mathcal{T}_{|\partial P} = \mathrm{LR}_P(g)$. But g extends uniquely into $\widetilde{g} = (\widetilde{g}_i) \in Z^1(\mathcal{S}, \mathrm{St}_{\mathcal{M}}^{<D})$. Thus, $\mathcal{T}$ is the Stokes torsor associated to $\widetilde{g}$. Hence, travelling down the diagram (3.3.3) produces the torsor over ∂Q associated to $\widetilde{g}_{|\partial Q} \in Z^1(\mathcal{S} \cap \partial Q, \mathrm{St}_{\mathcal{M}}^{<D})$. On the other hand, we observe that the parallel transport of g as a section of the relative Stokes sheaf is $(\widetilde{g}_{i,\alpha_i(Q)})_i$. Since $\mathrm{LR}_Q((\widetilde{g}_{i,\alpha_i(Q)})_i)$ is the Stokes torsor associated to $\widetilde{g}_{|\partial Q}$, the commutativity of (3.3.3) is proved.

□

3.4. One level at a singular point of D. — In this paragraph, we restrict our attention to what happens at a point $P \in \mathrm{Sing}(D)$.

Proposition 3.4.1. — *Suppose that $\mathcal{M}$ admits a unique level at P. Then, there is a canonical isomorphism*

$$\Gamma(\partial P, \mathbb{S}to_{\mathcal{M}}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}})$$

Proof. — By Galois descent, we can suppose that $\mathcal{M}$ is unramified. We denote by $\mathbf{m}$ its level. Let us choose local coordinates centred at P and let us denote by C the diagonal. Then, $\mathcal{M}_C$ admits only one level. Let $\underline{I}$ be a cover of S^1_C à la Loday-Richaud for $\mathcal{M}_C$. By definition, this is a cover by intervals with non empty triple intersection such that any of these interval contains exactly two consecutive anti-Stokes directions. An element of $\Gamma(\partial P, \mathbb{S}to_{\mathcal{M}})$ defines a cocycle in $Z^1(\partial P(\mathbf{m}, \underline{I}), \mathrm{St}_{\mathcal{M}})$ from which we deduce a $\mathrm{St}_{\mathcal{M}}$-torsor on ∂P. The same construction holds on S^1_C. Hence, there is a commutative diagram

$$\begin{array}{ccc} \Gamma(\partial P, \mathbb{S}to_{\mathcal{M}}) & \longrightarrow & \Gamma(S^1_C, \mathbb{S}to_{\mathcal{M}_C}) \\ \downarrow & & \downarrow \\ H^1(\partial P, \mathrm{St}_{\mathcal{M}}) & \xrightarrow[\mathbf{res}_C]{} & H^1(S^1_C, \mathrm{St}_{\mathcal{M}_C}) \end{array} \tag{3.4.2}$$

From [**Lod94**], the right vertical map of (3.4.2) is an isomorphism. Taking the invariants under the action of $\pi_1(\mathbb{T}(\mathbf{m}, 0), 0)$ on the right part of (3.4.2) gives a commutative diagram

$$\begin{array}{ccc} \Gamma(\partial P, \mathbb{S}to_{\mathcal{M}}) & \xrightarrow{\sim} & \Gamma(S^1_C, \mathbb{S}to_{\mathcal{M}_C})^{\pi_1} \\ \downarrow & & \downarrow \wr \\ H^1(\partial P, \mathrm{St}_{\mathcal{M}}) & \longrightarrow & H^1(S^1_C, \mathrm{St}_{\mathcal{M}_C})^{\pi_1} \end{array} \tag{3.4.3}$$

From 1.7.1, the bottom arrow of (3.4.3) is an isomorphism. Hence, the left vertical arrow of (3.4.3) is an isomorphism. This concludes the proof of 3.4.1. □

In simple cases, the previous lemmas tell precisely what the moduli of Stokes torsor looks like.

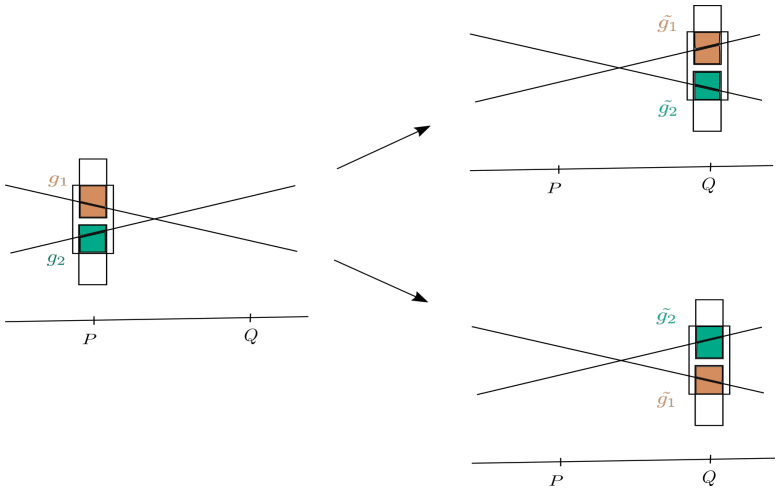
Corollary 3.4.4. — *Suppose that $\mathcal{M}$ has rank 2. Then, $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ is an affine space.*

Proof. — It is enough to show that the morphisms (2.5.2) and (2.5.1) are linear inclusions of affine spaces. Relative Stokes groups are sheaves of unipotent algebraic groups. The underlying scheme of a unipotent algebraic group is an affine space. Then, corollary 3.4.4 is an immediate consequence of the description 3.3.1 and 3.4.1 of Stokes torsors in terms of relative Stokes groups in the one level case. □

3.5. A remark on the multi-level case. — In this subsection, we restrict to the case where D is smooth. The question whether $R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D}$ and $p_*\mathbb{S}to_{\mathcal{M}}$ are isomorphic seems to be fruitful, since it would imply that *when D is smooth, moduli of Stokes torsors are affine spaces.* This last assertion is known in dimension 1 from [**BV89**]. We thus formulate the following

Conjecture. — *Suppose that D is smooth. Then, the local systems $R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D}$ and $p_*\mathbb{S}to_{\mathcal{M}}$ are isomorphic.*

In the several level case, the main difficulty comes from the fact that the parallel transports for $R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D}$ and $p_*\mathbb{S}to_{\mathcal{M}}$ produce different cocycles that are not equal on the nose, but might be cohomologous. The following picture illustrates this phenomenon. The picture on the left features part of our initial element of the Stokes



group above P. In this situation, two anti-Stokes lines L_1 and L_2 intersect once along the path joining P to Q. Let us call x the intersection point. Since anti-Stokes lines are parallel to Stokes lines, there is a neighbourhood Ω of x in ∂D not meeting any Stokes line coming from the differences of irregular values giving rise to L_1 and L_2. In particular, g_1 and g_2 extend uniquely into sections $\widetilde{g_1}$ and $\widetilde{g_2}$ of $\mathrm{St}_{\mathcal{M}}^{<D}$ over Ω. When applying the parallel transport for $R^1p_*\,\mathrm{St}_{\mathcal{M}}^{<D}$, we end up with the cocycle in the upper right picture. The bottom right picture represents the effect of the parallel transport for $p_*\mathbb{S}to_{\mathcal{M}}$. Finally, one passes from one cocycle to the other by permuting $\widetilde{g_1}$ and $\widetilde{g_2}$. Since the Stokes sheaf is not commutative, it is not a priori clear that these cocycles are cohomologous.

4. Reduction of Theorem 1 to extending the formal model

4.1. Reduction to the dimension 2 case. — In this subsection, we reduce the proof of Theorem 1 to the dimension 2 case. The main tool is André's goodness criterion [**And07**, 3.4.3] in terms of Newton polygons. This reduction does not seem superfluous. Of crucial importance for the sequel of the proof (see 4.3.1) will be indeed the fact that for an unramified meromorphic connection $\mathcal{M}$ with poles along a divisor D and for a point $0 \in D$, the formal model of $\mathcal{M}$ *splits on a small enough punctured disc around* 0. This fact is specific to dimension 2, since it pertains to the property that turning points of connections in dimension 2 are isolated.

Lemma 4.1.1. — *Theorem* 1 *is true in any dimension if it is true in dimension* 2.

Proof. — Take $n > 2$. We argue recursively by supposing that Theorem 1 holds in dimension strictly less than n and we prove that Theorem 1 holds in dimension n. Let $0 \in D$ and suppose that $\mathrm{Irr}^*_D \mathcal{M}$ and $\mathrm{Irr}^*_D \mathrm{End}\,\mathcal{M}$ are local systems in a neighbourhood of 0. If $j : X\backslash D \longrightarrow X$ and $i : D \longrightarrow X$ are the canonical inclusions, we have distinguished triangle

$$j_! L \longrightarrow \mathrm{Sol}\,\mathcal{M} \longrightarrow i_* \mathrm{Irr}^*_D \mathcal{M}$$

where L is a local system on the complement of D. Hence, the characteristic cycle of $\mathrm{Sol}\,\mathcal{M}$ is supported on the union of $T^*_X X$ with $T^*_D X$. From a theorem of Kashiwara and Schapira [**KS90**, 11.3.3], so does the characteristic cycle of $\mathcal{M}$. Hence, any smooth hypersurface transverse to D and passing through 0 is non characteristic with respect to $\mathcal{M}$ in a neighbourhood of 0. Let us choose such a hypersurface Z and let $i_Z : Z \longrightarrow X$ be the canonical inclusion. From [**And07**, 3.4.3], the turning point locus of $\mathcal{M}$ is a closed subset of D which is either empty or purely of codimension 1 in D. Since $n > 2$, the hypersurface Z can consequently be chosen such that $\mathcal{M}$ and $\mathrm{End}\,\mathcal{M}$ have good formal decomposition generically along $Z \cap D$. The connection $i_Z^+\mathcal{M}$ is a meromorphic connection with poles along $Z \cap D$. It satisfies the hypothesis of Theorem 1 at the point 0. Indeed by Kashiwara's restriction theorem [**Kas95**],

$$\mathrm{Irr}^*_{Z\cap D}\, i_Z^+\mathcal{M} = (\mathrm{Sol}\, i_Z^+\mathcal{M})_{|Z\cap D} \simeq (\mathrm{Sol}\,\mathcal{M})_{|Z\cap D}$$

and similarly for $\mathrm{End}\,\mathcal{M}$. Hence, $\mathrm{Irr}^*_{Z\cap D}\,\mathcal{M}$ and $\mathrm{Irr}^*_{Z\cap D}\,\mathrm{End}\,\mathcal{M}$ are local systems in a neighbourhood of 0 in $Z \cap D$. By recursion hypothesis, $i_Z^+\mathcal{M}$ is good at 0. In particular, the Newton polygon of $i_Z^+\mathcal{M}$ at 0 (which is also the Newton polygon of $\mathcal{M}$ at 0) is the generic Newton polygon of $i_Z^+\mathcal{M}$ along $Z \cap D$. From our choice for Z, the generic Newton polygon of $i_Z^+\mathcal{M}$ along $Z \cap D$ is the generic Newton polygon of $\mathcal{M}$ along D. Hence, the Newton polygon of $\mathcal{M}$ at 0 is the generic Newton polygon of $\mathcal{M}$ along D, and similarly with $\mathrm{End}\,\mathcal{M}$. By a theorem of André [**And07**, 3.4.1], we deduce that $\mathcal{M}$ has good formal decomposition at 0, which proves the reduction 4.1.1.

□

4.2. Setup and recollections. — From now on, we restrict to dimension 2. We use coordinates (x, y) on $\mathbb{A}^2$ and set $D_x := \{y = 0\}, D_y := \{x = 0\}$. Let D be a

neighbourhood of 0 in D_x and let $\mathbb{C}[D]$ be the coordinate ring of D. Set $D^* := D\backslash\{0\}$.

Let $\mathcal{M}$ be an algebraic meromorphic flat bundle on a neighbourhood of D in $\mathbb{A}^2$ with poles along D. In algebraic terms, $\mathcal{M}_{\widehat{D}}$ defines a $\mathbb{C}[D]((y))$-differential module. At the cost of shrinking D if necessary, we can suppose that the restriction $\mathcal{M}^*$ of $\mathcal{M}$ to a neighbourhood of D^* has good formal decomposition at every point of D^*.

There is a ramification $v = y^{1/d}$, $d \geqslant 1$ and a finite Galois extension $L/\mathbb{C}(x)$ such that the set $\mathcal{I}$ of generic irregular values for $\mathcal{M}$ lies in $\operatorname{Frac} L(v)$. If $p : D_L \longrightarrow D$ is the normalization of D in L, the generic irregular values of $\mathcal{M}$ are thus meromorphic functions on $D_L \times \mathbb{A}^1_v$. We have

$$L((v)) \otimes \mathcal{M} \simeq \bigoplus_{a\in\mathcal{I}} \mathcal{E}^a \otimes \mathcal{R}_a \tag{4.2.1}$$

where the $\mathcal{R}_a$ are regular. Following [**And07**, 3.2.4], we recall the following

Definition 4.2.2. — *We say that $\mathcal{M}$ is semi-stable at $P \in D$ if*

(1) We have $\mathcal{I} \subset \mathbb{C}[D_L]_P((v))$.

(2) The decomposition (4.2.1) *descends to $\mathbb{C}[D_L]_P((v)) \otimes \mathcal{M}$.*

In this definition, $\mathbb{C}[D_L]_P$ denotes the localization of $\mathbb{C}[D_L]$ above P. This is a semi-local ring. Let $\pi_a \in L((v)) \otimes \operatorname{End}\mathcal{M}$ be the projector on the factor $\mathcal{E}^a \otimes \mathcal{R}_a$. As explained in [**And07**, 3.2.2], the point P is stable if and only if the generic irregular values of $\mathcal{M}$ and the coefficients of the π_a in a basis of $\operatorname{End}\mathcal{M}$ belong to $\mathbb{C}[D_L]_P((v))$. Since $\mathcal{M}$ has good formal decomposition at any point of D^*, the generic irregular values of $\mathcal{M}$ and the coefficients of the π_a in a basis of $\operatorname{End}\mathcal{M}$ belong to $\mathbb{C}[D_L]_P((v))$ for every $P \in D^*$. Hence, they belong $\mathbb{C}[D_L^*]((v))$ where $D_L^* := D\backslash p^{-1}(0)$. Thus

$$\mathbb{C}[D_L^*]((v)) \otimes \mathcal{M} \simeq \mathbb{C}[D_L^*]((v)) \otimes \mathcal{N}_L^* \tag{4.2.3}$$

where

$$\mathcal{N}_L^* = \bigoplus_{a\in\mathcal{I}} \mathcal{E}^a \otimes \mathcal{R}_a$$

is a germ of meromorphic connection defined on a neighbourhood of D_L^* in $D_L \times \mathbb{A}^1_v$ and with poles along D_L^*. The action of

$$\operatorname{Gal}(L/\mathbb{C}(x)) \times \mathbb{Z}/d\mathbb{Z}$$

on the left-hand side of (4.2.3) induces an action on $\mathcal{N}_L^*$. Taking the invariants yields a meromorphic flat bundle $\mathcal{N}^*$ defined on a neighbourhood Ω of D^* in $\mathbb{A}^2$. By Galois descent, (4.2.3) descents to an isomorphism iso^* between the formalizations of $\mathcal{M}^*$ and $\mathcal{N}^*$ along D^*.

4.3. Reduction to the problem of extending the formal model. — The goal of this subsection is to show that Theorem 1 reduces to prove that the $\mathcal{M}^*$-marked connection $(\mathcal{N}^*, \mathrm{iso}^*)$ defined in 4.2 extends into a $\mathcal{M}$-marked connection in a neighbourhood of 0. To do this, we need three preliminary lemmas.

Lemma 4.3.1. — *Suppose that $\mathcal{N}^*$ extends into a meromorphic flat bundle $\mathcal{N}$ defined in a neighbourhood of D in $\mathbb{A}^2$ and with poles along D. Then, $\mathcal{N}$ is semi-stable at* 0.

Proof. — It is enough to treat the case where $K = \mathbb{C}(x)$ and $d = 1$. In that case, discussion 4.2 shows that on a neighbourhood Ω of D^* in $\mathbb{A}^2$, we have

$$\mathcal{N}^* = \bigoplus_{a\in\mathcal{I}} \mathcal{N}_a^*$$

where $\mathcal{N}_a^*$ is a meromorphic connection on Ω with poles along D^* and with single irregular value a. The open $D \times \mathbb{A}^1$ retracts on the small neighbourhood on which $\mathcal{N}$ is defined. Since $\mathcal{N}$ is smooth away from D, we deduce that $\mathcal{N}$ extends canonically into a meromorphic connection on $D \times \mathbb{A}^1$ with poles along D.

Let $a \in \mathcal{I}$. The restriction of the projector π_a to the complement of D^* in Ω is a flat section of $\operatorname{End}\mathcal{N}$. Since $D^* \times \mathbb{A}^1$ retracts on Ω, parallel transport allows to extend π_a canonically to $D^* \times \mathbb{A}^1$. We still denote by π_a this extension. Hence, $\mathcal{N}_a^*$ extends into a meromorphic connection on $D^* \times \mathbb{A}^1$ with poles along D^*. Let γ be a small loop in Ω going around the axis D_y. By assumption, the monodromy of $\mathcal{N}$ along γ is trivial. Thus, π_a is invariant under the monodromy of $\operatorname{End}\mathcal{N}$ along γ. Hence, π_a extends canonically to $(D \times \mathbb{A}^1)\backslash\{0\}$. By Hartog's property, it extends further into a section ϖ_a of $\operatorname{End}\mathcal{N}$ on $D \times \mathbb{A}^1$.

Set $\mathcal{N}_a := \varpi_a(\mathcal{N}) \subset \mathcal{N}$ for every $a \in \mathcal{I}$. We have $\varpi_a^2 = \varpi_a$ and $\sum_{a\in\mathcal{I}} \varpi_a = \mathrm{Id}_{\mathcal{N}}$ because these equalities hold on a non empty open set. Hence, $\mathcal{N} = \oplus_{a\in\mathcal{I}}\mathcal{N}_a$. Since ϖ_a is flat, the connection on $\mathcal{N}$ preserves each $\mathcal{N}_a$. Let us prove that the $\mathcal{N}_a$ are locally free as $\mathcal{O}_{D\times\mathbb{A}^1}(*D)$-modules.

Let E be a Deligne-Malgrange lattice [**Mal96**] for $\mathcal{N}$. Since we work in dimension 2, we know from [**Mal96**, 3.3.2] that E is a vector bundle. We observe that ϖ_a stabilizes E away from 0. By Hartog's property, we deduce that ϖ_a stabilizes E. Hence, $\varpi_a(E)$ is a direct factor of E. So $\varpi_a(E)$ is a vector bundle. Thus,

$$\mathcal{N}_a = \varpi_a(\mathcal{N}) = \varpi_a(E(*D)) = (\varpi_a(E))(*D)$$

is a locally free $\mathcal{O}_{D\times\mathbb{A}^1}(*D)$-module of finite rank with connection extending $\mathcal{N}_a^*$. To prove 4.3.1, we are thus left to consider the case where $\mathcal{I} = \{a\}$.

If $\mathcal{I} = \{a\}$, then [**And07**, 3.3.1] implies $a \in \mathbb{C}[D]((y))$. Hence, $\mathcal{R} := \mathcal{E}^{-a} \otimes \mathcal{N}_{\widehat{D}}$ is a formal meromorphic connection with poles along D. By assumption, $\mathcal{R}$ is generically regular along D. From [**Del70**, 4.1], we deduce that $\mathcal{R}$ is regular. Hence, $\mathcal{N}_{\widehat{D}} = \mathcal{E}^a \otimes \mathcal{R}$ with $\mathcal{R}$ regular, which concludes the proof of 4.3.1. □

Lemma 4.3.2. — *Let $\mathcal{N}$ be a meromorphic flat connection with poles along D. We suppose that $\mathcal{N}$ is semi-stable at 0 and that $\operatorname{Irr}_D^*\mathcal{N}$ and $\operatorname{Irr}_D^*\operatorname{End}\mathcal{N}$ are local systems in a neighbourhood of 0. Then, $\mathcal{N}$ has good formal decomposition at 0.*

Proof. — Let $\mathcal{I}$ be the set of irregular values of $\mathcal{N}$ at 0. There is a ramification $v = y^{1/d}$, $d \geqslant 1$ and a finite Galois extension $L/\mathbb{C}(x)$ such that $\mathcal{I} \subset L((v))$. Let $D_L \longrightarrow D$ be the normalization of D in L. At the cost of shrinking D, we can suppose that every point of D is semi-stable for $\mathcal{N}$. Hence, $\mathcal{I} \subset \mathbb{C}[D_L]((v))$ and

$$\mathbb{C}[D_L]((v)) \otimes \mathcal{N} = \bigoplus_{a\in\mathcal{I}} \mathcal{E}^a \otimes \mathcal{R}_a$$

where the connections $\mathcal{R}_a$ are regular. As seen in the proof of 4.1.1, the assumption on Irr_D^* implies that any smooth curve transverse to D is non characteristic for $\mathcal{N}$.

Taking the axis D_y yields

$$\dim \mathcal{H}^1 \operatorname{Irr}_0^* \mathcal{N}_{|D_y} = \dim(\mathcal{H}^1 \operatorname{Irr}_D^* \mathcal{N})_0 = \sum_{a\in\mathcal{I}} (\operatorname{ord}_y a) \operatorname{rk} \mathcal{R}_a$$

On the other hand, choose a point $P \in D_L$ above 0. Then, the irregular values of $\mathcal{N}_{|D_y}$ are the $a(P)$, $a \in \mathcal{I}$. Thus,

$$\mathcal{H}^1 \operatorname{Irr}_0^* \mathcal{N}_{|D_y} = \sum_{a\in\mathcal{I}} \operatorname{ord}_y a(P) \operatorname{rk} \mathcal{R}_a$$

Hence, $\operatorname{ord}_y a(P) = \operatorname{ord}_y a$ for every $a \in \mathcal{I}$. In particular, the coefficient function of the highest power of $1/v$ contributing to $a \in \mathcal{I}$ does not vanish at P. Arguing similarly for $\operatorname{End}\mathcal{N}$, we obtain that $\mathcal{N}$ has good formal decomposition at 0. □

Lemma 4.3.3. — *Suppose that* $\operatorname{Irr}_D^* \mathcal{M}$ *is a local system. For every* $\mathcal{M}$*-marked connection* $(\mathcal{N}, \mathrm{iso})$*, the complex* $\operatorname{Irr}_D^* \mathcal{N}$ *is a local system.*

Proof. — Let $\chi(D, \operatorname{Irr}_D^* \mathcal{M}) : D \longrightarrow \mathbb{Z}$ be the local Euler-Poincaré characteristic of $\operatorname{Irr}_D^* \mathcal{M}$. By local index theorem [**Kas73**][**Mal81**], the value of $\chi(D, \operatorname{Irr}_D^* \mathcal{M})$ at $P \in D$ only depends on the multiplicities of the components of the characteristic cycle of $\mathcal{M}$ passing through P. These multiplicities can be computed at the level of the formal neighbourhood of P in $\mathbb{C}^2$. Since $\mathcal{M}$ and $\mathcal{N}$ are formally isomorphic at P, we have

$$\chi(D, \operatorname{Irr}_D^* \mathcal{M}) = \chi(D, \operatorname{Irr}_D^* \mathcal{N})$$

Hence, $\chi(D, \operatorname{Irr}_D^* \mathcal{N})$ is constant. On the other hand, we know from [**Meb90**] that $\operatorname{Irr}_D^* \mathcal{N}$ is perverse. We conclude with the fact that a perverse sheaf with constant local Euler-Poincaré characteristic is a local system [**Tey13**, 13.1.6]. □

Using notations from 4.2, we are now in position to prove the sought-after

Proposition 4.3.4. — *Suppose that* $\operatorname{Irr}_D^* \mathcal{M}$ *and* $\operatorname{Irr}_D^* \operatorname{End}\mathcal{M}$ *are local systems in a neighbourhood of* 0*. If* $(\mathcal{N}^*, \mathrm{iso}^*)$ *extends into a* $\mathcal{M}$*-marked connection* $(\mathcal{N}, \mathrm{iso})$*, then* $\mathcal{M}$ *has good formal decomposition at* 0*.*

Proof. — From 4.3.1, the extension $\mathcal{N}$ is semi-stable at 0. From 4.3.3, we know that $\operatorname{Irr}_D^* \mathcal{N}$ and $\operatorname{Irr}_D^* \operatorname{End}\mathcal{N}$ are local systems in a neighbourhood of 0. From 4.3.2, we deduce that $\mathcal{N}$ has good formal decomposition at 0. Hence, so does $\mathcal{M}$. □

5. Extension via moduli of Stokes torsors

5.1. A geometric extension criterion. — We keep notations from 4.2. We first relate moduli of Stokes torsors to the problem of extending marked connections. Let $\pi : X \longrightarrow \mathbb{C}^2$ be a resolution of the turning point 0 for $\mathcal{M}$. Such a resolution exists by works of Kedlaya [**Ked10**] and Mochizuki [**Moc09**]. Set $E := \pi^{-1}(D)$ and pick $P \in D^*$. Let

$$\Phi : H^1(\partial E, \operatorname{St}^{<E}_{\pi^+\mathcal{M}}) \longrightarrow H^1(\partial P, \operatorname{St}^{<D}_{\mathcal{M}})$$

be the restriction morphism of Stokes torsors.

Lemma 5.1.1. — *Let* $(\mathcal{N}^*, \mathrm{iso}^*)$ *be a* $\mathcal{M}^*$*-marked connection such that* $(\mathcal{N}^*_P, \mathrm{iso}^*_P)$ *lies in the image of* Φ*. Then,* $(\mathcal{N}^*, \mathrm{iso}^*)$ *extends into an* $\mathcal{M}$*-marked connection in a neighbourhood of* 0*.*

Proof. — From 2.6.1, any $\mathbb{C}$-point of $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ comes from a unique $\pi^+\mathcal{M}$-marked connection. Hence, there exists $(\mathcal{N}', \mathrm{iso}') \in H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ such that $\Phi(\mathcal{N}', \mathrm{iso}') = (\mathcal{N}^*_P, \mathrm{iso}^*_P)$. From [**Meb04**, 3.6-4], the $\mathcal{D}$-module $\mathcal{N} := \pi_+\mathcal{N}'$ is a meromorphic connection with poles along D. By flat base change

$$\begin{aligned}
\mathcal{N}_{\widehat{D}} &\simeq \mathcal{O}_{\widehat{\mathbb{C}^2|D}} \otimes R\pi_*(\mathcal{D}_{X\to\mathbb{C}^2} \otimes \mathcal{N}') \\
&\simeq R\pi_*(\mathcal{O}_{\widehat{X|E}} \otimes \mathcal{D}_{X\to\mathbb{C}^2} \otimes \mathcal{N}') \\
&\simeq R\pi_*(\mathcal{D}_{X\to\mathbb{C}^2} \otimes \mathcal{N}'_{\widehat{E}}) \\
&\simeq \pi_+\mathcal{N}'_{\widehat{E}}
\end{aligned}$$

and similarly $\mathcal{M}_{\widehat{D}} \simeq \pi_+(\pi^+\mathcal{M})_{\widehat{E}}$. Hence, $\mathrm{iso} := \pi_+\,\mathrm{iso}'$ defines an isomorphism between $\mathcal{N}_{\widehat{D}}$ and $\mathcal{M}_{\widehat{D}}$. So $(\mathcal{N}, \mathrm{iso})$ is a $\mathcal{M}$-marked connection. By definition, the germ of $(\mathcal{N}, \mathrm{iso})$ at P is $(\mathcal{N}^*_P, \mathrm{iso}^*_P)$. Since $Rp_*\,\mathrm{St}^{<D}_{\mathcal{M}}$ is a local system on D^*, we deduce

$$(\mathcal{N}_{|D^*}, \mathrm{iso}_{|D^*}) = (\mathcal{N}^*, \mathrm{iso}^*)$$

Hence, $(\mathcal{N}, \mathrm{iso})$ extends $(\mathcal{N}^*, \mathrm{iso}^*)$ in a neighbourhood of 0. So 5.1.1 is proved. □

Combining 4.3.4 with 5.1.1 and the following proposition will finish the proof of Theorem 1.

Proposition 5.1.2. — *If the perverse complex* $\mathrm{Irr}^*_D \operatorname{End}\mathcal{M}$ *is a local system on* D*, then* Φ *induces an isomorphism between each irreducible component of* $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ *and* $H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}})$*.*

Proof. — From [**BV89**], we know that $H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}})$ is an affine space. Since affine spaces in characteristic 0 have no non trivial finite étale covers, it is enough to prove that Φ is finite étale. From 2.5.18, the morphism Φ is a closed immersion. We are thus left to show that Φ is étale.

Etale morphisms between smooth schemes of finite type over $\mathbb{C}$ are those morphisms inducing isomorphisms on the tangent spaces. Hence, we are left to prove that $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ is smooth and that Φ induces isomorphisms on the tangent spaces. Let (M, iso) be a $\pi^+\mathcal{M}$-marked connection. From 2.8.3, an obstruction theory to lifting infinitesimally the Stokes torsor of (M, iso) is given by

$$H^2(E, \mathrm{Irr}^*_E \operatorname{End} M) \simeq H^2(D, \mathrm{Irr}^*_D\, \pi_+ \operatorname{End} M) \simeq 0$$

The first identification expresses the compatibility of irregularity with proper push-forward. From 4.3.3 applied to the $\operatorname{End}\mathcal{M}$-marked connection $(\pi_+ \operatorname{End} M, \pi_+\,\mathrm{iso})$, the perverse complex $\mathrm{Irr}^*_D\, \pi_+ \operatorname{End} M$ is a local system in a neighbourhood of 0 and concentrated in degree 1. This implies the vanishing. Hence, $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ is smooth at (M, iso). From 2.6.1, any $\mathbb{C}$-point of $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ is of the form (M, iso).

Thus, $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ is smooth.

Furthermore, we have a commutative diagram

$$\begin{array}{ccc}
T_{(M,\mathrm{iso})}H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}}) & \longrightarrow & T_{(M_P,\mathrm{iso}_P)}H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}}) \\
\wr\downarrow & & \downarrow\wr \\
H^1(E, \mathrm{Irr}^*_E \operatorname{End} M) & \longrightarrow & (\mathcal{H}^1 \mathrm{Irr}^*_D \operatorname{End} M)_P \\
\wr\downarrow & & \downarrow| \\
H^1(D, \mathrm{Irr}^*_D \pi_+ \operatorname{End} M) & \longrightarrow & (\mathcal{H}^1 \mathrm{Irr}^*_D \operatorname{End} M)_P \\
\wr\downarrow & & \downarrow| \\
H^0(D, \mathcal{H}^1 \mathrm{Irr}^*_D \pi_+ \operatorname{End} M) & \longrightarrow & (\mathcal{H}^1 \mathrm{Irr}^*_D \operatorname{End} M)_P
\end{array}$$

The first vertical maps are isomorphisms by 2.8.4. As already proved, $\mathrm{Irr}^*_D \pi_+ \operatorname{End} M$ is a local system concentrated in degree 1. Hence, the last vertical and the bottom arrows are isomorphisms. Thus, the tangent map of Φ at (M, iso) is an isomorphism. This finishes the proof of proposition 5.1.2, and thus the proof of Theorem 1.

□

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