Moduli of Stokes torsors and singularities of differential equations

Jean-Baptiste Teyssier

MODULI OF STOKES TORSORS AND SINGULARITIES OF DIFFERENTIAL EQUATIONS

by

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Abstract. — Let $\mathcal{M}$ be a meromorphic connection with poles along a smooth divisor D in a smooth algebraic variety. Let $\operatorname{Sol}\mathcal{M}$ be the solution complex of $\mathcal{M}$. We prove that the good formal structure locus of $\mathcal{M}$ coincides with the locus where the restrictions to D of $\operatorname{Sol}\mathcal{M}$ and $\operatorname{Sol}\operatorname{End}\mathcal{M}$ are local systems. By contrast to the very different natures of these loci (the first one is defined via algebra, the second one is defined via analysis), the proof of their coincidence is geometric. It relies on the moduli of Stokes torsors.

The main problematic of this paper is to understand how the geometry of the Stokes phenomenon in any dimension sheds light on the interplay between the singularities of a differential equation and the singularities of its solutions.

Consider an algebraic linear system $\mathcal{M}$ of differential equations with n variables

$$\frac{\partial X}{\partial x_i} = \Omega_i X \qquad i = 1, \dots, n$$

where Ω_i is a square matrix of size r with coefficients into the ring $\mathbb{C}[x_1, \dots, x_n][x_n^{-1}]$ of Laurent polynomials with poles along the hyperplane D in $\mathbb{C}^n$ given by $x_n = 0$. At a point away from D, the holomorphic solutions of the system $\mathcal{M}$ are fully understood by means of Cauchy's theorem. At a point of D, the situation is much more complicated. It is still the source of challenging unsolved problems. We call D *the singular locus of* $\mathcal{M}$. Two distinguished open subsets of D where the singularities of $\mathcal{M}$ are mild can be defined.

First, the set $\mathrm{Good}(\mathcal{M})$ of *good formal structure points of* $\mathcal{M}$ is the subset of D consisting of points P at the formal neighbourhood of which $\mathcal{M}$ admits a *good formal structure*. For P being the origin, and modulo ramification issues that will be neglected in this introduction, this means roughly that there exists a base change with coefficients in $\mathbb{C}[\![x_1, \dots, x_n]\!][x_n^{-1}]$ splitting $\mathcal{M}$ as a direct sum of well-understood systems easier to work with.

Good formal structure can always be achieved in the one variable case [**Sv00**]. It is desirable in general because it provides a concrete description of the system, at least formally at a point. In the higher variable case however, it was observed in [**Sab00**] that $\mathcal{M}$ may not have good formal structure at every point of D. Thus, the set $\mathrm{Good}(\mathcal{M})$ is a non trivial invariant of $\mathcal{M}$. As proved by André [**And07**], the set $\mathrm{Good}(\mathcal{M})$ is the complement in D of a Zariski closed subset F of D either purely of codimension 1 in D or empty. Traditionally, F is called the *Turning locus of* $\mathcal{M}$, by reference to the way the Stokes directions of $\mathcal{M}$ move along a small circle in D going around a turning point. In a sense, **the good formal structure locus of $\mathcal{M}$ is the open subset of D where the singularities of the system $\mathcal{M}$ are as simple as possible**.

To define the second distinguished subset of D associated to $\mathcal{M}$, let us view $\mathcal{M}$ as a $\mathcal{D}$-module, that is a module over the Weyl algebra of differential operators. Let us denote by $\mathrm{Sol}\,\mathcal{M}$ the solution complex of the analytification of $\mathcal{M}$. Concretely, $\mathcal{H}^0\,\mathrm{Sol}\,\mathcal{M}$ encodes the holomorphic solutions of our differential system while the higher cohomologies of $\mathrm{Sol}\,\mathcal{M}$ keep track of higher Ext groups in the category of $\mathcal{D}$-modules. As proved by Kashiwara [**Kas75**], the complex $\mathrm{Sol}\,\mathcal{M}$ is perverse. From a theorem of Mebkhout [**Meb90**], the restriction of $\mathrm{Sol}\,\mathcal{M}$ to D, that is, the *irregularity complex of* $\mathcal{M}$ *along* D, denoted by $\mathrm{Irr}^*_D\,\mathcal{M}$ in this paper, is also perverse. In particular, $(\mathrm{Sol}\,\mathcal{M})_{|D}$ is a local system on D away from a closed analytic subset of D. The *smooth locus of* $(\mathrm{Sol}\,\mathcal{M})_{|D}$ denotes the biggest open in D on which $(\mathrm{Sol}\,\mathcal{M})_{|D}$ is a local system. In a sense, **the smooth locus of $(\mathrm{Sol}\,\mathcal{M})_{|D}$ is the open subset of D where the singularities of the (derived) solutions of $\mathcal{M}$ are as simple as possible**.

As observed in [**Tey13**], the open set $\mathrm{Good}(\mathcal{M})$ is included in the smooth locus of $(\mathrm{Sol}\,\mathcal{M})_{|D}$ and $(\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_{|D}$. The reverse inclusion was conjectured in [**Tey13**, 15.0.5]. Coincidence of $\mathrm{Good}(\mathcal{M})$ with the smooth locus of $(\mathrm{Sol}\,\mathcal{M})_{|D}$ and $(\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_{|D}$ seems surprising at first sight, since goodness is an *algebraic* notion whereas $\mathrm{Sol}\,\mathcal{M}$ is *transcendental*. The main goal of this paper is to prove via *geometric* means the following

Theorem 1. — *Let X be a smooth complex algebraic variety. Let D be a smooth divisor in X. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Then, the good formal structure locus of $\mathcal{M}$ is the locus of D where* $(\mathrm{Sol}\,\mathcal{M})_{|D}$ *and* $(\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_{|D}$ *are local systems.*

Other criteria detecting good points of meromorphic connections are available in the literature. Let us mention André's criterion [**And07**, 3.4.1] in terms of specialisations of Newton polygons. Let us also mention Kedlaya's criterion [**Ked10**, 4.4.2] in terms of the variation of spectral norms under varying Gauss norms on the ring of formal power series. This criterion is numerical in nature. By contrast, the new criterion given by Theorem 1 is *transcendental*. Its sheaf theoretic flavour makes it possible to track the turning points in the cohomology of the irregularity complex. For an application of this observation, let us refer to Theorem 2 below.

The main tool at stake in the proof of Theorem 1 is geometric, via *moduli of Stokes torsors* [**Tey19**]. For a detailed explanation of the line of thoughts that brought them into the picture, let us refer to 2.1. Before stating an application of Theorem 1 (see Theorem 2 below), we explain how these moduli are used by giving the main ingredients of the proof of Theorem 1 in dimension 2. In that case, we have to show the goodness of a point $0 \in D$ given that $(\operatorname{Sol}\mathcal{M})_{|D}$ and $(\operatorname{Sol}\operatorname{End}\mathcal{M})_{|D}$ are local systems in a neighbourhood of 0. The main problem is to extend the good formal structure of $\mathcal{M}$ across 0. This decomposition can be seen as a system of linear differential equations $\mathcal{N}$ defined in a neighbourhood of a small disc Δ^* of D punctured at 0.

To show that $\mathcal{N}$ extends across 0, we first construct via Stokes torsors a moduli space $\mathcal{X}$ parametrizing very roughly systems defined in a neighbourhood of Δ and formally isomorphic to $\mathcal{M}$ along Δ. A distinguished point of $\mathcal{X}$ is given by $\mathcal{M}$ itself. Similarly, we construct a moduli space $\mathcal{Y}$ parametrizing roughly systems defined in a neighbourhood of Δ^* and formally isomorphic to $\mathcal{M}_{|\Delta^*}$ along Δ^*. Two distinguished points of $\mathcal{Y}$ are $\mathcal{M}_{|\Delta^*}$ and $\mathcal{N}$. Restriction from Δ to Δ^* provides a morphism of algebraic varieties $\mathbf{res} : \mathcal{X} \longrightarrow \mathcal{Y}$. The problem of extending $\mathcal{N}$ is then the problem of proving that **res** hits $\mathcal{N}$. The moduli $\mathcal{X}$ and $\mathcal{Y}$ have the wonderful property that the tangent map $T_{\mathcal{M}}\,\mathbf{res}$ of **res** at $\mathcal{M}$ is exactly the map

$$\Gamma(\Delta, \mathcal{H}^1 \operatorname{Sol}\operatorname{End}\mathcal{M}) \longrightarrow \Gamma(\Delta^*, \mathcal{H}^1 \operatorname{Sol}\operatorname{End}\mathcal{M})$$

associating to $s \in \Gamma(\Delta, \mathcal{H}^1 \operatorname{Sol}\operatorname{End}\mathcal{M})$ the restriction of s to Δ^*. In this geometric picture, the smoothness of $(\mathcal{H}^1 \operatorname{Sol}\operatorname{End}\mathcal{M})_{|D}$ around 0 thus translates into the fact that $T_{\mathcal{M}}\,\mathbf{res}$ *is an isomorphism of vector spaces.* Since $\mathcal{X}$ and $\mathcal{Y}$ are smooth, we deduce that **res** is *étale* at the point $\mathcal{M}$. Thus, the image of **res** in $\mathcal{Y}$ contains a non empty open set. We prove furthermore (see Theorem 3 below) that **res** is a closed immersion, so its image is closed in $\mathcal{Y}$. Since $\mathcal{Y}$ is irreducible, we conclude that **res** is surjective, which proves the existence of the sought-after extension of $\mathcal{N}$.

Let us now describe an application of Theorem 1. Let X be a smooth variety over a finite field of characteristic $p > 0$. Let $\ell \neq p$ be a prime number. As proved by Deligne [**EK12**], there is only a finite number of semi-simple ℓ-adic local systems on X with prescribed rank, bounded ramification at infinity and up to a twist by a character coming from the base field. A natural question is to look for a differential analogue of this finiteness result. Let X be a smooth complex proper algebraic variety. Let $\mathcal{M}$ be a meromorphic connection on X. In this situation, H. Esnault and A. Langer asked whether it is possible to control the resolution of turning points of $\mathcal{M}$ by means of X, the rank of $\mathcal{M}$ and the irregularity of $\mathcal{M}$. In dimension 2, this question amounts to bound the number of blow-ups needed to eliminate the turning points of $\mathcal{M}$. To the author's knowledge, this question is still widely open. If such a bound exists in dimension 2, the number of turning points of $\mathcal{M}$ should in particular be bounded by a quantity depending only on the surface X, the rank of $\mathcal{M}$ and the irregularity of $\mathcal{M}$. As an application of Theorem 1, we give such a bound in a relative situation, thus providing the first evidence for a positive answer to H. Esnault and A. Langer's question. This is the following

Theorem 2. — *Let S be a smooth complex algebraic curve. Let $0 \in S$. Let $p : \mathcal{C} \longrightarrow S$ be a relative smooth proper curve of genus g. Let $\mathcal{M}$ be a meromorphic connection on $\mathcal{C}$ with poles along the fibre $\mathcal{C}_0$ of p above 0. Let $r_D(\mathcal{M})$ be the highest generic* *slope of $\mathcal{M}$ along $\mathcal{C}_0$. Then, the number of turning points of $\mathcal{M}$ along $\mathcal{C}_0$ is bounded by* $8(\operatorname{rank}\mathcal{M})^2(g+1)r_D(\mathcal{M})$.

To prove Theorem 2, the main tools are Theorem 1 and a new boundedness result for nearby slopes [**Tey16**] suggested by the ℓ-adic picture [**HT21**]. See remark 7.1.4 for details.

A crucial step in the proof of Theorem 1 is to understand the geometry of the restriction map for Stokes torsors. This is achieved by Theorem 3 below. To state it, let X be a smooth complex algebraic variety. Let D be a normal crossing divisor in X. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure in the sense of Mochizuki 1.5. Let $p_D : \widetilde{X} \longrightarrow X$ be the fibre product of the real blow-ups of X along the components of D. For every subset $A \subset D$, put $\partial A := p_D^{-1}(A)$. Let $\mathrm{St}_{\mathcal{M}}^{<D}$ be the Stokes sheaf of $\mathcal{M}$ (see section 2.3 for details). This is a sheaf of complex unipotent algebraic groups on ∂D. Then, we have the following

Theorem 3. — *Let $U \subset V \subset D$ be non empty open subsets in D such that V is connected. Then, the natural morphism*

$$H^1(\partial V, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D})$$

is a closed immersion of affine schemes of finite type over $\mathbb{C}$.

Let us finally give an application of Theorem 3 to degenerations of irregular singularities. Let X be a smooth algebraic variety and let D be a germ of smooth divisor at $0 \in X$. Let $\mathcal{M}$ be a germ of meromorphic connection defined in a neighbourhood of D in X and with poles along D. Motivated by Dubrovin's conjecture and the study of Frobenius manifolds, Cotti, Dubrovin and Guzzetti [**CDG19**] studied how much information on the Stokes data of $\mathcal{M}$ can be retrieved from the restriction of $\mathcal{M}$ to a smooth curve C transverse to D and passing through 0.

Under the assumption that $\mathcal{M}_{\hat{D}}$ splits as a direct sum of regular connections twisted by meromorphic functions $a_1, \ldots, a_n \in \mathcal{O}_X(*D)$ with simple poles along D, they proved that *the Stokes data of the restriction $\mathcal{M}_{|C}$ determine in a bijective way the Stokes data of $\mathcal{M}$ in a small neighbourhood of* 0 *in* D. This is striking, since the numerators of the $a_i - a_j$ may vanish at 0, thus inducing a discontinuity at 0 in the configuration of the Stokes directions. Using different methods, this was reproved by Sabbah in [**Sab**, Th 1.4]. In this paper, we give a short conceptual proof of a stronger version of Cotti, Dubrovin and Guzzetti's injectivity theorem: we don't make any assumption on the shape of $\mathcal{M}_{\hat{D}}$, nor do we suppose that D is smooth, nor do we assume that C is transverse to D. The price to pay for this generality is the use of resolution of turning points, as proved in the fundamental work of Kedlaya [**Ked11**] and Mochizuki [**Moc11b**]. The intuition that the techniques developed in this paper could be applied to the questions considered by Cotti, Dubrovin and Guzzetti is due to C. Sabbah.

To state our result, let us recall that a *$\mathcal{M}$-marked connection* is the data of a pair (M, iso) where M is a germ of meromorphic connection with poles along D defined in a neighbourhood of D in X, and where iso : $M_{\widehat{D}} \longrightarrow \mathcal{M}_{\widehat{D}}$ is an isomorphism of formal connections.

Theorem 4. — *Let X be a germ of smooth algebraic variety around a point 0. Let D be a germ of divisor passing through 0. Let $\mathcal{M}$ be a germ of meromorphic connection at 0 with poles along D. Let C be a smooth curve passing through 0 and not contained in any of the irreducible components of D. If (M_1, iso_1) and (M_2, iso_2) are $\mathcal{M}$-marked connections such that*

$$(M_1, \mathrm{iso}_1)_{|C} \simeq (M_2, \mathrm{iso}_2)_{|C}$$

then (M_1, iso_1) and (M_2, iso_2) are isomorphic in a neighbourhood of 0.

Let us give an outline of the paper. In section 1, we introduce some background material on asymptotic analysis and on the Stokes sheaf. In section 2, we introduce the sheaf of relative Stokes torsors and prove its constructibility. In section 3, we prove the representability of the moduli of Stokes torsors. We then prove Theorem 3. In section 4, we interpret the tangent spaces and the obstruction theory for these moduli in a transcendental way via the solution complexes for connections. We then prove Theorem 4. In section 5, we show how to reduce the proof of Theorem 1 to extending the good formal model of $\mathcal{M}$ across the point 0 under study. In section 6, we show that the sought-after extension exists provided that the moduli of Stokes torsors associated to a resolution of the turning point 0 for $\mathcal{M}$ satisfies suitable geometric conditions. Finally, we show that these geometric conditions are always satisfied when the hypothesis of Theorem 1 are satisfied, thus concluding the proof of Theorem 1. Section 7 is devoted to the proof of Theorem 2. We collect in an appendix some elementary facts about torsors and Stokes filtered local systems. Note that our use of Stokes filtered local systems in this paper is a purely technical detour to obtain the triviality criterion 8.4.1.

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1. The Stokes sheaf. Global aspects

1.1. Geometric setup. — In this subsection, we introduce basic notations. In this paper, a *regular pair* (X, D) will be the data of a smooth complex algebraic variety

X and of a strict normal crossing divisor D in X. For a quasi-coherent sheaf $\mathcal{F}$ on X, we denote by $\mathcal{F}_{|D}$ the sheaf of germs of sections of $\mathcal{F}$ along D. Let $D_1, \dots, D_m$ be the irreducible components of D. For $I \subset [\![1, m]\!]$, set

$$D_I := \bigcap_{i \in I} D_i \text{ and } D_I^\circ := D_I \backslash \bigcup_{i \notin I} D_i$$

1.2. Functions with asymptotic expansion along D. — For $i = 1, \dots, m$, let $\widetilde{X}_i \longrightarrow X$ be the real blow-up of X along D_i. Let $p_D : \widetilde{X} \longrightarrow X$ be the fibre product of the $\widetilde{X}_i$, $i = 1, \dots, m$ above X. For every subset $A \subset D$, put $\partial A := p_D^{-1}(A)$. Let $\iota_A : \partial A \longrightarrow \partial D$ be the canonical inclusion.

Let $\mathcal{A}$ be the sheaf of functions on ∂D admitting an asymptotic expansion along D [**Sab00**]. For a closed subset Z in D, let $\mathcal{A}_{\widehat{Z}}$ be the completion of $\mathcal{A}$ along the pull-back by p_D of the ideal sheaf of Z. Put $\mathcal{A}^{<Z} := \mathrm{Ker}(\mathcal{A} \longrightarrow \mathcal{A}_{\widehat{Z}})$. When $Z = D$, the sheaf $\mathcal{A}^{<D}$ can be concretely described locally as follows (see [**Sab00**, II 1.1.11] for a proof). Let $(x_1, \dots, x_n)$ be local coordinates centred at $0 \in D$ such that D is defined around 0 by $x_1 \cdots x_l = 0$ for some $l \in [\![1, m]\!]$. Then, the germ of $\mathcal{A}^{<D}$ at $Q \in \partial 0$ is given by those holomorphic functions u defined over the trace on $X \backslash D$ of a neighbourhood Ω of Q in $\widetilde{X}$, and such that for every compact $K \subset \Omega$, for every $(N_1, \dots, N_l) \in \mathbb{N}^l$, there exists a constant $C_{K,N} > 0$ satisfying

$$|u(x)| \leqslant C_{K,N} |x_1|^{N_1} \cdots |x_l|^{N_l} \text{ for every } x \in K \cap (X \backslash D) \tag{1.2.1}$$

From the formula (1.2.1), we deduce the following elementary lemma.

Lemma 1.2.2. — *Let $\rho : Y \longrightarrow X$ be a cyclic Galois cover of X ramified along D. Put $E = \rho^{-1}(D)$. Let $\widetilde{\rho} : \widetilde{Y} \longrightarrow \widetilde{X}$ be the map induced by ρ at the level of the real blow-up. Then, the canonical map $\mathcal{A}^{<D} \longrightarrow \widetilde{\rho}_* \mathcal{A}^{<E}$ induces an isomorphism between $\mathcal{A}^{<D}$ and the sheaf of invariants of $\widetilde{\rho}_* \mathcal{A}^{<E}$ under the Galois group of ρ.*

1.3. Good formal structure. — Let (X, D) be a regular pair. Meromorphic connections in this paper will be supposed to be flat. Let P be a point in D. An *elementary local model at P* denotes a meromorphic connection $\mathcal{N}$ of the form

$$\mathcal{N} = \bigoplus_{a \in I_P} \mathcal{E}^a \otimes \mathcal{R}_a$$

where I_P is a finite set in $\mathcal{O}_{X,P}(*D)$, where $\mathcal{E}^a = (\mathcal{O}_{X,P}(*D), d - da)$ and where $\mathcal{R}_a$ is a regular singular meromorphic connection on X with poles along D. If furthermore the following conditions are satisfied

(1) For an element a in I_P, if a does not belong to $\mathcal{O}_{X,P}$, then the divisor of a is anti-effective with support in D,

(2) For an element a, b in I_P, if $a - b$ does not belong to $\mathcal{O}_{X,P}$, then the divisor of $a - b$ is anti-effective with support in D,

we say that $\mathcal{N}$ is a *good elementary local model at P*. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Let Y be the stratum of D containing P. Following [**Ked10**, 6.2.3], we say that $\mathcal{M}$ has an *elementary local model at P* if at the cost of shrinking X, there exists an elementary local model $\mathcal{N}$ at P and an isomorphism of connections

$$\mathcal{O}_{\widehat{X|Y}} \otimes_{\mathcal{O}_X} \mathcal{M} \simeq \mathcal{O}_{\widehat{X|Y}} \otimes_{\mathcal{O}_X} \mathcal{N} \tag{1.3.1}$$

If furthermore $\mathcal{N}$ is a good elementary local model at P, we say that $\mathcal{M}$ has a *good elementary local model at* P. We say that $\mathcal{M}$ has a *good formal structure at* P if there exists a cyclic Galois cover of some neighbourhood of P, ramified along D, on which the pull-back of $\mathcal{M}$ admits a good elementary local model at some inverse image of P. If this is true for every point P in D, we say that $\mathcal{M}$ has a *good formal structure*.

1.4. Irregular values. — Let (X, D) be a regular pair. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. If $\mathcal{M}$ has a good elementary local model at every point P in D, the images by $\mathcal{O}_X(*D) \longrightarrow \mathcal{O}_X(*D)/\mathcal{O}_X$ of the finite sets $\mathcal{I}_P$ appearing in 1.3 organize into a subsheaf of $\mathcal{O}_X(*D)/\mathcal{O}_X$ on D. This is the *sheaf of irregular values of* $\mathcal{M}$. Let us denote it by $\mathcal{I}$. We say that $\mathcal{I}$ is *very good* if for every point P in D, the difference of any two distinct elements of $\mathcal{I}_P$ has poles along *every* component of D passing through P. These definitions extend in a straighforward way to the case where $\mathcal{M}$ has good formal structure. See [**Sab12**, 9.c] for details.

1.5. Mochizuki's definition of good formal structure and its use in the paper. — As pointed out in [**Ked10**, 4.3.3], there is a small discrepancy between the notion of good formal structure in [**Ked10**] and that from [**Moc11b**]. Mochizuki further requires that the sets I_P appearing in the decomposition (1.3.1) satisfy the extra assumption that the $\mathcal{O}_X$-modules generated by the differences $a-b, a, b \in I_P$ not lying in $\mathcal{O}_X$ are totally ordered under containment. If $\mathcal{M}$ has good formal structure in the sense of Mochizuki, then $\mathcal{M}$ has good formal structure in the sense of Kedlaya. Due to [**Ked10**, 4.3.1], if $\mathcal{M}$ and $\operatorname{End}\mathcal{M}$ have good formal structure in the sense of Kedlaya, then $\mathcal{M}$ has good formal structure in the sense of Mochizuki.

Note that these notions coincide when D is smooth. Hence, this discrepancy is invisible in the statement of Theorem 1. However, the notion of goodness used in Theorem 3, Theorem 5 and Theorem 6 is that of Mochizuki. It will be needed to ensure that for every point P in D, there exists a component Z of D passing through P such that for every $a, b \in I_P$ distinct, $a - b$ has poles along Z. If not explicitly mentioned otherwise, good formal structure will be taken in the sense of Kedlaya.

1.6. The Stokes sheaf. — Let (X, D) be a regular pair. Let $\mathcal{M}$ be a meromorphic connection defined on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure. We set

$$\partial\mathcal{M} = \mathcal{A} \otimes_{p_D^{-1}\mathcal{O}_{X|D}} p_D^{-1}\mathcal{M}$$

and

$$\partial\mathcal{M}_{\widehat{D}} = \mathcal{A}_{\widehat{D}} \otimes_{p_D^{-1}\mathcal{O}_{X|D}} p_D^{-1}\mathcal{M}$$

Let $\mathcal{D}_X$ be the sheaf of differential operators on X. The sheaf $\mathcal{A}$ is endowed with an action of $p_D^{-1}\mathcal{D}_{X|D}$. Hence, so does $\partial\mathcal{M}$. We can thus form the De Rham complex of

$\mathcal{M}$ with coefficients in $\mathcal{A}$ as

$$\partial\mathcal{M} \longrightarrow p_D^{-1}\Omega^1_{X|D} \otimes_{p_D^{-1}\mathcal{O}_{X|D}} \partial\mathcal{M} \longrightarrow \cdots \longrightarrow p_D^{-1}\Omega^n_{X|D} \otimes_{p_D^{-1}\mathcal{O}_{X|D}} \partial\mathcal{M}$$

It is denoted by $\mathrm{DR}\,\partial\mathcal{M}$. Similarly, we denote by $\mathrm{DR}^{<D}\mathcal{M}$ the De Rham complex of $\mathcal{M}$ with coefficients in $\mathcal{A}^{<D}$.

Let Z be a closed subset of D. Let $\mathrm{St}^{<Z}_{\mathcal{M}}$ be the subsheaf of $\mathcal{H}^0\,\mathrm{DR}\,\partial\,\mathrm{End}\,\mathcal{M}$ of sections asymptotic to the Identity along Z, that is of the form $\mathrm{Id}+f$ where f has coefficients in $\mathcal{A}^{<Z}$. The sheaf $\mathrm{St}^{<Z}_{\mathcal{M}}$ is a sheaf of complex unipotent algebraic groups on ∂Z. This is *the Stokes sheaf of $\mathcal{M}$ along Z*. For every $\mathbb{C}$-algebra R, the sheaf of R-points of $\mathrm{St}^{<Z}_{\mathcal{M}}$ is a sheaf of groups on ∂Z. It is denoted by $\mathrm{St}^{<Z}_{\mathcal{M}}(R)$. This is the *Stokes sheaf of $\mathcal{M}$ along Z relative to R*.

1.7. The Stokes locus. — Let (X, D) be a regular pair. Let $p_D : \widetilde{X} \longrightarrow X$ be the real blow-up of X along D. Let $\mathcal{M}$ be a meromorphic connection defined on X with poles along D. Suppose that $\mathcal{M}$ has a good elementary local model at every point of D. Let $\mathcal{I}$ be the sheaf of irregular values of $\mathcal{M}$. Let P be a point in D. Let $a, b \in \mathcal{I}_P$ distinct. Put $F_{a,b} := \mathrm{Re}(a-b)|x^{-\operatorname{ord}(a-b)}|$ where $(x_1, \ldots, x_n)$ are local coordinates centred at P such that D is given by $x_1 \cdots x_m = 0$. By definition, the *Stokes locus of* (a, b) is defined as $F_{a,b} \circ p_D = 0$. The *Stokes locus of $\mathcal{M}$* is the union of the loci of the form $F_{a,b} \circ p_D = 0$, where a, b are as above. If $\mathcal{M}$ is ramified, then at the cost of shrinking X, there exists a cyclic Galois cover $\rho : Y \longrightarrow X$ ramified along D such that $\rho^+\mathcal{M}$ is unramified. Let $\widetilde{\rho} : \widetilde{Y} \longrightarrow \widetilde{X}$ be the map induced by ρ at the level of the real blow-up. Then the Galois group G of ρ acts on $\mathcal{I}$. Hence, the action of G on $\widetilde{Y}$ preserves the Stokes locus of $\rho^+\mathcal{M}$. Thus, the Stokes locus of $\rho^+\mathcal{M}$ descends to a closed subset in ∂D, called the *Stokes locus of $\mathcal{M}$*. This locus depends only on $\mathcal{M}$ and not on the choice of ρ.

1.8. Some facts on the Stokes sheaf. — Let (X, D) be a regular pair. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure. As a consequence of lemma 1.2.2, we have the following

Lemma 1.8.1. — *Let $\rho : Y \longrightarrow X$ be a cyclic Galois cover of X ramified along D. Put $E = \rho^{-1}(D)$. Let $\widetilde{\rho} : \widetilde{Y} \longrightarrow \widetilde{X}$ be the map induced by ρ at the level of the real blow-up. Then, $\widetilde{\rho}^{-1}\,\mathrm{St}^{<D}_{\mathcal{M}} \simeq \mathrm{St}^{<E}_{\rho^+\mathcal{M}}$ and the canonical map $\mathrm{St}^{<D}_{\mathcal{M}} \longrightarrow \widetilde{\rho}_*\,\mathrm{St}^{<E}_{\rho^+\mathcal{M}}$ induces an isomorphism between $\mathrm{St}^{<D}_{\mathcal{M}}$ and the sheaf of invariants of $\widetilde{\rho}_*\,\mathrm{St}^{<E}_{\rho^+\mathcal{M}}$ under the Galois group of ρ.*

Lemma 1.8.2. — *The Stokes sheaf of $\mathcal{M}$ is constructible with respect to the stratification of ∂D induced by the Stokes locus of $\mathcal{M}$.*

Proof. — From lemma 1.8.1, we can suppose that $\mathcal{M}$ is unramified. The question is local on ∂D. From Mochizuki's asymptotic development theorem [**Sab12**, 12.5], we can further suppose that $\mathcal{M}$ is a good elementary local model at a point P. In that case, let us write

$$\mathcal{M} = \bigoplus_{a\in\mathcal{I}} \mathcal{E}^a \otimes \mathcal{R}_a$$

where $\mathcal{I}$ is a good set of irregular values at P, where $\mathcal{E}^a = (\mathcal{O}_X(*D), d - da)$ and where $\mathcal{R}_a$ is regular with poles along D. Let $i_a : \mathcal{E}^a \otimes \mathcal{R}_a \longrightarrow \mathcal{N}$ be the canonical inclusion and let $p_a : \mathcal{N} \longrightarrow \mathcal{E}^a \otimes \mathcal{R}_a$ be the canonical projection. Sections of $\mathrm{St}^{<D}_{\mathcal{M}}$ on an open set $\mathcal{S}$ are automorphisms of $\mathcal{M}$ on $\mathcal{S} \cap (X \backslash D)$ of the form $\mathrm{Id} + f$ where $p_a f i_b = 0$ unless

$$e^{a-b} \in \Gamma(\mathcal{S}, \mathcal{A}^{<D}) \tag{1.8.3}$$

Lemma 1.8.2 then follows from the observation that the condition (1.8.3) is constant on each stratum of the stratification of ∂D induced by the Stokes locus of (a, b). □

Lemma 1.8.4. — *Let (X, D) be a germ of regular pair at a point P. Let $D_1, \dots, D_m$ be the components of D. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure. Let $\rho : Y \longrightarrow X$ be a cyclic Galois cover of X ramified along D such that $\rho^+\mathcal{M}$ is unramified. Suppose that the difference of any two distinct irregular values for $\mathcal{M}$ at P has poles along $\rho^{-1}(D_m)$. Put $I = [\![1, m-1]\!]$. Then, the adjunction morphism*

$$\iota_{D_I}^{-1} \mathrm{St}^{<D}_{\mathcal{M}} \longrightarrow \iota_{D_I}^{-1} \iota_{D_I^\circ *} \iota_{D_I^\circ}^{-1} \mathrm{St}^{<D}_{\mathcal{M}} \tag{1.8.5}$$

is an isomorphism.

Proof. — From lemma 1.8.1, we can suppose that $\mathcal{M}$ is unramified. Injectivity is obvious, so we are left to prove surjectivity at a point Q in ∂P. This is a local question around Q. From Mochizuki's asymptotic development theorem [**Sab12**, 12.5], we can thus suppose that $\mathcal{M}$ is a good elementary local model. Let us write

$$\mathcal{M} = \bigoplus_{a \in \mathcal{I}} \mathcal{E}^a \otimes \mathcal{R}_a$$

where $\mathcal{I}$ is a good set of irregular values at P, where $\mathcal{E}^a = (\mathcal{O}_X(*D), d - da)$ and where $\mathcal{R}_a$ is regular with poles along D. Let $i_a : \mathcal{E}^a \otimes \mathcal{R}_a \longrightarrow \mathcal{N}$ be the canonical inclusion and let $p_a : \mathcal{N} \longrightarrow \mathcal{E}^a \otimes \mathcal{R}_a$ be the canonical projection. Let $\mathcal{S}$ be a neighbourhood of Q in ∂X of the form

$$([0, r[\times I_1) \times \cdots \times ([0, r[\times I_m) \times \Delta$$

where $r > 0$, where Δ is a ball in $\mathbb{C}^{n-m}$ centred at 0 and where $I_1, \dots, I_m$ are closed intervals in S^1. To prove the surjectivity of (1.8.5) at Q, it is enough to show that at the cost of shrinking $\mathcal{S}$, the restriction morphism

$$\Gamma(\mathcal{S} \cap \partial D, \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow \Gamma(\mathcal{S} \cap \partial D_I^\circ, \mathrm{St}^{<D}_{\mathcal{M}})$$

is a bijection. Sections of $\mathrm{St}^{<D}_{\mathcal{M}}$ on $\mathcal{S} \cap \partial D$ are automorphisms of $\mathcal{M}$ on $\mathcal{S} \cap (X \backslash D)$ of the form $\mathrm{Id} + f$ where $p_a f i_b = 0$ unless

$$e^{a-b} \in \Gamma(\mathcal{S} \cap \partial D, \mathcal{A}^{<D}) \tag{1.8.6}$$

Sections of $\mathrm{St}^{<D}_{\mathcal{M}}$ on

$$\mathcal{S} \cap \partial D_I^\circ = (\{0\} \times I_1) \times \cdots \times (\{0\} \times I_{m-1}) \times (]0, r[\times I_m) \times \Delta$$

are automorphisms of $\mathcal{M}$ on $\mathcal{S} \cap (X \backslash D)$ of the form $\mathrm{Id} + f$ where $p_a f i_b = 0$ unless

$$e^{a-b} \in \Gamma(\mathcal{S} \cap \partial D_I^\circ, \mathcal{A}^{<D}) \tag{1.8.7}$$

We thus have to show that for every distinct $a, b \in \mathcal{I}$, the conditions (1.8.6) and (1.8.7) are equivalent for a small enough choice of $\mathcal{S}$. A change of variable reduces the problem to the case where $a - b = 1/x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ where $(\alpha_1, \dots, \alpha_m) \in \mathbb{N}^{m-1} \times \mathbb{N}^*$. Note that condition (1.8.6) trivially implies condition (1.8.7). Suppose that $e^{1/x_1^{\alpha_1} \cdots x_m^{\alpha_m}}$ lies in $\Gamma(\mathcal{S} \cap \partial D_I^\circ, \mathcal{A}^{<D})$. At the cost of shrinking $\mathcal{S}$, we can suppose that there exists a constant $C > 0$ such that for every

$$(x_1, \dots, x_n) \in (]0, r[\times I_1) \times \cdots \times (]0, r[\times I_{m-1}) \times ([\frac{r}{2}, r[\times I_m) \times \Delta$$

we have

$$|e^{1/x_1^{\alpha_1} \cdots x_m^{\alpha_m}}| \leqslant C|x_1| \cdots |x_{m-1}|$$

Writing $x_i = (r_i, \theta_i)$ for $i = 1, \dots, m$, this means

$$e^{\cos(\alpha_1\theta_1 + \cdots + \alpha_m\theta_m)/r_1^{\alpha_1} \cdots r_m^{\alpha_m}} \leqslant Cr_1 \cdots r_{m-1}$$

In particular, $\alpha_i > 0$ for $i = 1, \dots, m-1$ and $\cos(\alpha_1\theta_1 + \cdots + \alpha_m\theta_m) < 0$ for every $(\theta_1, \dots, \theta_m) \in I_1 \times \cdots \times I_m$. At the cost of shrinking $\mathcal{S}$ further, there exists $c > 0$ such that $\cos(\alpha_1\theta_1 + \cdots + \alpha_m\theta_m) < -c$ on $I_1 \times \cdots \times I_m$. Then, we have

$$|e^{1/x_1^{\alpha_1} \cdots x_m^{\alpha_m}}| \leqslant e^{-c/|x_1|^{\alpha_1} \cdots |x_m|^{\alpha_m}}$$

on $\mathcal{S}$. Since $\alpha_i > 0$ for $i = 1, \dots, m$, we deduce that (1.8.7) holds. This proves the equivalence between conditions (1.8.6) and (1.8.7) and thus finishes the proof of lemma 1.8.4. □

2. Stokes torsors

2.1. Why moduli of Stokes torsors?— Let us explain in this subsection how the moduli of Stokes torsors were found to be relevant to the proof of Theorem 1. We use the notations from the introduction and work in dimension 2. We suppose that $0 \in D$ lies in the smooth locus of $(\mathrm{Sol}\,\mathcal{M})_{|D}$ and $(\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_{|D}$, and we want to prove that 0 is a good formal structure point for $\mathcal{M}$.

From a theorem of Kedlaya [**Ked10**][**Ked11**] and Mochizuki [**Moc09**][**Moc11b**], our connection $\mathcal{M}$ acquires good formal structure at any point after pulling-back by a suitable sequence of blow-ups above D. To test the validity of the conjecture [**Tey13**, 15.0.5], a natural case to consider was the case where only one blow-up is needed. Using results of André [**And07**], it was shown in [**Tey14**] that the conjecture reduces in this case to the following

Question. — *Given two good meromorphic connections $\mathcal{M}$ and $\mathcal{N}$ with poles along the coordinate axis in $\mathbb{C}^2$ and formally isomorphic at 0, is it true that*

$$\dim(\mathcal{H}^1\,\mathrm{Sol}\,\mathrm{End}\,\mathcal{M})_0 = \dim(\mathcal{H}^1\,\mathrm{Sol}\,\mathrm{End}\,\mathcal{N})_0 \quad ? \tag{2.1.1}$$

It turns out that each side of (2.1.1) appeared as dimensions of moduli spaces of Stokes torsors constructed by Babbitt-Varadarajan in [**BV89**]. These moduli were associated with germs of meromorphic connections in dimension 1. Babbitt and Varadarajan proved that they are affine spaces. This suggested the existence of a moduli $\mathcal{X}$ with two points $P, Q \in \mathcal{X}$ such that the left-hand side of (2.1.1) would be

$\dim T_P\mathcal{X}$ and the right-hand side of (2.1.1) would be $\dim T_Q\mathcal{X}$. The equality (2.1.1) would then follow from the smoothness and connectedness of the putative moduli. This is what led to [**Tey19**], but the question of smoothness and connectedness was left open. In the meantime, a positive answer to the above question was given by purely analytic means by C. Sabbah in [**Sab17**].

2.2. Relation with [Tey19]. — In [**Tey19**], a moduli for local Stokes torsors was constructed in any dimension. This moduli suffers two drawbacks in view of the proof of Theorem 1. First, the Stokes sheaf used in [**Tey19**] only makes sense at a neigbourhood of a point, whereas our situation will be global as soon as we apply Kedlaya-Mochizuki's resolution of turning points. Second, the relation between Irregularity and the tangent spaces of the moduli from [**Tey19**] only holds in particular cases. To convert the hypothesis on Irregularity appearing in Theorem 1 into a geometric statement pertaining to moduli of torsors, we need to replace the Stokes sheaf $\mathrm{St}_{\mathcal{M}}$ of a connection $\mathcal{M}$ by a subsheaf denoted by $\mathrm{St}_{\mathcal{M}}^{<D}$. We will abuse terminology be also calling the torsors under $\mathrm{St}_{\mathcal{M}}^{<D}$ Stokes torsors. The sheaf $\mathrm{St}_{\mathcal{M}}^{<D}$ has the advantage of being globally defined when $\mathcal{M}$ is globally defined. Along the smooth locus of D, the sheaf $\mathrm{St}_{\mathcal{M}}^{<D}$ is the usual Stokes sheaf. The only difference between $\mathrm{St}_{\mathcal{M}}$ and $\mathrm{St}_{\mathcal{M}}^{<D}$ appears at a singular point of D.

2.3. The functor of relative Stokes torsors. — We use the notations from 1.6. Let R be a $\mathbb{C}$-algebra. Torsors under $\mathrm{St}_{\mathcal{M}}^{<Z}(R)$ are the *Stokes torsors along Z relative to R*. For every subset $A \subset Z$, let $H^1(\partial A, \mathrm{St}_{\mathcal{M}}^{<Z})$ be the functor

$$\begin{array}{ccc} \mathbb{C}\text{-alg} & \longrightarrow & \mathrm{Set} \\ R & \longrightarrow & H^1(\partial A, \mathrm{St}_{\mathcal{M}}^{<Z}(R)) \end{array}$$

From [**Tey19**, Th. 1], the functor $H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<P})$ is an affine scheme of finite type over $\mathbb{C}$ for every point P in D.

Lemma 2.3.1. — *Let P be a point in D. Torsors under $\mathrm{St}_{\mathcal{M}}^{<P}$ on ∂P have no non trivial automorphisms.*

Proof. — The lemma 2.3.2 was proved in [**Tey19**, 1.8.1] in the case where $\mathcal{M}$ is a good elementary local model. An inspection of the proof, relying on Babbitt-Varadarajan representability theorem in dimension 1 as well as Malgrange-Sibuya theorem, shows that it carries over verbatim to the case of an arbitrary connection with good formal structure. □

Lemma 2.3.2. — *Let A be a subset in D. Torsors under $\mathrm{St}_{\mathcal{M}}^{<D}$ on ∂A have no non trivial automorphisms.*

Proof. — Let P be a point in A. It is enough to show that torsors under $\mathrm{St}_{\mathcal{M}}^{<D}$ on ∂P have no non trivial automorphisms. Let $\mathcal{T}$ be a $\mathrm{St}_{\mathcal{M}}^{<D}$-torsor on ∂P. Let $\phi : \mathcal{T} \longrightarrow \mathcal{T}$ be an automorphism of $\mathrm{St}_{\mathcal{M}}^{<D}$-torsors. Since $\mathcal{A}^{<D}$ is a subsheaf of $\mathcal{A}^{<P}$, there is an injection $\iota : \mathrm{St}_{\mathcal{M}}^{<D} \longrightarrow \mathrm{St}_{\mathcal{M}}^{<P}$. To show that ϕ is the identity of $\mathcal{T}$ amounts to show that the push-forward $\iota_*\phi : \iota_*\mathcal{T} \longrightarrow \iota_*\mathcal{T}$ is the identity of the $\mathrm{St}_{\mathcal{M}}^{<P}$-torsor $\iota_*\mathcal{T}$. This last assertion is a consequence of lemma 2.3.1. This finishes the proof of lemma 2.3.2. □

As a straighforward consequence of lemma 2.3.2, we deduce the following

Corollary 2.3.3. — *Let A be a subset in D. We endow A with the topology induced by D. Then, the presheaf of functors $R^1 p_{A*} \mathrm{St}_{\mathcal{M}}^{<D}$ defined as*

$$\begin{array}{ccc} \mathrm{Open}(A) & \longrightarrow & \mathrm{Set} \\ U & \longrightarrow & H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D}) \end{array}$$

is a sheaf of functors. That is, for every cover $\mathcal{U}$ of A by open subsets, the first arrow in the following diagram of pointed functors

$$H^1(\partial A, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow \prod_{U\in\mathcal{U}} H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D}) \rightrightarrows \prod_{U,V\in\mathcal{U}} H^1(\partial U \cap \partial V, \mathrm{St}_{\mathcal{M}}^{<D})$$

is an equalizer.

Remark 2.3.4. — *Observe that the sheaf condition in corollary* 2.3.3 *is still satisfied if one takes instead of a cover by open subsets in A a cover $\mathcal{K}$ by compact subsets $K \in \mathcal{K}$ such that the associated family of open subsets $\overset{\circ}{K}$ form a cover of A.*

As a consequence of 1.8.1, 2.3.4 and 8.1.1, we have the following

Corollary 2.3.5. — *Let (X, D) be a regular pair. Let $\rho : Y \longrightarrow X$ be a cyclic Galois cover of X ramified along D. Let G be the Galois group of ρ. Put $E = \rho^{-1}(D)$. Let A be a subset in D. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure. Then, the canonical morphism of functors*

$$H^1(\partial A, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial\rho^{-1}(A), \mathrm{St}_{\rho^+\mathcal{M}}^{<E})^G$$

is an isomorphism.

2.4. Dévissage of the sheaf of relative Stokes torsors. — The goal of this subsection is to relate Stokes torsors on a stratum with Stokes torsors on a suitably chosen stratum which is less deep. This will be done in proposition 2.4.2.

Lemma 2.4.1. — *Let (X, D) be a germ of regular pair at a point P. Let $D_1, \ldots, D_m$ be the components of D. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure. Let I be a subset in $[\![1, m]\!]$. For every element $\mathcal{T}$ in $H^1(\partial D_I^\circ, \mathrm{St}_{\mathcal{M}}^{<D})$, the sheaf $\iota_{D_I^\circ *}\mathcal{T}$ is a $\iota_{D_I^\circ *}\iota_{D_I^\circ}^{-1} \mathrm{St}_{\mathcal{M}}^{<D}$-torsor on ∂D.*

Proof. — Let Q be a point in ∂D. From lemma 8.1.2, we have to show the existence of a neighbourhood $\mathcal{S}$ of Q in $\widetilde{X}$ such that the Stokes torsors on $\mathcal{S} \cap \partial D_I^\circ$ are trivial. To do this, we can suppose that Q lies in ∂P. Let $\rho : Y \longrightarrow X$ be a cyclic Galois cover of X ramified along D such that $\rho^+\mathcal{M}$ is unramified. Let $\widetilde{\rho} : \widetilde{Y} \longrightarrow \widetilde{X}$ be the Galois cover induced by ρ at the level of the real blow-up. Let U be a neighbourhood of Q that decomposes $\widetilde{\rho}$. By pulling-back the situation to a connected component of $\widetilde{\rho}^{-1}(U)$, we reduce to the case where $\mathcal{M}$ is unramified. Since we are working in a neighbourhood of Q, Mochizuki's asymptotic development theorem [**Sab12**, 12.5] reduces the proof of lemma 2.4.1 to the case where $\mathcal{M}$ is a good elementary local model. Then, lemma 2.4.1 is a consequence of the triviality criterion 8.4.1. $\square$

Proposition 2.4.2. — *Let (X, D) be a germ of regular pair at a point P. Let $D_1, \ldots, D_m$ be the components of D. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure. Let $\rho : Y \longrightarrow X$ be a cyclic Galois cover of X ramified along D such that $\rho^+\mathcal{M}$ is unramified. Suppose that the difference of any two distinct irregular values for $\mathcal{M}$ at P has poles along $\rho^{-1}(D_m)$. Put $I = [\![1, m-1]\!]$. Then, the restriction morphism*

$$H^1(\partial D_I, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial D_I^\circ, \mathrm{St}_{\mathcal{M}}^{<D}) \tag{2.4.3}$$

is an isomorphism of functors.

Proof. — From lemma 2.4.1 and lemma 1.8.4, the functor $\iota_{D_I}^{-1}\iota_{D_I^\circ *}$ induces a well-defined morphism $H^1(\partial D_I^\circ, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial D_I, \mathrm{St}_{\mathcal{M}}^{<D})$, providing an inverse for (2.4.3). □

2.5. The stalks of the sheaf of Stokes torsors. —

Lemma 2.5.1. — *Let (X, D) be a regular pair. Let $D_1, \ldots, D_m$ be the irreducible components of D. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure. Let I, J be subsets in $[\![1, m]\!]$ with $J \subset I$. Let Z be a manifold in D_I°. Then Z admits a fundamental system of neighbourhood U in D_J such that the restriction morphism*

$$H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial Z, \mathrm{St}_{\mathcal{M}}^{<D}) \tag{2.5.2}$$

is an isomorphism.

Proof. — Let $\mathcal{I}$ be the sheaf of irregular values of $\mathcal{M}$. It is enough to prove that Stokes torsors on ∂Z and their morphisms extend uniquely over a neighbourhood of Z in D_J depending only on $\mathcal{I}$. To do this, we can suppose that Z is a point P and that (X, D) is a germ of regular pair at P. Similarly as in [**Tey19**, 1.9.1], the constructibility of $\mathrm{St}_{\mathcal{M}}^{<D}$ allows to construct a ball U in D_J of radius $r > 0$ centred at P and a cover $\mathcal{V}$ of ∂U by subsets V depending only on $\mathcal{I}$, of the form

$$\prod_{i \in J}(\{0\} \times I_i) \times \prod_{i \notin J}([0, r[\times I_i) \times \Delta$$

where $I_1, \ldots, I_m$ are closed intervals in S^1, where Δ is the ball of radius r centred at 0, and such that V trivializes every torsor under $\mathrm{St}_{\mathcal{M}}^{<D}$. At the cost of shrinking $\mathcal{V}$, the constructibility of $\mathrm{St}_{\mathcal{M}}^{<D}$ further allows to suppose that for every $V, W \in \mathcal{V}$, the maps

$$\Gamma(V, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow \Gamma(V \cap \partial P, \mathrm{St}_{\mathcal{M}}^{<D})$$

and

$$\Gamma(V \cap W, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow \Gamma(V \cap W \cap \partial P, \mathrm{St}_{\mathcal{M}}^{<D})$$

are bijective. For the above choice of ball U, the bijectivity of (2.5.2) follows. □

Corollary 2.5.3. — *Let (X, D) be a regular pair. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure. Let I be a subset in $[\![1, m]\!]$. Let $i_Z : Z \longrightarrow D_I^\circ$ be a manifold in D_I°. Then, the canonical morphism*

$$i_Z^{-1} R^1 p_{D*} \operatorname{St}_{\mathcal{M}}^{<D} \longrightarrow R^1 p_{Z*} \operatorname{St}_{\mathcal{M}}^{<D}$$

is an isomorphism.

Proof. — By definition, the sheaf $i_Z^{-1} R^1 p_{D*} \operatorname{St}_{\mathcal{M}}^{<D}$ is the sheaf associated to the presheaf

$$\begin{array}{rcl} \mathcal{F}_Z : \operatorname{Open}(Z) & \longrightarrow & \operatorname{Set} \\ V & \longrightarrow & \varinjlim\limits_{U \supset V} \Gamma(U, R^1 p_{D*} \operatorname{St}_{\mathcal{M}}^{<D}) \end{array}$$

By taking J to be the empty set in lemma 2.5.1, we observe that the above inductive limit identifies canonically with $H^1(\partial V, \operatorname{St}_{\mathcal{M}}^{<D}) = \Gamma(V, R^1 p_{Z*} \operatorname{St}_{\mathcal{M}}^{<D})$. Hence, we have

$$\mathcal{F}_Z \simeq R^1 p_{Z*} \operatorname{St}_{\mathcal{M}}^{<D}$$

From lemma 2.3.3, the presheaf $R^1 p_{Z*} \operatorname{St}_{\mathcal{M}}^{<D}$ is a sheaf. Corollary 2.5.3 thus follows. □

Corollary 2.5.4. — *Let (X, D) be a regular pair. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure. Let P be a point in D. Then, the stalk of $R^1 p_{D*} \operatorname{St}_{\mathcal{M}}^{<D}$ at P is canonically identified with $H^1(\partial P, \operatorname{St}_{\mathcal{M}}^{<D})$.*

2.6. Constructibility of the sheaf of Stokes torsors. —

Theorem 5. — *Let (X, D) be a regular pair. Let $D_1, \dots, D_m$ be the irreducible components of D. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure in the sense of Mochizuki. Then, the sheaf $R^1 p_{D*} \operatorname{St}_{\mathcal{M}}^{<D}$ is constructible on D. More precisely, for every subset $I \subset [\![1, m]\!]$, the restriction of $R^1 p_{D*} \operatorname{St}_{\mathcal{M}}^{<D}$ to D_I° is locally constant.*

Proof. — The statement is local along D. Hence, we can suppose that (X, D) is a germ of regular pair at a point P and that $I = [\![1, m]\!]$. From lemma 2.3.5, we can suppose that $\mathcal{M}$ is unramified. We argue recursively on m. The case where $m = 1$ will be treated last. Suppose that $m \geqslant 2$. Since $R^1 p_{D*} \operatorname{St}_{\mathcal{M}}^{<D}$ is a sheaf, to prove that its restriction to $D_I^\circ = D_I$ is a local system, it is enough to find a connected open neighbourhood U_I of P in $D_I^\circ = D_I$ such that any point Q in U_I admits a fundamental system of connected open neighbourhoods V_I such that

$$\Gamma(U_I, R^1 p_{D*} \operatorname{St}_{\mathcal{M}}^{<D}) \longrightarrow \Gamma(V_I, R^1 p_{D*} \operatorname{St}_{\mathcal{M}}^{<D})$$

is an isomorphism. From corollary 2.5.3, we have to find a connected open neighbourhood U_I of P in $D_I^\circ = D_I$ such that any point Q in U_I admits a fundamental system of connected open neighbourhoods V_I such that

$$H^1(\partial U_I, \operatorname{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial V_I, \operatorname{St}_{\mathcal{M}}^{<D})$$

is an isomorphism. Since $\mathcal{M}$ has good formal structure in the sense of Mochizuki, we can suppose that the difference of any two distinct irregular values of $\mathcal{M}$ at P has poles along D_m. Put $J = [\![1, m-1]\!]$. Let U_I be a small ball in D_I centred at P. From lemma 2.5.1, we can choose a ball U in D_J centred at P and containing U_I such that

$$H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial U_I, \mathrm{St}_{\mathcal{M}}^{<D})$$

is an isomorphism. Let Q be a point in U_I. Let V_I be a ball in U_I centred at Q. From lemma 2.5.1 again, we can choose a ball V in D_J centred at Q with $V_I \subset V \subset U_I$ and such that the morphism

$$H^1(\partial V, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial V_I, \mathrm{St}_{\mathcal{M}}^{<D})$$

is an isomorphism. Put $U_J^\circ = U \cap D_J^\circ$ and $V_J^\circ = V \cap D_J^\circ$. Then, we are left to prove that the middle vertical arrow in the commutative diagram

$$\begin{tikzcd} H^1(\partial U_I, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[d] & H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[l, "\sim"'] \arrow[r] \arrow[d] & H^1(\partial U_J^\circ, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[d] \\ H^1(\partial V_I, \mathrm{St}_{\mathcal{M}}^{<D}) & H^1(\partial V, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[l, "\sim"'] \arrow[r] & H^1(\partial V_J^\circ, \mathrm{St}_{\mathcal{M}}^{<D}) \end{tikzcd} \tag{2.6.1}$$

is an isomorphism. From lemma 2.4.2, the right horizontal arrows in (2.6.1) are isomorphisms. Hence, we are left to prove that the right vertical arrow in (2.6.1) is an isomorphism. By recursion assumption, the restriction of $R^1 p_{D*} \mathrm{St}_{\mathcal{M}}^{<D}$ to D_J° is a local system. We observe that the map $V_J^\circ \longrightarrow U_J^\circ$ is a product of the inclusion of two discs $\Delta_1^* \subset \Delta_2^*$ punctured at 0 with the inclusion of two balls $B_1 \subset B_2$ in $\mathbb{C}^{n-m}$. In particular, $V_J^\circ \longrightarrow U_J^\circ$ is a homotopy equivalence. Hence the right vertical arrow in (2.6.1) is an isomorphism. This concludes the reduction of the proof of Theorem 5 to the case where D is smooth. We now treat the case where D is smooth. The question is again local on D. Hence, we can suppose that (X, D) is a germ of smooth divisor at a point P. From lemma 2.3.5, we can suppose that $\mathcal{M}$ is unramified. This case was treated in [**Sab02**, II 6.3]. Alternatively, since D is smooth, the sheaf of irregular values for $\mathcal{M}$ is very good. Hence, lemma 8.3.2 reduces the proof of Theorem 5 with D smooth to the analogous statement for marked Stokes filtered local systems. This case follows from Mochizuki's extension Theorem 4.13 in [**Moc11a**]. □

3. The geometry of the moduli of Stokes torsors

3.1. Representability by a scheme. — The first goal of this section is to prove the following representability Theorem:

Theorem 6. — *Let (X, D) be a regular pair. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure in the sense of Mochizuki* 1.5. *Then, the functor $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ is representable by an affine scheme of finite type over $\mathbb{C}$.*

Proof. — The idea is to analyse separately the contributions coming from each stratum of D. Let $D_1, \ldots, D_m$ be the components of D. We argue by recursion on the depth of the deepest stratum of D. The case where D is smooth will be treated last. Let Z be the deepest stratum of D. From lemma 2.5.1, there is an open neighbourhood U of Z in ∂D such that the restriction morphism

$$H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial Z, \mathrm{St}^{<D}_{\mathcal{M}})$$

is an isomorphism. Put $V = D\backslash Z$. From corollary 2.3.3, we have

$$H^1(\partial D, \mathrm{St}^{<D}_{\mathcal{M}}) = H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}}) \times_{H^1(\partial(U\cap V), \mathrm{St}^{<D}_{\mathcal{M}})} H^1(\partial V, \mathrm{St}^{<D}_{\mathcal{M}})$$

By recursion assumption, the functors $H^1(\partial V, \mathrm{St}^{<D}_{\mathcal{M}})$ and $H^1(\partial(U\cap V), \mathrm{St}^{<D}_{\mathcal{M}})$ are affine schemes of finite type over $\mathbb{C}$. Hence, we are left to prove that $H^1(\partial Z, \mathrm{St}^{<D}_{\mathcal{M}})$ is an affine scheme of finite type over $\mathbb{C}$. To do this, we can suppose that Z is connected. Hence, at the cost of shrinking the situation to a small enough open neighbourhood of a connected component of Z, we can suppose that Z is $D_I^\circ = D_I$ for $I = [\![1, m]\!]$. From lemma 2.5.3, $H^1(\partial D_I^\circ, \mathrm{St}^{<D}_{\mathcal{M}})$ is the space of sections of the sheaf $R^1 p_{D*} \mathrm{St}^{<D}_{\mathcal{M}}$ on ∂D_I°. From Theorem 5, the restriction of $R^1 p_{D*} \mathrm{St}^{<D}_{\mathcal{M}}$ to D_I° is a local system. Hence, if B is a small ball in D_I° centred at point P, the functor $H^1(\partial D_I^\circ, \mathrm{St}^{<D}_{\mathcal{M}})$ is the functor of invariants for the action of $\pi_1(D_I^\circ, P)$ on $H^1(\partial B, \mathrm{St}^{<D}_{\mathcal{M}})$. That is, if $(\gamma_1, \dots, \gamma_N)$ denotes a set of generators for $\pi_1(D_I^\circ, P)$, the following diagram of functors

$$\begin{tikzcd}
H^1(\partial D_I^\circ, \mathrm{St}^{<D}_{\mathcal{M}}) \arrow[r] \arrow[d] & H^1(\partial B, \mathrm{St}^{<D}_{\mathcal{M}}) \arrow[d, "(\mathrm{Id}, \gamma_1, \dots, \gamma_N)"] \\
H^1(\partial B, \mathrm{St}^{<D}_{\mathcal{M}}) \arrow[r, "\text{Diagonal}"'] & H^1(\partial B, \mathrm{St}^{<D}_{\mathcal{M}})^{N+1}
\end{tikzcd} \tag{3.1.1}$$

is cartesian. To prove Theorem 6, we are thus left to prove that $H^1(\partial B, \mathrm{St}^{<D}_{\mathcal{M}})$ is an affine scheme of finite type over $\mathbb{C}$. In particular, we can suppose that (X, D) is a germ of regular pair at P. Since $\mathcal{M}$ is good in the sense of Mochizuki, the conditions of proposition 2.4.2 are satisfied. Put $J = [\![1, m-1]\!]$. From lemma 2.5.1, there is a small ball U in D_J centred at P such that

$$H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial B, \mathrm{St}^{<D}_{\mathcal{M}})$$

is an isomorphism. Hence we are left to prove that $H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}})$ is an affine scheme of finite type over $\mathbb{C}$. Hence, if we put $U_J^\circ = U \cap D_J^\circ$, proposition 2.4.2 implies that the restriction morphism

$$H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial U_J^\circ, \mathrm{St}^{<D}_{\mathcal{M}})$$

is an isomorphism. By recursion assumption, the functor $H^1(\partial U_J^\circ, \mathrm{St}^{<D}_{\mathcal{M}})$ is an affine scheme of finite type over $\mathbb{C}$. This concludes the reduction of Theorem 6 to the case where D is smooth. If D is smooth, we reduce using (3.1.1) and corollary 2.5.1 to prove that for a point P in D, the functor $H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}})$ is a scheme of affine type over $\mathbb{C}$. Let $i : C \longrightarrow X$ be a smooth curve in X transverse to D at P. Let $\iota : \widetilde{C} \longrightarrow \widetilde{X}$ be the morphism induced by C at the level of the real blow-up. Observe that ι induces an isomorphism above P. Since $\iota^{-1} \mathrm{St}^{<D}_{\mathcal{M}} \simeq \mathrm{St}^{<P}_{i^+\mathcal{M}}$, we deduce that

$$H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}}) \simeq H^1(\partial P, \mathrm{St}^{<P}_{i^+\mathcal{M}})$$

Hence, we are left to prove Theorem 6 in the one dimensional case. This case was treated by Babbit-Varadarajan [**BV89**]. This finishes the proof of Theorem 6. □

Remark 3.1.2. — *Note that in the case where $\mathcal{M}$ has rank two, the moduli of Stokes torsors whose existence is asserted by Theorem* 6 *is known to be an affine space* [**Tey20**].

The diagram (3.1.1) in the proof of Theorem 6 gives the following

Proposition 3.1.3. — *Let (X, D) be a regular pair. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure in the sense of Mochizuki* 1.5. *Let $D_1, \dots, D_m$ be the components of D. Let I be a subset of $[\![1, m]\!]$. Suppose that D_I° is connected. Let P be a point in D_I°. Then, the natural morphism*

$$H^1(\partial D_I^\circ, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D})$$

is a closed immersion.

We store the following immediate corollary of proposition 3.1.3 for later use.

Corollary 3.1.4. — *Let (X, D) be a regular pair. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure in the sense of Mochizuki* 1.5. *Let $D_1, \dots, D_m$ be the components of D. Let I be a subset of $[\![1, m]\!]$. Let $U \subset V \subset D_I^\circ$ be non empty open subsets in D_I° such that V is connected. Then, the natural morphism*

$$H^1(\partial V, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D})$$

is a closed immersion.

Proof. — Choose a point P in U. Then, there is a factorization

$$\begin{tikzcd} H^1(\partial V, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[r] \arrow[dr] & H^1(\partial U, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[d] \\ & H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \end{tikzcd}$$

From proposition 3.1.3, the diagonal arrow is a closed immersion between affine schemes. Hence, the horizontal arrow is a closed immersion. □

3.2. Passing from one stratum to an other stratum is a closed immersion. — The next proposition is the technical core of this paper.

Proposition 3.2.1. — *Let (X, D) be a germ of regular pair at a point P. Let $D_1, \dots, D_m$ be the components of D. Put $I = [\![1, m]\!]$. Let $i \in I$. Then, for a small enough ball Δ in D_i centred at P, the morphism of schemes*

$$H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial \Delta^*, \mathrm{St}_{\mathcal{M}}^{<D}) \tag{3.2.2}$$

is a closed immersion, where $\Delta^* = \Delta \setminus \bigcup_{j \in I \setminus \{i\}} D_j \subset D_i^\circ$.

Proof. — Let us first construct the morphism (3.2.2). From lemma 2.5.1, for a small enough ball Δ in D_i centred at P, the restriction morphism

$$H^1(\partial\Delta, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D})$$

is an isomorphism. Then, the morphism (3.2.2) is defined as the composition

$$H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \xleftarrow{\sim} H^1(\partial\Delta, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial\Delta^*, \mathrm{St}_{\mathcal{M}}^{<D})$$

Note that both functors appearing in (3.2.2) are affine schemes as a consequence of Theorem 6. Let $j : \partial\Delta^* \longrightarrow \partial D$ be the canonical inclusion. The sheaf of algebraic groups $\iota_{D_i}^{-1}\mathrm{St}_{\mathcal{M}}^{<D}$ is distinguished in $\mathrm{St}_{\mathcal{M}}^{<D_i}$. We thus have an exact sequence of sheaves of algebraic groups on ∂D_i

$$1 \longrightarrow \iota_{D_i}^{-1}\mathrm{St}_{\mathcal{M}}^{<D} \xrightarrow{\iota} \mathrm{St}_{\mathcal{M}}^{<D_i} \longrightarrow \mathcal{Q} \longrightarrow 1$$

There is an adjunction morphism

$$\iota_P^{-1}\mathrm{St}_{\mathcal{M}}^{<D_i} \longrightarrow \iota_P^{-1}j_*j^{-1}\mathrm{St}_{\mathcal{M}}^{<D_i} = \iota_P^{-1}j_*j^{-1}\mathrm{St}_{\mathcal{M}}^{<D} \tag{3.2.3}$$

Hence, there is a factorization

$$\begin{tikzcd} H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[r, "\iota_*"] \arrow[dr] & H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D_i}) \arrow[d] \\ & H^1(\partial\Delta^*, \mathrm{St}_{\mathcal{M}}^{<D}) \end{tikzcd} \tag{3.2.4}$$

From a similar argument to that in lemma 1.8.4, the adjunction morphism (3.2.3) is an isomorphism of sheaves on ∂P. Hence, the vertical arrow in (3.2.4) is an isomorphism of functors. Hence, $H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D_i})$ is an affine scheme of finite type over $\mathbb{C}$ and to prove proposition 3.2.1, it is enough to prove that

$$\iota_* : H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D_i})$$

is a closed immersion. From [**Fre57**, I.2], there is an exact sequence of pointed functors

$$H^0(\partial P, \mathcal{Q}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \xrightarrow{\iota_*} H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D_i}) \longrightarrow H^1(\partial P, \mathcal{Q}) \tag{3.2.5}$$

Let us prove that $H^0(\partial P, \mathcal{Q})$ is trivial. The complex of sheaves

$$\mathrm{St}_{\mathcal{M}}^{<D_i} \longrightarrow \partial\,\mathrm{End}\,\mathcal{M}_{\widehat{D}} \longrightarrow \partial\,\mathrm{End}\,\mathcal{M}_{\widehat{D_i}}$$

induces a sequence of sheaves

$$\mathcal{Q} \hookrightarrow \partial\,\mathrm{End}\,\mathcal{M}_{\widehat{D}} \longrightarrow \partial\,\mathrm{End}\,\mathcal{M}_{\widehat{D_i}} \tag{3.2.6}$$

By applying p_{D*} and then looking at the germs at P, we deduce from [**Sab00**, p44] the following sequence

$$0 \longrightarrow H^0(\partial P, \mathcal{Q}) \longrightarrow \mathrm{End}\,\mathcal{M}_{\widehat{D},P} \longrightarrow \mathrm{End}\,\mathcal{M}_{\widehat{D_i},P} \tag{3.2.7}$$

By flatness of $\mathrm{End}\,\mathcal{M}$ over $\mathcal{O}_X$, the second map in (3.2.7) is injective. Hence, $H^0(\partial P, \mathcal{Q})$ is trivial. Thus, the following diagram of functors

$$\begin{tikzcd} H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[r] \arrow[d, "\iota_*"'] & * \arrow[d] \\ H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D_i}) \arrow[r] & H^1(\partial P, \mathcal{Q}) \end{tikzcd} \tag{3.2.8}$$

is cartesian, where $*$ denotes the trivial $\mathcal{Q}$-torsor. If we knew that $H^1(\partial P, \mathcal{Q})$ is a scheme, we would directly obtain that ι_* is a closed immersion. This question does not seem to follow from the use of skeletons in [**Tey19**]. We will circumvent this problem with a group theoretic argument.

From lemma 1.8.2, the sheaf $\mathrm{St}_{\mathcal{M}}^{<D_i}$ is constructible with respect to the stratification of ∂D induced by the Stokes locus of $\mathcal{M}$. Hence, the same argument as in [**Tey19**, 1.9.1] applies. In particular, there exists a cover $\mathcal{U}$ of ∂P by open subsets such that the morphism of affine schemes

$$Z^1(\mathcal{U}, \mathrm{St}_{\mathcal{M}}^{<D_i}) \longrightarrow H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D_i}) \tag{3.2.9}$$

is surjective at the level of R-points for every $\mathbb{C}$-algebra R. From [**BV89**, 2.7.3], the morphism (3.2.9) admits a section. Composing this section with

$$Z^1(\mathcal{U}, \mathrm{St}_{\mathcal{M}}^{<D_i}) \longrightarrow Z^1(\mathcal{U}, \mathcal{Q})$$

gives rise to a commutative triangle of functors

$$\begin{tikzcd} H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D_i}) \arrow[r] \arrow[dr] & H^1(\partial P, \mathcal{Q}) \\ & Z^1(\mathcal{U}, \mathcal{Q}) \arrow[u] \end{tikzcd} \tag{3.2.10}$$

The algebraic group

$$G_{\mathcal{U}} := \prod\nolimits_{U \in \mathcal{U}} \Gamma(U, \mathcal{Q})$$

acts on $Z^1(\mathcal{U}, \mathcal{Q})$. Let

$$G_{\mathcal{U}} \longrightarrow Z^1(\mathcal{U}, \mathcal{Q}) \tag{3.2.11}$$

be the morphism of schemes obtained by restricting the action of $G_{\mathcal{U}}$ to the trivial cocycle. Since $H^0(\partial P, \mathcal{Q}) \simeq 0$, the morphism (3.2.11) is a monomorphism. There is a commutative diagram

$$\begin{tikzcd} H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D}) \arrow[d, "\iota_*"'] & G_{\mathcal{U}} \arrow[r] \arrow[d] & * \arrow[d] \\ H^1(\partial P, \mathrm{St}_{\mathcal{M}}^{<D_i}) \arrow[r] & Z^1(\mathcal{U}, \mathcal{Q}) \arrow[r] & H^1(\partial P, \mathcal{Q}) \end{tikzcd} \tag{3.2.12}$$

We would like to reduce the problem of proving that ι_* is a closed immersion to the problem of proving that (3.2.11) is a closed immersion. To do this, we would like to fill the left diagram in (3.2.12) into a cartesian square. Note that the right square in (3.2.12) may not be cartesian since there may be cocycles in $Z^1(\mathcal{U},\mathcal{Q})$ that are cohomologous to the trivial cocycle only after passing to a refinement of $\mathcal{U}$. To treat this problem, we argue by using the universal torsor under $\mathrm{St}_{\mathcal{M}}^{<D}$ on ∂P.

Let $\mathcal{T}^{\mathrm{univ}}$ be the universal torsor under $\mathrm{St}_{\mathcal{M}}^{<D}$ on ∂P. Let A be the ring of functions of $H^1(\partial P,\mathrm{St}_{\mathcal{M}}^{<D})$. From the commutativity of (3.2.10), the image γ of $\mathcal{T}^{\mathrm{univ}}$ in $Z^1(\mathcal{U},\mathcal{Q}(A))$ induces the trivial $\mathcal{Q}(A)$-torsor. Hence, there exists a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\gamma_{|\mathcal{V}}$ is cohomologous to the trivial cocycle, that is, such that $\gamma_{|\mathcal{V}}$ lies in the image of $G_{\mathcal{V}}(A)\longrightarrow Z^1(\mathcal{V},\mathcal{Q}(A))$. Hence, there is a commutative square

$$
\begin{tikzcd}
H^1(\partial P,\mathrm{St}_{\mathcal{M}}^{<D}) \arrow[r] \arrow[d, "\iota_*"'] & G_{\mathcal{V}} \arrow[d] \\
H^1(\partial P,\mathrm{St}_{\mathcal{M}}^{<D_i}) \arrow[r] & Z^1(\mathcal{V},\mathcal{Q})
\end{tikzcd}
\tag{3.2.13}
$$

This square is cartesian. Indeed, let F be the fibre product of $H^1(\partial P,\mathrm{St}_{\mathcal{M}}^{<D_i})$ with $G_{\mathcal{V}}$ over $Z^1(\mathcal{V},\mathcal{Q})$. By definition, there is a commutative diagram of functors

$$
\begin{tikzcd}
H^1(\partial P,\mathrm{St}_{\mathcal{M}}^{<D}) \arrow[r] \arrow[dr, "\iota_*"'] & F \arrow[d] \\
 & H^1(\partial P,\mathrm{St}_{\mathcal{M}}^{<D_i})
\end{tikzcd}
\tag{3.2.14}
$$

Since the right vertical arrow in (3.2.13) is a monomorphism, F is a sub-functor of $H^1(\partial P,\mathrm{St}_{\mathcal{M}}^{<D_i})$. Hence, all maps in (3.2.14) are inclusions of functors. We are thus left to prove that F is a sub-functor of $H^1(\partial P,\mathrm{St}_{\mathcal{M}}^{<D})$. This is an immediate consequence of the fact that $H^1(\partial P,\mathrm{St}_{\mathcal{M}}^{<D})$ is the functor of torsors $\mathcal{T}\in H^1(\partial P,\mathrm{St}_{\mathcal{M}}^{<D_i})$ inducing the trivial $\mathcal{Q}$-torsor.

Hence, to prove that ι_* is a closed immersion, we are left to show that (3.2.11) for $\mathcal{V}$ is a closed immersion. From the general theory of algebraic group actions, the map (3.2.11) factors as

$$G_{\mathcal{V}}\xrightarrow{\alpha}\mathcal{O}\xrightarrow{\beta}Z^1(\mathcal{V},\mathcal{Q})$$

where α is faithfully flat, where $\mathcal{O}$ is the orbit of the trivial cocycle under $G_{\mathcal{V}}$ and where β is an immersion of schemes. Since smoothness is a local property for the fppf topology [**SPD**, 05B5], the smoothness of $G_{\mathcal{V}}$ implies that $\mathcal{O}$ is smooth. By definition, α is an isomorphism at the level of $\mathbb{C}$-points. Hence, α is an isomorphism of varieties. We are thus left to show that $\mathcal{O}$ is closed in $Z^1(\mathcal{V},\mathcal{Q})$. It is enough to show that $\mathcal{O}$ is closed in $Z^1(\mathcal{V},\mathcal{Q})^{\mathrm{red}}$. From Kostant-Rosenlicht theorem [**Bor91**, I 4.10], it is enough to show that $G_{\mathcal{V}}$ is a unipotent algebraic group, which is a consequence of the fact that the Stokes sheaves are sheaves of unipotent algebraic groups. This concludes the proof of proposition 3.2.1.

□

3.3. Proof of Theorem 3. — Let $U \subset V \subset D$ be non empty open subsets in D such that V is connected. We want to show that the natural morphism

$$H^1(\partial V, \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}}) \tag{3.3.1}$$

is a closed immersion of affine schemes of finite type over $\mathbb{C}$. Let A be the set of open subsets U' in V containing U and such that the natural morphism

$$H^1(\partial U', \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}})$$

is a closed immersion. We want to show that A contains V. Note that A is not empty since it contains U. Let A' be a subset of A which is totally ordered for the inclusion. Let R be the ring of functions of $H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}})$. For $U' \in A'$, let $\mathcal{I}_{U'}$ be the ideal of functions of $H^1(\partial U', \mathrm{St}^{<D}_{\mathcal{M}})$ in $H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}})$. By assumption on A', the family of ideals $(\mathcal{I}_{U'})_{U' \in A'}$ is totally ordered for the inclusion. Hence, $\mathcal{I} := \bigcup_{U' \in \mathcal{I}_{U'}}$ is an ideal in R. Since R is noetherian, there exists $U'_0 \in A'$ such that $\mathcal{I} = \mathcal{I}_{U'_0}$. In particular, $\mathcal{I}_{U'} = \mathcal{I}_{U'_0}$ for every $U' \in A'$ containing U'_0. Set $V' := \bigcup_{U' \in A'} U'$. From lemma 2.3.3, we deduce

$$\begin{aligned} H^1(\partial V', \mathrm{St}^{<D}_{\mathcal{M}}) &\simeq \lim_{U' \in A'} H^1(\partial U', \mathrm{St}^{<D}_{\mathcal{M}}) \\ &\simeq \lim_{U' \in A', U'_0 \subset U'} H^1(\partial U', \mathrm{St}^{<D}_{\mathcal{M}}) \\ &\simeq H^1(\partial U'_0, \mathrm{St}^{<D}_{\mathcal{M}}) \end{aligned}$$

Thus, $V' \in A$. From Zorn lemma, we deduce that A admits a maximal element W. If W is closed in V, then we have $W = V$ by connectedness of V. Suppose now that W is not closed in V. Let $P \in \overline{W} \backslash W$ and let B be a small ball in V containing P and such that $H^1(\partial B, \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial P, \mathrm{St}^{<D}_{\mathcal{M}})$ is an isomorphism. Set $W' := W \cup B \subset V$. We are going to show that $W' \in A$, which contradicts the fact that W is maximal in A. From the factorization

$$H^1(\partial W', \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial W, \mathrm{St}^{<D}_{\mathcal{M}}) \longrightarrow H^1(\partial U, \mathrm{St}^{<D}_{\mathcal{M}}) \tag{3.3.2}$$

we are left to show that the first arrow in (3.3.2) is a closed immersion. From lemma 2.3.3, the following diagram

$$\begin{tikzcd} H^1(\partial W', \mathrm{St}^{<D}_{\mathcal{M}}) \arrow[r] \arrow[d] & H^1(\partial B, \mathrm{St}^{<D}_{\mathcal{M}}) \arrow[d] \\ H^1(\partial W, \mathrm{St}^{<D}_{\mathcal{M}}) \arrow[r] & H^1(\partial (W \cap B), \mathrm{St}^{<D}_{\mathcal{M}}) \end{tikzcd} \tag{3.3.3}$$

is cartesian. Hence, it is enough to show that the right vertical arrow in (3.3.3) is a closed immersion. Let $(P_n)_{n \in \mathbb{N}}$ be a sequence of points in W converging to P. Since W is open, the sequence $(P_n)_{n \in \mathbb{N}}$ can be supposed to lie in some D_i° for $i \in [\![1, m]\!]$. Let $\Delta \subset B$ be a small enough neighbourhood of P in D_i. Set $\Delta^* = \Delta \backslash \bigcup_{j \in I \backslash \{i\}} D_j \subset D_i^\circ$. From our choice for i, the open set $W \cap \Delta^*$ is not empty. We have the following commutative diagram

$$
\begin{tikzcd}
H^1(\partial B, \mathrm{St}^{<D}_{\mathcal{M}}) \arrow[r] \arrow[d] & H^1(\partial\Delta^*, \mathrm{St}^{<D}_{\mathcal{M}}) \arrow[d] \\
H^1(\partial(W\cap B), \mathrm{St}^{<D}_{\mathcal{M}}) \arrow[r] & H^1(\partial(W\cap\Delta^*), \mathrm{St}^{<D}_{\mathcal{M}})
\end{tikzcd}
\tag{3.3.4}
$$

From proposition 3.2.1, the top horizontal arrow in (3.3.4) is a closed immersion. From corollary 3.1.4, the right vertical arrow in (3.3.4) is a closed immersion. Hence, the left vertical arrow in (3.3.4) is a closed immersion. Hence, $W' \in A$, which contradicts the fact that W is maximal in A. Thus, $W = V \in A$, which finishes the proof of Theorem 3.

4. Stokes torsors and marked connections

4.1. Notations. — For a morphism of smooth complex varieties $\pi : Y \longrightarrow X$, we denote by π^+ the inverse image functor for $\mathcal{D}$-modules and by π_+ the direct image functor for $\mathcal{D}$-modules. For precise definitions, let us refer to [**HTT00**].

In this section, (X, D) will denote a regular pair. Let $\mathcal{M}$ be a connection on X with poles along D. Suppose that $\mathcal{M}$ has good formal structure.

4.2. Definition of marked connections and relation with Stokes torsors. — Let us recall that a $\mathcal{M}$*-marked connection* is the data of a pair (M, iso) where M is a germ of meromorphic connection with poles along D defined in a neighbourhood of D in X, and where $\mathrm{iso} : M_{\hat{D}} \longrightarrow \mathcal{M}_{\hat{D}}$ is an isomorphism of formal connections. We denote by $\mathrm{Isom}_{\mathrm{iso}}(M, \mathcal{M})$ the $\mathrm{St}^{<D}_{\mathcal{M}}(\mathbb{C})$-torsor of isomorphisms between ∂M and $\partial\mathcal{M}$ which are asymptotic to iso along D. The proof of the following statement was suggested to me by T. Mochizuki. I thank him for kindly sharing it. When D is smooth, it was known to Malgrange [**Mal83a**]. See also [**Sab02**, II 6.3].

Lemma 4.2.1. — *The map associating to every isomorphism class of $\mathcal{M}$-marked connection (M, iso) the $\mathrm{St}^{<D}_{\mathcal{M}}(\mathbb{C})$-torsor $\mathrm{Isom}_{\mathrm{iso}}(M, \mathcal{M})$ is bijective.*

Proof. — Let us construct an inverse. Take $\mathcal{T} \in \mathrm{St}^{<D}_{\mathcal{M}}(\mathbb{C})$ and let $g = (g_{ij})$ be a cocycle for $\mathcal{T}$ associated to a cover $(U_i)_{i\in I}$ of ∂D. Let $\mathcal{L}$ be the Stokes filtered local system on ∂D associated to $\mathcal{M}$. Set $\mathcal{L}_i := \mathcal{L}_{|U_i}$. Then, g allows to glue the $\mathcal{L}_i$ into a Stokes filtered local system $\mathcal{L}_{\mathcal{T}}$ on ∂D independent of the choice of g. From the irregular Riemann-Hilbert correspondence [**Moc11a**, 4.11], $\mathcal{L}_{\mathcal{T}}$ is the Stokes filtered local system associated to a unique (up to isomorphism) good meromorphic connection $\mathcal{M}_{\mathcal{T}}$ defined in a neighbourhood of D and with poles along D. By construction, the isomorphism $\mathcal{L}_{\mathcal{T}|U_i} \longrightarrow \mathcal{L}_{|U_i}$ corresponds to an isomorphism $\partial\mathcal{M}_{\mathcal{T}|U_i} \longrightarrow \partial\mathcal{M}_{|U_i}$. We thus obtain a formal isomorphism $\mathrm{iso}_i : \partial\mathcal{M}_{\mathcal{T},\hat{D}|U_i}, \longrightarrow \partial\mathcal{M}_{\hat{D}|U_i}$. On U_{ij}, the discrepancy between iso_i and iso_j is measured by the asymptotic of g_{ij} along D. By definition, this asymptotic is Id. Hence, the iso_i glue into a globally defined isomorphism $\partial\mathcal{M}_{\mathcal{T},\hat{D}} \longrightarrow \partial\mathcal{M}_{\hat{D}}$. Applying p_{D*} thus yields an isomorphism iso :

$\mathcal{M}_{\mathcal{T},\widehat{D}} \longrightarrow \mathcal{M}_{\widehat{D}}$. It is then standard to check that the map $\mathcal{T} \longrightarrow (\mathcal{M}_{\mathcal{T}}, \mathrm{iso})$ is the sought-after inverse. □

4.3. Proof of Theorem 4. — We are now in position to prove Theorem 4. Let X be a germ of smooth algebraic variety around a point 0. Let D be a germ of divisor passing through 0. Let $\mathcal{M}$ be a germ of meromorphic connection at 0 with poles along D. Let C be a smooth curve passing through 0 and not contained in any of the irreducible components of D. Let (M_1, iso_1) and (M_2, iso_2) be $\mathcal{M}$-marked connections such that

$$(M_1, \mathrm{iso}_1)_{|C} \simeq (M_2, \mathrm{iso}_2)_{|C}$$

We want to show that (M_1, iso_1) and (M_2, iso_2) are isomorphic in a neighbourhood of 0. Let $\pi : Y \longrightarrow X$ be a resolution of turning points for $\mathcal{M}$ around 0. Such a resolution exists by works of Kedlaya [**Ked11**] and Mochizuki [**Moc11b**]. Set $E := \pi^{-1}(D)$. At the cost of blowing up further, we can suppose that the strict transform C' of C is transverse to E at a point P in the smooth locus of E. Note that E is connected. From lemma 4.2.1, the $\pi^+\mathcal{M}$-marked connections $(\pi^+ M_1, \pi^+ \mathrm{iso}_1)$ and $(\pi^+ M_2, \pi^+ \mathrm{iso}_2)$ define two $\mathbb{C}$-points of $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$. For $i = 1, 2$, the cone of the canonical comparison morphism

$$\pi_+\pi^+ M_i \longrightarrow M_i \tag{4.3.1}$$

is supported on D. Note that the right-hand side of (4.3.1) is localized along D. From [**Meb04**, 3.6-4], the left-hand side of (4.3.1) is localized along D. Hence, the morphism (4.3.1) is an isomorphism. Thus

$$(\pi_+\pi^+ M_i, \pi_+\pi^+ \mathrm{iso}_i) \simeq (M_i, \mathrm{iso}_i)$$

Hence, it is enough to show $(\pi^+ M_1, \pi^+ \mathrm{iso}_1) \simeq (\pi^+ M_2, \pi^+ \mathrm{iso}_2)$. By assumption,

$$\begin{aligned}(\pi^+ M_1, \pi^+ \mathrm{iso}_1)_{|C'} &\simeq (M_1, \mathrm{iso}_1)_{|C} \\ &\simeq (M_2, \mathrm{iso}_2)_{|C} \\ &\simeq (\pi^+ M_2, \pi^+ \mathrm{iso}_2)_{|C'}\end{aligned}$$

Hence, $(\pi^+ M_1, \pi^+ \mathrm{iso}_1)_{|C'}$ and $(\pi^+ M_2, \pi^+ \mathrm{iso}_2)_{|C'}$ define the same $\mathbb{C}$-point in $H^1(\partial P, \mathrm{St}^{<P}_{(\pi^+\mathcal{M})_{|C'}})$. Let $\iota : \widetilde{C'} \longrightarrow \widetilde{Y}$ be the morphism induced by $C' \longrightarrow Y$ at the level of the real blow-up. Observe that ι induces an isomorphism above P. Since $\iota^{-1}\mathrm{St}^{<E}_{\pi^+\mathcal{M}} \simeq \mathrm{St}^{<P}_{(\pi^+\mathcal{M})_{|C'}}$, we have

$$H^1(\partial P, \mathrm{St}^{<P}_{(\pi^+\mathcal{M})_{|C'}}) \simeq H^1(\partial P, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$$

Hence, the image of $(\pi^+ M_1, \pi^+ \mathrm{iso}_1)$ and $(\pi^+ M_2, \pi^+ \mathrm{iso}_2)$ by the restriction map

$$H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}}) \longrightarrow H^1(\partial P, \mathrm{St}^{<E}_{\pi^+\mathcal{M}}) \tag{4.3.2}$$

are the same. From Theorem 3, the map (4.3.2) is a closed immersion. Hence, $(\pi^+ M_1, \pi^+ \mathrm{iso}_1) \simeq (\pi^+ M_2, \pi^+ \mathrm{iso}_2)$, which concludes the proof of Theorem 4.

4.4. Obstruction theory and tangent space. — We use the notations from 4.1. Let us compute the obstruction theory of $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ at a point $\mathcal{T}_0 \in H^1(\partial D, \mathrm{St}_{\mathcal{N}}^{<D}(\mathbb{C}))$. We fix a morphism of infinitesimal extensions of $\mathbb{C}$-algebras

$$R' \longrightarrow R \longrightarrow \mathbb{C}, \qquad I := \operatorname{Ker} R' \longrightarrow R$$

such that I is annihilated by $\operatorname{Ker} R' \longrightarrow \mathbb{C}$. In particular, $I^2 = 0$ and I is endowed with a structure of $\mathbb{C}$-vector space, which we suppose to be finite dimensional. Let $\mathcal{T} \in H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D}(R))$ lifting $\mathcal{T}_0$. Choose a cover $\mathcal{U} = (U_i)_{i\in K}$ of ∂D such that $\mathcal{T}$ comes from a cocycle $g = (g_{ij})_{i,j\in K}$. Set $L_i(R) := \operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(R)_{|U_i}$. The identifications

$$\begin{array}{ccc} L_i(R)_{|U_{ij}} & \xrightarrow{\sim} & L_j(R)_{|U_{ij}} \\ M & \longrightarrow & g_{ij}^{-1} M g_{ij} \end{array}$$

allow to glue the $L_i(R)$ into a sheaf of R-Lie algebras over ∂D denoted by $\operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(R)^{\mathcal{T}}$ and depending only on $\mathcal{T}$ and not on g. For $t = (t_{ijk}) \in \check{C}^2(\mathcal{U}, \operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(R)^{\mathcal{T}})$, we denote by s_{ijk} the unique representative of t_{ijk} in $\Gamma(U_{ijk}, L_i(R))$. Then

$$\begin{aligned} (dt)_{ijkl} &= t_{jkl} - t_{ikl} + t_{ijl} - t_{ijk} \\ &= [g_{ij} s_{jkl} g_{ij}^{-1} - s_{ikl} + s_{ijl} - s_{ijk}] \end{aligned}$$

We have the following

Lemma 4.4.1. — *There exists*

$$\mathrm{ob}(\mathcal{T}) \in I \otimes_{\mathbb{C}} \check{H}^2(\partial D, \operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}_0})$$

such that $\mathrm{ob}(\mathcal{T}) = 0$ *if and only if* $\mathcal{T}$ *lifts to* $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D}(R'))$.

Proof. — For every $i, j \in K$, let $h_{ij} \in \Gamma(U_{ij}, \mathrm{St}_{\mathcal{M}}^{<D}(R'))$ be an arbitrary lift of g_{ij} to R'. We can always choose the h_{ij} to satisfy $h_{ii} = \mathrm{Id}$ and $h_{ij}h_{ji} = \mathrm{Id}$. Since $\operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(R')$ is locally free,

$$I \cdot \operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(R') \simeq I \otimes_{R'} \operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(R') \simeq I \otimes_{\mathbb{C}} \operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})$$

We will use both descriptions without mention. We set

$$s_{ijk} := h_{ij} h_{jk} h_{ki} - \mathrm{Id} \in \Gamma(U_{ijk}, I \cdot \operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(R'))$$

We see s_{ijk} as a section of $I \otimes_{\mathbb{C}} L_i(\mathbb{C})$ over U_{ijk} and denote by $[s_{ijk}]$ its class in $I \otimes_{\mathbb{C}} \operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}_0}$. We want to prove that the $[s_{ijk}]$ define a cocycle. As seen above, this amounts to prove the following equality in $\Gamma(U_{ijk}, I \otimes_{\mathbb{C}} \operatorname{Lie} \mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C}))$

$$g_{ij}(0) s_{jkl} g_{ij}^{-1}(0) - s_{ikl} + s_{ijl} - s_{ijk} = 0 \tag{4.4.2}$$

Where $g_{ij}(0)$ is the image of g_{ij} by $R \longrightarrow \mathbb{C}$. We have

$$\begin{aligned}
g_{ij}(0)s_{jkl}g_{ij}^{-1}(0) &= h_{ij}h_{jk}h_{kl}h_{lj}h_{ji} - \mathrm{Id} \\
&= (h_{ij}h_{jk} - h_{ik} + h_{ik})h_{kl}h_{lj}h_{ji} - \mathrm{Id} \\
&= (h_{ij}h_{jk} - h_{ik})g_{kl}(0)g_{lj}(0)g_{ji}(0) + h_{ik}h_{kl}h_{lj}h_{ji} - \mathrm{Id} \\
&= (h_{ij}h_{jk} - h_{ik})g_{ki}(0) + h_{ik}h_{kl}h_{lj}h_{ji} - \mathrm{Id} \\
&= (h_{ij}h_{jk} - h_{ik})h_{ki} + h_{ik}h_{kl}h_{lj}h_{ji} - \mathrm{Id} \\
&= h_{ij}h_{jk}h_{ki} + h_{ik}h_{kl}h_{lj}h_{ji} - 2\,\mathrm{Id}
\end{aligned}$$

We now see how the second term of the last line above interacts with the second term of the left-hand side of (4.4.2).

$$\begin{aligned}
h_{ik}h_{kl}h_{lj}h_{ji} - s_{ikl} &= h_{ik}h_{kl}h_{lj}h_{ji} - h_{ik}h_{kl}h_{li} + \mathrm{Id} \\
&= h_{ik}h_{kl}(h_{lj}h_{ji} - h_{li}) + \mathrm{Id} \\
&= g_{ik}(0)g_{kl}(0)(h_{lj}h_{ji} - h_{li}) + \mathrm{Id} \\
&= g_{il}(0)(h_{lj}h_{ji} - h_{li}) + \mathrm{Id} \\
&= h_{il}h_{lj}h_{ji}
\end{aligned}$$

Hence,

$$\begin{aligned}
g_{ij}(0)s_{jkl}g_{ij}^{-1}(0) - s_{ikl} + s_{ijl} - s_{ijk} &= h_{il}h_{lj}h_{ji} + h_{ij}h_{jl}h_{li} - 2\,\mathrm{Id} \\
&= (h_{ij}h_{jl}h_{li})^{-1} + h_{ij}h_{jl}h_{li} - 2\,\mathrm{Id} \\
&= (h_{ij}h_{jl}h_{li})^{-1}((h_{ij}h_{jl}h_{li})^2 - 2h_{ij}h_{jl}h_{li} + \mathrm{Id}) \\
&= (h_{ij}h_{jl}h_{li})^{-1}s_{ijl}^2 \\
&= 0
\end{aligned}$$

where the last equality comes from $I^2 = 0$. Hence, the $[s_{ijk}]$ define a cocycle of $I \otimes_{\mathbb{C}} \mathrm{Lie}\,\mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}_0}$. An other choice of lift gives rise to homologous cocycles. We denote by $\mathrm{ob}(\mathcal{T})$ the class of $([s_{ijk}])_{ijk}$ in $\check{H}^2(\partial D, I \otimes_{\mathbb{C}} \mathrm{Lie}\,\mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}_0})$. It is standard to check that $\mathrm{ob}(\mathcal{T})$ has the sought-after property. $\square$

Corollary 4.4.3. — *Let (M, iso) be a $\mathcal{M}$-marked connection. Then, the space $H^2(D, \mathrm{Irr}_D^* \operatorname{End} M))$ is an obstruction theory for $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ at $\mathrm{Isom}_{\mathrm{iso}}(M, \mathcal{M})$.*

Proof. — Set $\mathcal{T} := \mathrm{Isom}_{\mathrm{iso}}(M, \mathcal{M})$. As observed in [**Tey19**, 5.2], the canonical identification

$$\mathcal{H}^0\,\mathrm{DR}^{<D}\operatorname{End}\mathcal{M} \xrightarrow{\sim} \mathrm{Lie}\,\mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}}$$

induces

$$\begin{aligned}
\check{H}^i(\partial D, \mathrm{Lie}\,\mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}}) &\simeq H^i(\partial D, \mathrm{Lie}\,\mathrm{St}_{\mathcal{M}}^{<D}(\mathbb{C})^{\mathcal{T}}) \\
&\simeq H^i(\partial D, \mathcal{H}^0\,\mathrm{DR}^{<D}\operatorname{End} M) \\
&\simeq H^i(\partial D, \mathrm{DR}^{<D}\operatorname{End} M) \\
&\simeq H^i(D, \mathrm{Irr}_D^*\operatorname{End} M)
\end{aligned}$$

The second identification comes from the fact [**Hie09**, Prop. 1] that $\mathrm{DR}^{<D}\operatorname{End}\mathcal{M}$ is concentrated in degree 0. The third identification comes from [**Sab17**, 2.2]. Then, corollary 4.4.3 follows from lemma 4.4.1. □

Reasoning exactly as in [**Tey19**, 5.2.1], we prove the following

Lemma 4.4.4. — *For every $\mathcal{M}$-marked connection (M, iso), the tangent space of $H^1(\partial D, \mathrm{St}_{\mathcal{M}}^{<D})$ at (M, iso) identifies canonically with $H^1(D, \mathrm{Irr}_D^* \operatorname{End} M)$.*

5. Reduction of Theorem 1 to extending the formal model

5.1. Reduction to the dimension 2 case. — In this subsection, we reduce the proof of Theorem 1 to the dimension 2 case. The main tool is André's goodness criterion [**And07**, 3.4.3] in terms of Newton polygons. This reduction does not seem superfluous. Of crucial importance for the sequel of the proof will be indeed the fact that when X is an algebraic surface and D a smooth divisor in X, then for every point $0 \in D$ and every meromorphic connection $\mathcal{M}$ on X with poles along D, the formal model of $\mathcal{M}$ *splits on a small enough punctured disc around* 0. This fact is specific to dimension 2, since it pertains to the property that turning points in dimension 2 are isolated.

Lemma 5.1.1. — *The converse inclusion in Theorem* 1 *is true in any dimension if it is true in dimension* 2.

Proof. — Take $n > 2$. We argue recursively by supposing that Theorem 1 holds in dimension strictly less than n and we prove that Theorem 1 holds in dimension n. Let X be a smooth complex algebraic variety of dimension n. Let D be a smooth divisor in X. Let $\mathcal{M}$ be an algebraic meromorphic connection on X with poles along D. Let $0 \in D$ and suppose that $\mathrm{Irr}_D^* \mathcal{M}$ and $\mathrm{Irr}_D^* \operatorname{End}\mathcal{M}$ are local systems in a neighbourhood of 0. If $j : X \backslash D \longrightarrow X$ and $i : D \longrightarrow X$ are the canonical inclusions, we have a distinguished triangle

$$j_! L \longrightarrow \mathrm{Sol}\,\mathcal{M} \longrightarrow i_* \mathrm{Irr}_D^* \mathcal{M}$$

where L is a local system on the complement of D. Hence, the characteristic cycle of $\mathrm{Sol}\,\mathcal{M}$ is supported on the union of $T_X^* X$ with $T_D^* X$. From a theorem of Kashiwara and Schapira [**KS90**, 11.3.3], so does the characteristic cycle of $\mathcal{M}$. Hence, any smooth hypersurface transverse to D and passing through 0 is non characteristic with respect to $\mathcal{M}$ in a neighbourhood of 0. Let us choose such a hypersurface Z and let $i_Z : Z \longrightarrow X$ be the canonical inclusion. From [**And07**, 3.4.3], the turning locus of $\mathcal{M}$ is a closed subset of D which is either empty or purely of codimension 1 in D. Since $n > 2$, the hypersurface Z can consequently be chosen such that $\mathcal{M}$ and $\operatorname{End}\mathcal{M}$ have good formal structure generically along $Z \cap D$. The connection $i_Z^+ \mathcal{M}$ is a meromorphic connection with poles along $Z \cap D$. It satisfies the hypothesis of Theorem 1 at the point 0. Indeed by Kashiwara's restriction theorem [**Kas95**],

$$\mathrm{Irr}_{Z\cap D}^* i_Z^+ \mathcal{M} = (\mathrm{Sol}\, i_Z^+ \mathcal{M})_{|Z\cap D} \simeq (\mathrm{Sol}\,\mathcal{M})_{|Z\cap D}$$

and similarly for $\operatorname{End}\mathcal{M}$. Hence, $\operatorname{Irr}^*_{Z\cap D} i^+_Z\mathcal{M}$ and $\operatorname{Irr}^*_{Z\cap D}\operatorname{End} i^+_Z\mathcal{M}$ are local systems in a neighbourhood of 0 in $Z\cap D$. By recursion hypothesis, $i^+_Z\mathcal{M}$ is good at 0. In particular, the Newton polygon of $i^+_Z\mathcal{M}$ at 0 (which is also the Newton polygon of $\mathcal{M}$ at 0) is the generic Newton polygon of $i^+_Z\mathcal{M}$ along $Z\cap D$. From our choice for Z, the generic Newton polygon of $i^+_Z\mathcal{M}$ along $Z\cap D$ is the generic Newton polygon of $\mathcal{M}$ along D. Hence, the Newton polygon of $\mathcal{M}$ at 0 is the generic Newton polygon of $\mathcal{M}$ along D, and similarly with $\operatorname{End}\mathcal{M}$. By a theorem of André [**And07**, 3.4.1], we deduce that $\mathcal{M}$ has good formal structure at 0, which proves lemma 5.1.1.

□

5.2. Setup and recollections. — From now on, we restrict the situation to dimension 2. We use coordinates (x,y) on $\mathbb{A}^2_{\mathbb{C}}$ and set $D_x := \{y=0\}, D_y := \{x=0\}$. Let D be a neighbourhood of 0 in D_x and let $\mathbb{C}[D]$ be the coordinate ring of D. Set $D^* := D\backslash\{0\}$.

Let $\mathcal{M}$ be an algebraic meromorphic flat bundle on a neighbourhood of D in $\mathbb{A}^2_{\mathbb{C}}$ with poles along D. In algebraic terms, $\mathcal{M}_{\hat{D}}$ defines a $\mathbb{C}[D]((y))$-differential module. At the cost of shrinking D if necessary, we can suppose that the restriction $\mathcal{M}^*$ of $\mathcal{M}$ to a neighbourhood of D^* has good formal structure at every point of D^*.

There is a ramification $v = y^{1/d}$, $d\geqslant 1$ and a finite Galois extension $L/\mathbb{C}(x)$ such that the set $\mathcal{I}$ of generic irregular values for $\mathcal{M}$ lies in $\operatorname{Frac} L(v)$. If $p: D_L \longrightarrow D$ is the normalization of D in L, the generic irregular values of $\mathcal{M}$ are thus meromorphic functions on $D_L\times\mathbb{A}^1_v$. We have

$$L((v))\otimes\mathcal{M}\simeq\bigoplus_{a\in\mathcal{I}}\mathcal{E}^a\otimes\mathcal{R}_a \tag{5.2.1}$$

where the $\mathcal{R}_a$ are regular. Following [**And07**, 3.2.4], we recall the following

Definition 5.2.2. — *We say that $\mathcal{M}$ is semi-stable at $P\in D$ if*
(1) We have $\mathcal{I}\subset\mathbb{C}[D_L]_P((v))$.
(2) The decomposition (5.2.1) *descends to $\mathbb{C}[D_L]_P((v))\otimes\mathcal{M}$.*

In this definition, $\mathbb{C}[D_L]_P$ denotes the localization of $\mathbb{C}[D_L]$ above P. This is a semi-local ring. Let $\pi_a\in L((v))\otimes\operatorname{End}\mathcal{M}$ be the projector on the factor $\mathcal{E}^a\otimes\mathcal{R}_a$. As explained in [**And07**, 3.2.2], the point P is stable if and only if the generic irregular values of $\mathcal{M}$ and the coefficients of the π_a in a basis of $\operatorname{End}\mathcal{M}$ belong to $\mathbb{C}[D_L]_P((v))$. Since $\mathcal{M}$ has good formal structure at any point of D^*, the generic irregular values of $\mathcal{M}$ and the coefficients of the π_a in a basis of $\operatorname{End}\mathcal{M}$ belong to $\mathbb{C}[D_L]_P((v))$ for every $P\in D^*$. Hence, they belong $\mathbb{C}[D_L^*]((v))$ where $D_L^* := D\backslash p^{-1}(0)$. Thus

$$\mathbb{C}[D_L^*]((v))\otimes\mathcal{M}\simeq\mathbb{C}[D_L^*]((v))\otimes\mathcal{N}_L^* \tag{5.2.3}$$

where

$$\mathcal{N}_L^* = \bigoplus_{a\in\mathcal{I}}\mathcal{E}^a\otimes\mathcal{R}_a$$

is a germ of meromorphic connection defined on a neighbourhood of D_L^* in $D_L\times\mathbb{A}^1_v$ and with poles along D_L^*. The action of

$$\operatorname{Gal}(L/\mathbb{C}(x))\times\mathbb{Z}/d\mathbb{Z}$$

on the left-hand side of (5.2.3) induces an action on $\mathcal{N}_L^*$. Taking the invariants yields a meromorphic flat bundle $\mathcal{N}^*$ defined on a neighbourhood Ω of D^* in $\mathbb{A}^2_{\mathbb{C}}$. By Galois descent, (5.2.3) descents to an isomorphism iso* between the formalizations of $\mathcal{M}^*$ and $\mathcal{N}^*$ along D^*.

5.3. Reduction to the problem of extending the formal model. — The goal of this subsection is to show that Theorem 1 reduces to prove that the $\mathcal{M}^*$-marked connection $(\mathcal{N}^*, \mathrm{iso}^*)$ defined in 5.2 extends into a $\mathcal{M}$-marked connection in a neighbourhood of 0. To do this, we need three preliminary lemmas. The notations and constructions from 5.2 are in use.

Lemma 5.3.1. — *Suppose that $\mathcal{N}^*$ extends into a meromorphic flat bundle $\mathcal{N}$ defined in a neighbourhood of D in $\mathbb{A}^2_{\mathbb{C}}$ and with poles along D. Then, $\mathcal{N}$ is semi-stable at* 0.

Proof. — It is enough to treat the case where $K = \mathbb{C}(x)$ and $d = 1$. In that case, discussion 5.2 shows that on a neighbourhood Ω of D^* in $\mathbb{A}^2_{\mathbb{C}}$, we have

$$\mathcal{N}^* = \bigoplus_{a\in\mathcal{I}} \mathcal{N}_a^*$$

where $\mathcal{N}_a^*$ is a meromorphic connection on Ω with poles along D^* and with single irregular value a. The open $D \times \mathbb{A}^1_{\mathbb{C}}$ retracts on the small neighbourhood on which $\mathcal{N}$ is defined. Since $\mathcal{N}$ is smooth away from D, we deduce that $\mathcal{N}$ extends canonically into a meromorphic connection on $D \times \mathbb{A}^1_{\mathbb{C}}$ with poles along D.

Let $a \in \mathcal{I}$. The restriction of the projector π_a to the complement of D^* in Ω is a flat section of $\mathrm{End}\,\mathcal{N}$. Since $D^* \times \mathbb{A}^1_{\mathbb{C}}$ retracts on Ω, parallel transport allows to extend π_a canonically with $D^* \times \mathbb{A}^1_{\mathbb{C}}$. We still denote by π_a this extension. Hence, $\mathcal{N}_a^*$ extends into a meromorphic connection on $D^* \times \mathbb{A}^1_{\mathbb{C}}$ with poles along D^*. Let γ be a small loop in Ω going around the axis D_y. By assumption, the monodromy of $\mathcal{N}$ along γ is trivial. Thus, π_a is invariant under the monodromy of $\mathrm{End}\,\mathcal{N}$ along γ. Hence, π_a extends canonically to $(D \times \mathbb{A}^1_{\mathbb{C}})\backslash\{0\}$. By Hartog's property, it extends further into a section ϖ_a of $\mathrm{End}\,\mathcal{N}$ on $D \times \mathbb{A}^1_{\mathbb{C}}$.

Set $\mathcal{N}_a := \varpi_a(\mathcal{N}) \subset \mathcal{N}$ for every $a \in \mathcal{I}$. We have $\varpi_a^2 = \varpi_a$ and $\sum_{a\in\mathcal{I}} \varpi_a = \mathrm{Id}_{\mathcal{N}}$ because these equalities hold on a non empty open set. Hence, $\mathcal{N} = \oplus_{a\in\mathcal{I}}\mathcal{N}_a$. Since ϖ_a is flat, the connection on $\mathcal{N}$ preserves each $\mathcal{N}_a$. Let us prove that the $\mathcal{N}_a$ are locally free as $\mathcal{O}_{D\times\mathbb{A}^1_{\mathbb{C}}}(*D)$-modules.

Let E be a Deligne-Malgrange lattice [**Mal96**] for $\mathcal{N}$. Since we work in dimension 2, we know from [**Mal96**, 3.3.2] that E is a vector bundle. We observe that ϖ_a stabilizes E away from 0. By Hartog's property, we deduce that ϖ_a stabilizes E. Hence, $\varpi_a(E)$ is a direct factor of E. So $\varpi_a(E)$ is a vector bundle. Thus,

$$\mathcal{N}_a = \varpi_a(\mathcal{N}) = \varpi_a(E(*D)) = (\varpi_a(E))(*D)$$

is a locally free $\mathcal{O}_{D\times\mathbb{A}^1_{\mathbb{C}}}(*D)$-module of finite rank with connection extending $\mathcal{N}_a^*$. To prove lemma 5.3.1, we are thus left to consider the case where $\mathcal{I} = \{a\}$.

If $\mathcal{I} = \{a\}$, then [**And07**, 3.3.1] implies $a \in \mathbb{C}[D]((y))$. Hence, $\mathcal{R} := \mathcal{E}^{-a} \otimes \mathcal{N}_{\hat{D}}$ is a formal meromorphic connection with poles along D. By assumption, $\mathcal{R}$ is generically regular along D. From [**Del70**, 4.1], we deduce that $\mathcal{R}$ is regular. Hence, $\mathcal{N}_{\widehat{D}} = \mathcal{E}^a \otimes \mathcal{R}$ with $\mathcal{R}$ regular, which concludes the proof of lemma 5.3.1. □

Lemma 5.3.2. — *Let $\mathcal{N}$ be a meromorphic flat connection with poles along D. Suppose that $\mathcal{N}$ is semi-stable at* 0 *and that* $\operatorname{Irr}_D^* \mathcal{N}$ *and* $\operatorname{Irr}_D^* \operatorname{End} \mathcal{N}$ *are local systems in a neighbourhood of* 0*. Then, $\mathcal{N}$ has good formal structure at* 0.

Proof. — Let $\mathcal{I}$ be the set of irregular values of $\mathcal{N}$ at 0. There is a ramification $v = y^{1/d}$, $d \geqslant 1$ and a finite Galois extension $L/\mathbb{C}(x)$ such that $\mathcal{I} \subset L((v))$. Let $D_L \longrightarrow D$ be the normalization of D in L. At the cost of shrinking D, we can suppose that every point of D is semi-stable for $\mathcal{N}$. Hence, $\mathcal{I} \subset \mathbb{C}[D_L]((v))$ and

$$\mathbb{C}[D_L]((v)) \otimes \mathcal{N} = \bigoplus_{a \in \mathcal{I}} \mathcal{E}^a \otimes \mathcal{R}_a$$

where the connections $\mathcal{R}_a$ are regular. As seen in the proof of lemma 5.1.1, the assumption on Irr_D^* implies that any smooth curve transverse to D is non characteristic for $\mathcal{N}$. Taking the axis D_y yields

$$\dim \mathcal{H}^1 \operatorname{Irr}_0^* \mathcal{N}_{|D_y} = \dim(\mathcal{H}^1 \operatorname{Irr}_D^* \mathcal{N})_0 = \sum_{a \in \mathcal{I}} (\operatorname{ord}_y a) \operatorname{rk} \mathcal{R}_a$$

On the other hand, choose a point $P \in D_L$ above 0. Then, the irregular values of $\mathcal{N}_{|D_y}$ are the $a(P)$, $a \in \mathcal{I}$. Thus,

$$\mathcal{H}^1 \operatorname{Irr}_0^* \mathcal{N}_{|D_y} = \sum_{a \in \mathcal{I}} \operatorname{ord}_y a(P) \operatorname{rk} \mathcal{R}_a$$

Hence, $\operatorname{ord}_y a(P) = \operatorname{ord}_y a$ for every $a \in \mathcal{I}$. In particular, the coefficient function of the highest power of $1/v$ contributing to $a \in \mathcal{I}$ does not vanish at P. Arguing similarly for $\operatorname{End} \mathcal{N}$, we obtain that $\mathcal{N}$ has good formal structure at 0. □

Lemma 5.3.3. — *Suppose that* $\operatorname{Irr}_D^* \mathcal{M}$ *is a local system. For every $\mathcal{M}$-marked connection* $(\mathcal{N}, \mathrm{iso})$*, the complex* $\operatorname{Irr}_D^* \mathcal{N}$ *is a local system.*

Proof. — From [**Meb90**], the complex $\operatorname{Irr}_D^* \mathcal{N}$ is perverse. To prove that it is a local system, it is thus enough to prove that the local Euler Poincaré characteristic $\chi(D, \operatorname{Irr}_D^* \mathcal{N}) : D \longrightarrow \mathbb{Z}$ of $\operatorname{Irr}_D^* \mathcal{N}$ is constant. From the local index theorem [**Kas73**][**Mal81**], the local Euler Poincaré characteristic of $\operatorname{Irr}_D^* \mathcal{N}$ depends only on the characteristic cycle of $\mathcal{N}$. Since the characteristic cycle of $\mathcal{N}$ depends only on $\mathcal{N}$ via $\mathcal{N}_{\widehat{D}}$, we have

$$\chi(D, \operatorname{Irr}_D^* \mathcal{N}) = \chi(D, \operatorname{Irr}_D^* \mathcal{M})$$

By assumption, $\chi(D, \operatorname{Irr}_D^* \mathcal{M})$ is constant. Hence, $\chi(D, \operatorname{Irr}_D^* \mathcal{N})$ is constant, which finishes the proof of lemma 5.3.3. □

Proposition 5.3.4. — *Let D be an open neighbourhood of* 0 *in an hyperplane of* $\mathbb{A}^2_{\mathbb{C}}$. *Let $\mathcal{M}$ be an algebraic meromorphic flat bundle on a neighbourhood of D with poles along D. Set $D^* = D \backslash \{0\}$ and let $\mathcal{M}^*$ be the restriction of $\mathcal{M}$ to a neighbourhood of D^*. Let* $(\mathcal{N}^*, \mathrm{iso}^*)$ *be the $\mathcal{M}^*$-marked connection constructed in* 5.2*. Suppose that* $\operatorname{Irr}_D^* \mathcal{M}$ *and* $\operatorname{Irr}_D^* \operatorname{End} \mathcal{M}$ *are local systems in a neighbourhood of* 0*. Then, if* $(\mathcal{N}^*, \mathrm{iso}^*)$ *extends into a $\mathcal{M}$-marked connection, $\mathcal{M}$ has good formal structure at* 0.

Proof. — Let $(\mathcal{N}, \mathrm{iso})$ be a $\mathcal{M}$-marked connection extending $(\mathcal{N}^*, \mathrm{iso}^*)$. From lemma 5.3.1, the extension $\mathcal{N}$ is semi-stable at 0. From lemma 5.3.3, we know that $\mathrm{Irr}_D^* \mathcal{N}$ and $\mathrm{Irr}_D^* \operatorname{End} \mathcal{N}$ are local systems in a neighbourhood of 0. From lemma 5.3.2, we deduce that $\mathcal{N}$ has good formal structure at 0. Hence, so does $\mathcal{M}$. □

6. Extension via moduli of Stokes torsors

6.1. A geometric extension criterion. — In this subsection, we relate the moduli of Stokes torsors to the problem of extending marked connections. Let D be an open subset of a hyperplane in $\mathbb{A}^2_{\mathbb{C}}$. Pick $P \in D$. Set $D^* := D \backslash \{P\}$. Let $\mathcal{M}$ be an algebraic meromorphic flat bundle in a neighbourhood U of D in $\mathbb{A}^2_{\mathbb{C}}$ and with poles along D. Let $\mathcal{M}^*$ be the restriction of $\mathcal{M}$ to $U \backslash \{P\}$. Let $\pi : Y \longrightarrow \mathbb{A}^2_{\mathbb{C}}$ be a resolution of the turning point P for $\mathcal{M}$. Such a resolution exists by works of Kedlaya [**Ked10**] and Mochizuki [**Moc09**]. Let Δ be an open disc of D containing P. Set $\Delta^* = \Delta \backslash \{P\}$. Set $E := \pi^{-1}(\Delta)$ and pick $Q \in \Delta^*$. Let

$$\Phi : H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}}) \longrightarrow H^1(\partial Q, \mathrm{St}^{<\Delta}_{\mathcal{M}})$$

be the restriction morphism of Stokes torsors.

Lemma 6.1.1. — *Let $(\mathcal{N}^*, \mathrm{iso}^*)$ be a $\mathcal{M}^*$-marked connection such that $(\mathcal{N}^*_Q, \mathrm{iso}^*_Q)$ lies in the image of Φ. Then, $(\mathcal{N}^*, \mathrm{iso}^*)$ extends into an $\mathcal{M}$-marked connection.*

Proof. — From lemma 4.2.1, any $\mathbb{C}$-point of $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ comes from a unique $\pi^+\mathcal{M}$-marked connection. Hence, there exists $(\mathcal{N}', \mathrm{iso}') \in H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ such that $\Phi(\mathcal{N}', \mathrm{iso}') = (\mathcal{N}^*_P, \mathrm{iso}^*_P)$. From [**Meb04**, 3.6-4], the $\mathcal{D}$-module $\mathcal{N} := \pi_+\mathcal{N}'$ is a meromorphic connection defined in a neighbourhood of Δ and and with poles along Δ. By flat base change

$$\begin{aligned}
\mathcal{N}_{\widehat{\Delta}} &\simeq \mathcal{O}_{\widehat{\mathbb{A}^2_{\mathbb{C}}|\Delta}} \otimes R\pi_*(\mathcal{D}_{X\to\mathbb{A}^2_{\mathbb{C}}} \otimes \mathcal{N}') \\
&\simeq R\pi_*(\mathcal{O}_{\widehat{X|E}} \otimes \mathcal{D}_{X\to\mathbb{A}^2_{\mathbb{C}}} \otimes \mathcal{N}') \\
&\simeq R\pi_*(\mathcal{D}_{X\to\mathbb{A}^2_{\mathbb{C}}} \otimes \mathcal{N}'_{\widehat{E}}) \\
&\simeq \pi_+\mathcal{N}'_{\widehat{E}}
\end{aligned}$$

and similarly $\mathcal{M}_{\widehat{\Delta}} \simeq \pi_+(\pi^+\mathcal{M})_{\widehat{E}}$. Hence, $\mathrm{iso} := \pi_+ \mathrm{iso}'$ defines an isomorphism between $\mathcal{N}_{\widehat{\Delta}}$ and $\mathcal{M}_{\widehat{\Delta}}$. So $(\mathcal{N}, \mathrm{iso})$ is a $\mathcal{M}$-marked connection in a neighbourhood of Δ. By definition, the germ of $(\mathcal{N}, \mathrm{iso})$ at Q is $(\mathcal{N}^*_Q, \mathrm{iso}^*_Q)$. Since $R^1 p_{\Delta *} \mathrm{St}^{<\Delta^*}_{\mathcal{M}}$ is a local system on Δ^*, we deduce

$$(\mathcal{N}, \mathrm{iso})_{|\Delta^*} = (\mathcal{N}^*, \mathrm{iso}^*)_{|\Delta^*}$$

Hence, the gluing of $(\mathcal{N}, \mathrm{iso})$ with $(\mathcal{N}^*, \mathrm{iso}^*)$ provides the sought-after extension of $(\mathcal{N}^*, \mathrm{iso}^*)$ into an $\mathcal{M}$-marked connection. So lemma 6.1.1 is proved. □

Let us now give a sufficient condition for the surjectivity of Φ in terms of the irregularity complex.

Proposition 6.1.2. — *With the notations from* 6.1*, suppose furthermore that the perverse complex* $\mathrm{Irr}_D^* \operatorname{End} \mathcal{M}$ *is a local system on* Δ*. Then* Φ *induces an isomorphism between each irreducible component of* $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ *and* $H^1(\partial Q, \mathrm{St}^{<\Delta}_{\mathcal{M}})$*.*

Proof. — From [**BV89**], we know that $H^1(\partial Q, \mathrm{St}^{<\Delta}_{\mathcal{M}})$ is an affine space. Since affine spaces in characteristic 0 have no non trivial finite étale covers, it is enough to prove that Φ is finite étale. From Theorem 3, the morphism Φ is a closed immersion. We are thus left to show that Φ is étale.

Etale morphisms between smooth schemes of finite type over $\mathbb{C}$ are those morphisms inducing isomorphisms on the tangent spaces. Hence, we are left to prove that $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ is smooth and that Φ induces isomorphisms on the tangent spaces. Let (M, iso) be a $\pi^+\mathcal{M}$-marked connection. From corollary 4.4.3, an obstruction theory to lifting infinitesimally the Stokes torsor of (M, iso) is given by

$$H^2(E, \mathrm{Irr}_E^* \operatorname{End} M) \simeq H^2(\Delta, \mathrm{Irr}_D^* \pi_+ \operatorname{End} M) \simeq 0 \tag{6.1.3}$$

The first identification expresses the compatibility of irregularity with proper push-forward. Furthermore, from lemma 5.3.3 applied to the $\operatorname{End} \mathcal{M}$-marked connection $(\pi_+ \operatorname{End} M, \pi_+ \mathrm{iso})$, the perverse complex $\mathrm{Irr}_D^* \pi_+ \operatorname{End} M$ is a local system concentrated in degree 1. This implies the vanishing (6.1.3). Hence, $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ is smooth at (M, iso). From lemma 4.2.1, any $\mathbb{C}$-point of $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ is of the form (M, iso). Thus, $H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}})$ is smooth. Furthermore, we have a commutative diagram

$$\begin{tikzcd}
T_{(M,\mathrm{iso})} H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}}) \arrow[r] \arrow[d, "\wr"'] & T_{(M_Q,\mathrm{iso}_Q)} H^1(\partial Q, \mathrm{St}^{<\Delta}_{\mathcal{M}}) \arrow[d, "\wr"] \\
H^1(E, \mathrm{Irr}_E^* \operatorname{End} M) \arrow[r] \arrow[d, "\wr"'] & (\mathcal{H}^1 \mathrm{Irr}_D^* \operatorname{End} M)_Q \arrow[d, "\vert"] \\
H^1(\Delta, \mathrm{Irr}_D^* \pi_+ \operatorname{End} M) \arrow[r] \arrow[d, "\wr"'] & (\mathcal{H}^1 \mathrm{Irr}_D^* \operatorname{End} M)_Q \arrow[d, "\vert"] \\
H^0(\Delta, \mathcal{H}^1 \mathrm{Irr}_D^* \pi_+ \operatorname{End} M) \arrow[r] & (\mathcal{H}^1 \mathrm{Irr}_D^* \operatorname{End} M)_Q
\end{tikzcd}$$

The first vertical maps are isomorphisms by lemma 4.4.4. As already seen, $\mathrm{Irr}_D^* \pi_+ \operatorname{End} M$ is a local system concentrated in degree 1. Hence, the last vertical and the bottom arrows are isomorphisms. Thus, the tangent map of Φ at (M, iso) is an isomorphism. This finishes the proof of proposition 6.1.2.

□

6.2. Proof of Theorem 1. — Let X be a smooth complex algebraic variety. Let D be a smooth divisor in X. Let $\mathcal{M}$ be an algebraic meromorphic connection with poles along D.

We first prove the direct inclusion in Theorem 1. Suppose that $\mathcal{M}$ has good formal structure at a closed point $P \in D$. Since the good formal structure locus of $\mathcal{M}$ is open in D [**And07**], we can suppose at the cost of restricting the situation that $\mathcal{M}$ has good formal structure along D. By Mebkhout's theorem [**Meb90**], the complexes $\mathrm{Irr}^*_D \mathcal{M}$ and $\mathrm{Irr}^*_D \operatorname{End} \mathcal{M}$ are perverse. To prove that they are local systems on D, it is thus enough to prove that their local Euler Poincaré characteristic is constant. From the local index theorem [**Kas73**][**Mal81**], the local Euler Poincaré characteristic of $\mathrm{Irr}^*_D \mathcal{M}$ depends only on the characteristic cycle of $\mathcal{M}$. Since the characteristic cycle of $\mathcal{M}$ depends only on $\mathcal{M}$ via $\mathcal{M}_{\hat{D}}$, we are reduced to treat the case where $\mathcal{M} = \mathcal{E}^a \otimes \mathcal{R}$ where $a \in \mathcal{O}_X(*D)$ is good and where $\mathcal{R}$ is a regular singular meromorphic connection with poles along D. Since Irr^*_D is exact, we can suppose further that the rank of $\mathcal{R}$ is one. In that case, a standard computation shows that the characteristic cycle of $\mathcal{M}$ is supported on the union of $T^*_X X$ with $T^*_D X$. Hence, any smooth transverse curve to D is non-characteristic for $\mathcal{M}$. Let $P \in D$ and let C be a smooth transverse curve to D passing through P. From [**Kas95**], we have

$$(\mathrm{Irr}^*_D \mathcal{M})_P \simeq \mathrm{Irr}^*_P \mathcal{M}_{|C} \simeq \mathbb{C}^{\operatorname{ord}_D a}[-1]$$

Hence, the local Euler-Poincaré characteristic of $\mathrm{Irr}^*_D \mathcal{M}$ is constant and similarly for $\mathrm{Irr}^*_D \operatorname{End} \mathcal{M}$. This finishes the proof of the direct inclusion in Theorem 1.

We now prove the converse inclusion in Theorem 1. From lemma 5.1.1, we can suppose that X is a surface. Let $P \in D$ such that $\mathrm{Irr}^*_D \mathcal{M}$ and $\mathrm{Irr}^*_D \operatorname{End} \mathcal{M}$ are local systems in a neighbourhood of P in D. At the cost of taking local coordinates around P, we can suppose that D is an open subset of a hyperplane in $\mathbb{A}^2_{\mathbb{C}}$. Put $D^* := D \backslash \{P\}$. Let $\mathcal{M}^*$ be the restriction of $\mathcal{M}$ to a small neighbourhood of D^* in X. Let $(\mathcal{N}^*, \mathrm{iso}^*)$ be the $\mathcal{M}^*$-marked connection defined in 5.2. Such a connection exists at the cost of replacing X by a small enough neighbourhood of P in X. From proposition 5.3.4, we are left to show that $(\mathcal{N}^*, \mathrm{iso}^*)$ extends into a $\mathcal{M}$-marked connection. Let Δ be a small enough disc in D containing P such that $\mathrm{Irr}^*_D \mathcal{M}$ and $\mathrm{Irr}^*_D \operatorname{End} \mathcal{M}$ are local systems on Δ. Put $\Delta^* := \Delta \backslash \{P\}$. Let $\pi : Y \longrightarrow X$ be a resolution of turning points for $\mathcal{M}$ at P. Set $E := \pi^{-1}(\Delta)$ and pick $Q \in \Delta^*$. Let

$$\Phi : H^1(\partial E, \mathrm{St}^{<E}_{\pi^+\mathcal{M}}) \longrightarrow H^1(\partial Q, \mathrm{St}^{<\Delta}_{\mathcal{M}})$$

be the restriction morphism of Stokes torsors. From lemma 6.1.1, to prove that $(\mathcal{N}^*, \mathrm{iso}^*)$ extends into a $\mathcal{M}$-marked connection, it is enough to prove that $(\mathcal{N}^*_Q, \mathrm{iso}^*_Q)$ lies in the image of Φ. This is indeed the case by lemma 6.1.2, which finishes the proof of Theorem 1.

7. A boundedness theorem for turning points

7.1. Nearby slopes. — Let X be a smooth complex algebraic variety and let $\mathcal{M}$ be an holonomic $\mathcal{D}_X$-module. Let $f \in \mathcal{O}_X$ be a non constant function. Let ψ_f be the nearby cycle functor associated to f [**Kas83**][**Mal83b**][**Meb89**][**MM04**]. Following [**Tey16**], we recall that the *nearby slopes of $\mathcal{M}$ associated to f* are the rational numbers $r \in \mathbb{Q}_{\geqslant 0}$ such that there exists a germ N of meromorphic connection at $0 \in \mathbb{A}^1_{\mathbb{C}}$ with slope r such that

$$\psi_f(\mathcal{M} \otimes f^+ N) \neq 0 \tag{7.1.1}$$

We denote by $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M})$ the set of nearby slopes of $\mathcal{M}$ associated to f. In dimension 1, the nearby slopes of $\mathcal{M}$ associated to a local coordinate centred at a point 0 are the usual slopes of $\mathcal{M}$ at 0. See [**Tey16**, 3.3.1] for a proof. In general, the set $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M})$ is finite [**Del07**]. If $\mathcal{M}$ is a meromorphic connection, an explicit bound for $\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M})$ is given in [**Tey16**] in terms of a resolution of turning points of $\mathcal{M}$. This bound behaves poorly with respect to restriction. We will need a sharper bound in the case where f is a smooth morphism. It will be provided by the following more general proposition.

Proposition 7.1.2. — *Let $\mathcal{M}$ be a germ of meromorphic connection at $0 \in \mathbb{A}^n$ with poles along the divisor D given by $f := x_1 \cdots x_d = 0$. Let r_i be the highest generic slope of $\mathcal{M}$ along $x_i = 0$. Put $r_D(\mathcal{M}) = \mathrm{Max}\{r_1, \dots, r_d\}$. Then,*

$$\mathrm{Sl}_f^{\mathrm{nb}}(\mathcal{M}) \subset [0, r_D(\mathcal{M})]$$

Proof. — To prove proposition 7.1.2, take $r > r_D(\mathcal{M})$ and let N be a germ of meromorphic connection at $0 \in \mathbb{A}^1_{\mathbb{C}}$ with slope r. We want to show the vanishing (7.1.1) in a neigbourhood of 0. By a standard Galois argument, one reduces to the case where r and the $r_i, i = 1, \dots, d$ are integers. Since ψ_f is a formal invariant, we can further suppose that $N = t^\alpha \mathcal{E}^{1/t^r}$ where $\alpha \in \mathbb{C}$. Let us accept for a moment that $\mathcal{M}$ is generated as a $\mathcal{D}_X$-module by a coherent $\mathcal{O}_X$-submodule F stable by $f^{r_D(\mathcal{M})} x_i \partial_{x_i}$, $i = 1, \dots, d$ and such that $\mathcal{M} = F(*D)$. Let $(e_1, \dots, e_N)$ be a generating family for F in a neighbourhood of 0. Then, the $f^\alpha e^{1/f^r} e_i$, $i = 1, \dots, N$ generate $\mathcal{M} \otimes f^+ N$ as a $\mathcal{D}_X$-module. Let $\iota : \mathbb{C}^n \longrightarrow \mathbb{C}^n \times \mathbb{C}_t$ be the graph of f. Set $\delta := \delta(t - f)$. Then, the $s_i = f^\alpha e^{1/f^r} e_i \delta$, $i = 1, \dots, N$ generate $\iota_+(\mathcal{M} \otimes f^+ N)$. To show that the germ of $\psi_f(\mathcal{M} \otimes f^+ N)$ at 0 vanishes, we are thus left to prove that s_i belongs to $V_{-1}(\mathcal{D}_{\mathbb{C}^n \times \mathbb{C}_t}) \iota_+ F$ for every $i = 1, \dots, N$, where $V_\bullet(\mathcal{D}_{\mathbb{C}^n \times \mathbb{C}_t})$ is the Kashiwara-Malgrange filtration on $\mathcal{D}_{\mathbb{C}^n \times \mathbb{C}_t}$. For $i = 1, \dots, N$, we have

$$f^{r_D(\mathcal{M})} x_1 \partial_{x_1} s_i = f^{r_D(\mathcal{M})} (\alpha - \frac{r}{f^r}) s_i + \sum_{j=1}^{d} g_j s_j - f^{r_D(\mathcal{M})+1} \partial_t s_i, \qquad g_j \in \mathcal{O}_X$$

Hence,

$$r s_i = \alpha t^r s_i + t^{r - r_D(\mathcal{M})} \sum_{j=1}^{d} g_j s_j - f^r x_1 \partial_{x_1} s_i - f^{r+1} \partial_t s_i$$

Since $r > r_D(\mathcal{M})$, we have

$$t^{r - r_D(\mathcal{M})} \sum_{j=1}^{d} g_j s_j \in V_{-1}(\mathcal{D}_{\mathbb{C}^n \times \mathbb{C}_t}) \iota_+ F$$

Note furthermore that

$$f^r x_1 \partial_{x_1} s_i = x_1 \partial_{x_1} f^r s_i - r f^r s_i = t^r (x_1 \partial_{x_1} - r) s_i \in V_{-1}(\mathcal{D}_{\mathbb{C}^n \times \mathbb{C}_t}) s_i$$

and that

$$f^{r+1} \partial_t s_i = \partial_t t^{r+1} s_i = (r+1) t^r s_i + t^r t \partial_t s_i \in V_{-1}(\mathcal{D}_{\mathbb{C}^n \times \mathbb{C}_t}) s_i$$

Hence, $s_i \in V_{-1}(\mathcal{D}_{\mathbb{C}^n \times \mathbb{C}_t}) \iota_+ F$, which proves the sought-after vanishing. We are thus left to prove the lemma 7.1.3 below. $\square$

Lemma 7.1.3. — *Let $\mathcal{M}$ be a germ of meromorphic connection at $0 \in \mathbb{A}^n_{\mathbb{C}}$ with poles along the divisor D given by $f := x_1, \cdots x_d = 0$. Let r_i be the highest generic slope of $\mathcal{M}$ along $x_i = 0$. Suppose that the r_i are integers and put $r_D(\mathcal{M}) = \mathrm{Max}\{r_1, \ldots, r_d\}$. Then, $\mathcal{M}$ is generated as a $\mathcal{D}_X$-module by a coherent $\mathcal{O}_X$-submodule F stable by $f^{r_D(\mathcal{M})} x_i \partial_{x_i}$ for every $i = 0, \ldots, d$ and such that $\mathcal{M} = F(*D)$.*

Proof. — Let E be a lattice in $\mathcal{M}$ as constructed by Malgrange in [**Mal96**]. By construction, $\mathcal{M} = E(*D)$. Since holonomic $\mathcal{D}_X$-modules are noetherian, $\mathcal{M} = \mathcal{D}_X f^{-k} E$ for k big enough. Let us show that $F = f^{-k}E$ fits our purpose. For $m \in E$, we have

$$f^{r_D(\mathcal{M})} x_i \partial_{x_i}(f^{-k}m) = -k f^{r_D(\mathcal{M})}(f^{-k}m) + f^{-k}(f^{r_D(\mathcal{M})} x_i \partial_{x_i} m)$$

Hence, it is enough to show that E is stable by $f^{r_D(\mathcal{M})} x_i \partial_{x_i}$, $i = 0, \ldots, d$. Since $\mathcal{O}_{X^{\mathrm{an}},x}$ is faithfully flat over $\mathcal{O}_{X,x}$ for every $x \in D$, we have $E = \mathcal{M} \cap E^{\mathrm{an}}$ in $\mathcal{M}^{\mathrm{an}}$. Hence, it is enough to show that E^{an} is stable by $f^{r_D(\mathcal{M})} x_i \partial_{x_i}$, $i = 0, \ldots, d$. Let $j : U \longrightarrow X^{\mathrm{an}}$ be the complement in X of the union of the singular locus of D with the turning locus of $\mathcal{M}$. By construction of the Deligne-Malgrange lattices, a section of $\mathcal{M}$ belongs to E^{an} if and only if its restriction to U belongs to $E^{\mathrm{an}}_{|U}$. Hence, we can suppose that D is smooth and that $\mathcal{M}$ has good formal structure along D. We can further suppose that $\mathcal{M}$ is unramified along D. Since $\mathcal{O}_{X^{\mathrm{an}},\hat{D}}$ is faithfully flat over $\mathcal{O}_{X^{\mathrm{an}},D}$, we can suppose that $\mathcal{M}$ is a good elementary local model, that is

$$\mathcal{M} = \bigoplus_{a \in \mathcal{O}_{X^{\mathrm{an}}}(r_D(\mathcal{M})D)} \mathcal{E}^a \otimes \mathcal{R}_a$$

where the $\mathcal{R}_a$ are regular meromorphic connections with poles along D. In that case, E is by definition a direct sum of the form $\bigoplus E_a$ where E_a is a Deligne lattice [**Del70**] in $\mathcal{R}_a$. In that case, the sought-after stability is obvious. This finishes the proof of lemma 7.1.3. □

Remark 7.1.4. — *The bound for nearby slopes proved in proposition* 7.1.2 *was suggested by the ℓ-adic picture* [**HT21**]. *In loc. it. indeed, a similar bound was obtained for ℓ-adic nearby slopes of smooth morphisms* [**Tey15**]. *In that setting, the main tools are Beilinson's and Saito's work on the singular support* [**Bei16**] *and the characteristic cycle* [**Sai17**] *for ℓ-adic sheaves, as well as semi-continuity properties* [**HY17**][**Hu17**] *for various ramification invariants produced by Abbes and Saito's ramification theory* [**AS02**]. *From this perspective, proposition* 7.1.2 *is a positive answer to a local variant for differential equations of a conjecture in* [**Lea16**] *on the ramification of the étale cohomology groups for local systems on the generic fiber of a strictly semi-stable pair. See Conjecture 5.8 from* [**HT21**] *for a precise statement.*

7.2. Boundedness of the turning locus in the case of smooth proper relative curves. — This subsection is devoted to the proof of Theorem 2. Let S be a smooth complex algebraic curve. Let $p : \mathcal{C} \longrightarrow S$ be a relative smooth proper curve of genus g. Let $\mathcal{M}$ be a meromorphic connection of rank r on $\mathcal{C}$ with poles along the fibre $\mathcal{C}_0$. Let $Z(\mathcal{M})$ be the subset of points in $\mathcal{C}_0$ at which $\mathcal{M}$ does not have good formal structure (that is, the turning locus of $\mathcal{M}$). Let $\operatorname{irr}_{\mathcal{C}_0}\mathcal{M}$ be the generic irregularity of $\mathcal{M}$ along $\mathcal{C}_0$. Let $r_D(\mathcal{M})$ be the highest generic slope of $\mathcal{M}$ along $\mathcal{C}_0$. We put

$$K := (\operatorname{Sol}\mathcal{M})_{|\mathcal{C}_0}[1] \oplus (\operatorname{Sol}\operatorname{End}\mathcal{M})_{|\mathcal{C}_0}[1]$$

Then, K is a complex of $\mathbb{C}$-vector spaces on $\mathcal{C}_0$ with constructible cohomology. It is concentrated in degree 0 and 1. The generic rank of K is

$$\begin{aligned} r_K &= \operatorname{irr}_{\mathcal{C}_0}\mathcal{M} + \operatorname{irr}_{\mathcal{C}_0}\operatorname{End}\mathcal{M} \\ &\leqslant r r_D(\mathcal{M}) + r^2 r_D(\operatorname{End}\mathcal{M}) \\ &\leqslant 2r^2 r_D(\mathcal{M}) \end{aligned}$$

where the last inequality comes from $r_D(\operatorname{End}\mathcal{M}) \leqslant r_D(\mathcal{M})$. The Euler-Poincaré characteristic formula [**Lau87**, Th. 2.2.1.2] applied to K gives

$$\chi(\mathcal{C}_0, K) = (2-2g)r_K - \sum_{x\in\operatorname{Sing}K}(r_K - \dim\mathcal{H}^0K_x) + \dim\mathcal{H}^1K_x$$

where $\operatorname{Sing}K$ denotes the singular locus of K, that is the subset of points in $\mathcal{C}_0$ in the neighbourhood of which K is not a local system concentrated in degree 0. From Mebkhout perversity theorem [**Meb90**], the complex K is perverse. In particular, $\mathcal{H}^0K$ does not have sections with punctual support. Thus,

$$r_K - \dim\mathcal{H}^0K_x \geqslant 0$$

for every $x \in \operatorname{Sing}K$. From perversity again [**Tey13**, 13.1.6], the local Euler-Poincaré characteristic of K at $x \in \operatorname{Sing}K$ differs from its generic value r_K. Hence, for $x \in \operatorname{Sing}K$, the quantity

$$(r_K - \dim\mathcal{H}^0K_x) + \dim\mathcal{H}^1K_x$$

is positive and non zero. It is thus strictly positive. Hence, we have a bound

$$|\operatorname{Sing}K| \leqslant (2-2g)r_K - \chi(\mathcal{C}_0, K)$$

From Theorem 1, the singular points of K are exactly the points in $\mathcal{C}_0$ at which $\mathcal{M}$ does not have good formal structure. Hence

$$|Z(\mathcal{M})| \leqslant 2r_K + |\chi(\mathcal{C}_0, K)|$$

We are now left to bound $\chi(\mathcal{C}_0, K)$. Since the irregularity complex is compatible with proper push-forward [**Meb04**, 3.6-6], we have

$$\begin{aligned} |\chi(\mathcal{C}_0, (\operatorname{Sol}\mathcal{M})_{|\mathcal{C}_0})| &= |\chi(0, Rp_*(\operatorname{Sol}\mathcal{M})_{|\mathcal{C}_0})| \\ &= |\chi(0, (\operatorname{Sol}p_+\mathcal{M})_{|0})| \\ &= |\sum_i(-1)^i \operatorname{irr}_0\mathcal{H}^ip_+\mathcal{M}| \\ &\leqslant \sum_i \operatorname{irr}_0\mathcal{H}^ip_+\mathcal{M} \\ &\leqslant \sum_i \operatorname{rk}\mathcal{H}^ip_+\mathcal{M} \times \operatorname{Max}\operatorname{Sl}_t^{\mathrm{nb}}(\mathcal{H}^ip_+\mathcal{M}) \\ &\leqslant \sum_i \operatorname{rk}\mathcal{H}^ip_+\mathcal{M} \times \operatorname{Max}\operatorname{Sl}_t^{\mathrm{nb}}(p_+\mathcal{M}) \end{aligned}$$

Since nearby slopes are compatible with proper push-forward [**Tey16**, Th. 3 (ii)], we have $\mathrm{Sl}^{\mathrm{nb}}_t(p_+\mathcal{M}) \subset \mathrm{Sl}^{\mathrm{nb}}_p(\mathcal{M})$. Since p is smooth, proposition 7.1.2 yields

$$|\chi(\mathcal{C}_0, (\mathrm{Sol}\,\mathcal{M})_{|\mathcal{C}_0})| \leqslant r_D(\mathcal{M}) \sum_i \mathrm{rk}\,\mathcal{H}^i p_+\mathcal{M}$$

For a generic point $s \in S$, we have furthermore

$$\begin{aligned}\sum_i \mathrm{rk}\,\mathcal{H}^i p_+\mathcal{M} &= \sum_i \dim(\mathrm{Sol}\,\mathcal{H}^i p_+\mathcal{M})_s\\ &= \sum_i \dim(\mathcal{H}^i\,\mathrm{Sol}\,p_+\mathcal{M})_s\\ &= \sum_i \dim(R^i p_*\,\mathrm{Sol}\,\mathcal{M})_s\\ &= \sum_i \dim H^i(\mathcal{C}_s, \mathcal{L})\end{aligned}$$

where $\mathcal{L}$ denotes the local system of solutions of $\mathcal{M}_{|\mathcal{C}_s}$. Then $H^i(\mathcal{C}_s, \mathcal{L}) = 0$ for every $i \neq 0, 1, 2$ and we have

$$\dim H^0(\mathcal{C}_s, \mathcal{L}) \leqslant \mathrm{rk}\,\mathcal{L} = \mathrm{rk}\,\mathcal{M}_{|\mathcal{C}_s} = r$$

From Poincaré-Verdier duality, we have

$$\dim H^2(\mathcal{C}_s, \mathcal{L}) = \dim H^0(\mathcal{C}_s, \mathcal{L}^*) \leqslant \mathrm{rk}\,\mathcal{L}^* = \mathrm{rk}\,\mathcal{M}_{|\mathcal{C}_s} = r$$

Finally,

$$\begin{aligned}\dim H^1(\mathcal{C}_s, \mathcal{L}) &= -\chi(\mathcal{C}_s, \mathcal{L}) + \dim H^0(\mathcal{C}_s, \mathcal{L}) + \dim H^2(\mathcal{C}_s, \mathcal{L})\\ &= -\chi(\mathcal{C}_s, \mathbb{C})\,\mathrm{rk}\,\mathcal{L} + \dim H^0(\mathcal{C}_s, \mathcal{L}) + \dim H^2(\mathcal{C}_s, \mathcal{L})\\ &\leqslant 2r(g+1)\end{aligned}$$

Putting everything together yields

$$|Z(\mathcal{M})| \leqslant 8r^2(g+1)r_D(\mathcal{M})$$

This finishes the proof of Theorem 2.

8. Appendix

8.1. Torsors. — In this subsection, we collect elementary facts and definitions on torsors under a sheaf of groups.

Let X be a topological space. Let $\mathcal{G}$ be a sheaf of groups on X. We recall that a torsor under $\mathcal{G}$ is a sheaf $\mathcal{F}$ on X endowed with a left action of $\mathcal{G}$ such that there exists a cover $\mathcal{U}$ by open subsets of X such that for every $U \in \mathcal{U}$, there exists an isomorphism of sheaves $\mathcal{F}_{|U} \simeq \mathcal{G}_{|U}$ commuting with the action of $\mathcal{G}$, where $\mathcal{G}$ acts on itself by multiplication on the left. We denote by $\mathrm{Tors}(X, \mathcal{G})$ the category of $\mathcal{G}$-torsors on X. It is a standard fact that isomorphism classes of $\mathcal{G}$-torsors are in bijection with $H^1(X, \mathcal{G})$, the set of non abelian cohomology classes of $\mathcal{G}$.

The following lemma is due to Babbit-Varadarajan [**BV89**, 1.3.3].

Lemma 8.1.1. — *Let $p : Y \longrightarrow X$ be a Galois finite covering of compact metric spaces with Galois group G. Let $\mathcal{G}$ be a sheaf of groups on X. Suppose that the $p^*\mathcal{G}$-torsors have no non trivial automorphisms. Then, the canonical morphism*

$$H^1(X, \mathcal{G}) \longrightarrow H^1(Y, p^*\mathcal{G})^G$$

is bijective.

In [**BV89**, 1.3.3], the condition on $p^*\mathcal{G}$ is expressed in terms of the triviality of the set of global sections of the twist [**BV89**, 1.3] of $p^*\mathcal{G}$ by a $p^*\mathcal{G}$-torsor $\mathcal{T}$. Observe that this set is also the set of automorphisms of $\mathcal{T}$.

Lemma 8.1.2. — *Let $p : Y \longrightarrow X$ be a morphism of topological spaces. Let $\mathcal{G}$ be a sheaf of groups on Y. Let $\mathcal{T}$ be a $\mathcal{G}$-torsor on Y. Suppose that X admits a cover $\mathcal{U}$ by open subsets such that for every $U \in \mathcal{U}$, the torsor $\mathcal{T}$ is trivial on $p^{-1}(U)$. Then $p_*\mathcal{T}$ is a $p_*\mathcal{G}$-torsor on X.*

8.2. Recollections on Stokes filtered local systems. — The goal of this subsection is to recall the notion of Stokes filtered local systems. Note that our use of Stokes filtered local systems in this paper is a purely technical detour to obtain the triviality criterion 8.4.1. Hence, we don't claim for completeness in these recollections and they can be omitted in a first reading. For more background material, let us refer to [**Moc11a**] and [**Sab12**].

We fix a germ of regular pair (X, D) at a point P and denote by $p_D : \widetilde{X} \longrightarrow X$ the real blow-up of X along D. Let $\mathcal{I}$ be a good sheaf of irregular values in $\mathcal{O}_X(*D)/\mathcal{O}_X$. Let us recall the following fact [**Moc11a**, 3.5].

Fact 8.2.1. — *For every point Q in ∂D, there is an open neighbourhood U_Q of Q in ∂D such that for every point Q' in U_Q, the induced map*

$$(\mathcal{I}_{p_D(Q)}, \leqslant_Q) \longrightarrow (\mathcal{I}_{p_D(Q')}, \leqslant_{Q'})$$

is well-defined and order preserving.

The following definition appears in the absolute case in [**Moc11a**, 3.6].

Definition 8.2.2. — *Let R be a ring. Let U be a locally connected subset in ∂D. An $\mathcal{I}$-Stokes filtered local system on U is the data of a local system L of projective R-modules of finite type on U, and for every point Q in U, a split $(\mathcal{I}_{p_D(Q)}, \leqslant_Q)$-filtration $L_{\leqslant,Q}$ on L_Q by projective submodules. We further require the following compatibility conditions when Q varies. For every point Q in U, for every neighbourhood U_Q of Q as in* 8.2.1 *such that $U_Q \cap U$ is connected, the filtration $L_{\leqslant,Q'}$ on $L_{Q'}$ is induced by that on L_Q via $(\mathcal{I}_{p_D(Q)}, \leqslant_Q) \longrightarrow (\mathcal{I}_{p_D(Q')}, \leqslant_{Q'})$.*

Remark 8.2.3. — *Let U be a locally connected subset in ∂D. Put $Z = p_D(U)$ and suppose that $\mathcal{I}_{|Z}$ is constant. Then, the notion of $\mathcal{I}_{|Z}$-graded local system on U makes sense. Observe that any $\mathcal{I}_{|Z}$-graded local system on U gives rise to an $\mathcal{I}_{|Z}$-Stokes filtered local system on U. On the other hand, the grading operation* Gr *from* [**Moc11a**, 3.6] *is a well defined functor converting $\mathcal{I}_{|Z}$-Stokes filtered local systems on*

U into $\mathcal{I}_{|Z}$-Stokes graded local systems on U. The case of interest to us will be the case where Z lies in a stratum of ∂D or when $\mathcal{I}$ is very good.

The following is a relative variant of Proposition 3.16 from [**Moc11a**] which can easily be deduced from it.

Proposition 8.2.4. — *Let (X, D) be a germ of regular pair at a point P. Let $D_1, \dots, D_m$ be the components of D. Let I be a subset in $[\![1, m]\!]$. Let $\mathcal{I} \subset \mathcal{O}_X(*D)/\mathcal{O}_X$ be a good sheaf of irregular values. Let Q be a point in ∂P. Let $\mathcal{S}$ be a neighbourhood of Q in $\widetilde{X}$ of the form*

$$\prod_{i=1}^{m}([0, r[\times I_i) \times \Delta$$

where $r > 0$, where $I_1, \dots, I_m$ are closed intervals in S^1, and where Δ is a ball in $\mathbb{C}^{n-m}$ centred at 0. We have

$$\mathcal{S} \cap \partial D_I^\circ = \prod_{i \in I}(\{0\} \times I_i) \times \prod_{i \notin I}(]0, r[\times I_i) \times \Delta$$

Let R be a ring. Then, at the cost of shrinking $\mathcal{S}$, every $\mathcal{I}$-Stokes filtered local system on $\mathcal{S} \cap \partial D_I^\circ$ relative to R comes from an $\mathcal{I}$-Stokes graded local system on $\mathcal{S} \cap \partial D_I^\circ$.

The relationship between Stokes filtered local systems and Stokes torsors will be made via the notion of marked Stokes filtered local systems, that we now introduce.

Definition 8.2.5. — *Let U be a locally connected subset in ∂D. Put $Z = p_D(U)$ and suppose that $\mathcal{I}_{|Z}$ is constant. Let R be a ring. Let $(L, L_\leqslant)$ be an $\mathcal{I}$-Stokes filtered local system on U relative to R. A $(L, L_\leqslant)$-marked Stokes filtered local system on U is the data of a pair $((\mathcal{L}, \mathcal{L}_\leqslant), \mathrm{iso})$ where $(\mathcal{L}, \mathcal{L}_\leqslant)$ is an $\mathcal{I}$-Stokes filtered local system on U and where* iso *is an isomorphism between* $\mathrm{Gr}\,\mathcal{L}$ *and* $\mathrm{Gr}\,L$.

8.3. Relation with Stokes torsors. — Let (X, D) be a germ of regular pair at a point P. Let $D_1, \dots, D_m$ be the components of D. Let I be a subset in $[\![1, m]\!]$. Let $\mathcal{M}$ be a good elementary local model on X with poles along D, that is

$$\mathcal{M} = \bigoplus_{a \in \mathcal{I}} \mathcal{E}^a \otimes \mathcal{R}_a$$

where $\mathcal{I}$ is a good set of irregular values at P, where $\mathcal{E}^a = (\mathcal{O}_X(*D), d - da)$ and where $\mathcal{R}_a$ is a regular singular meromorphic connection on X with poles along D. For $a \in \mathcal{I}$, we denote by $\xi_I(a)$ the truncation of a which consists in keeping only the monomials in a having poles along every component $D_i, i \in I$. Put

$$\mathcal{M}(I) = \bigoplus_{a \in \mathcal{I}} \mathcal{E}^{\xi_I(a)} \otimes \mathcal{R}_a$$

With the help of proposition 8.2.4, the lemma 8.3.1 that follows allows to transfer splitting statements for Stokes filtered local systems to triviality statements for Stokes torsors.

Lemma 8.3.1. — *Let $(L, L_{\leqslant})$ be the Stokes filtered local system on ∂D associated to $\mathcal{M}(I)$. Let U be a locally connected subset in ∂D_I°. Let R be a $\mathbb{C}$-algebra. Let $(L(R), L(R)_{\leqslant})$ be the Stokes filtered local system relative to R induced by $(L, L_{\leqslant})$. Then, there is a canonical bijection between $H^1(U, \mathrm{St}_{\mathcal{M}}^{<D}(R))$ and the set of isomorphisms classes of $(L(R), L(R)_{\leqslant})$-marked Stokes filtered local systems on U. Via this identification, the trivial torsor under $\mathrm{St}_{\mathcal{M}}^{<D}(R)$ corresponds to $((L(R), L(R)_{\leqslant}), \mathrm{id})$.*

Proof. — Observe that the restriction of $\mathrm{St}_{\mathcal{M}}^{<D}$ to ∂D_I° is the sheaf $\mathrm{St}_{\mathcal{M}(I)}^{<D}$. Hence, we are left to construct a bijection between $H^1(U, \mathrm{St}_{\mathcal{M}(I)}^{<D}(R))$ and the set of isomorphisms classes of $(L(R), L(R)_{\leqslant})$-marked Stokes filtered local systems on U. Note that the sheaf $\xi_I(\mathcal{I})$ of irregular values of $\mathcal{M}(I)$ is very good on D_I°. Then, lemma 8.3.1 is consequence of the next lemma 8.3.2. □

Lemma 8.3.2. — *Let (X, D) be a germ of regular pair at a point P. Let $\mathcal{M}$ be a meromorphic connection on X with poles along D. Suppose that $\mathcal{M}$ has a good elementary local model at every point. Suppose that the sheaf of irregular values of $\mathcal{M}$ is very good. Let $(L, L_{\leqslant})$ be the Stokes filtered local system on ∂D associated to $\mathcal{M}$. Let U be a locally connected subset in ∂D. Let R be a $\mathbb{C}$-algebra. Then, there is a canonical bijection between $H^1(U, \mathrm{St}_{\mathcal{M}}^{<D}(R))$ and the set of isomorphism classes of $(L(R), L(R)_{\leqslant})$-marked Stokes filtered local systems on U. Via this identification, the trivial torsor under $\mathrm{St}_{\mathcal{M}}^{<D}(R)$ corresponds to $((L(R), L(R)_{\leqslant}), \mathrm{id})$.*

Proof. — Let $((\mathcal{L}, \mathcal{L}_{\leqslant}), \mathrm{iso})$ be a $(L(R), L(R)_{\leqslant})$-marked Stokes filtered local system on U. Consider the sheaf $\mathrm{Isom}_{\mathrm{iso}}(\mathcal{L}, L(R))$ on U whose sections on an open subset V in U is the set of isomorphisms of Stokes filtered local systems $f : (\mathcal{L}, \mathcal{L}_{\leqslant}) \longrightarrow (L(R), L(R)_{\leqslant})$ on V such that $\mathrm{Gr}\, f = \mathrm{iso}$. Since $\mathcal{L}$ is locally isomorphic to the Stokes filtered local system associated to $\mathrm{Gr}\,\mathcal{L}$, and similarly with L, the sheaf $\mathrm{Isom}_{\mathrm{iso}}(\mathcal{L}, L(R))$ is a torsor for the action of $\mathrm{St}_{\mathcal{M}}^{<D}(R)$ on $\mathrm{Isom}_{\mathrm{iso}}(\mathcal{L}, L(R))$ by post-composition. On the other hand, let $\mathcal{T}$ be an element in $H^1(U, \mathrm{St}_{\mathcal{M}}^{<D}(R))$. Choose a cover $\mathcal{V} = (V_i)_{i\in K}$ of U such that $\mathcal{T}$ comes from a cocycle $g = (g_{ij})_{i,j\in K}$. Set $L_i := L(R)_{|U_i}$. The identifications $g_{ij} : L_{i|U_{ij}} \longrightarrow L_{j|U_{ij}}$ allow to glue the L_i into a Stokes filtered local system $(\mathcal{L}, \mathcal{L}_{\leqslant})$ on U. Since the g_{ij} lie in $\mathrm{St}_{\mathcal{M}}^{<D}(R)$, the graded of the identity morphisms $L_i \longrightarrow L(R)_{|U_i}$ glue into an isomorphism $\mathrm{iso} : \mathrm{Gr}\,\mathcal{L} \longrightarrow \mathrm{Gr}\, L$. Hence, $((\mathcal{L}, \mathcal{L}_{\leqslant}), \mathrm{iso})$ defines an $(L(R), L(R)_{\leqslant})$-marked Stokes filtered local systems on U whose isomorphism class does not depend on any choice. It is then a standard check to verify that the two constructions above are mutually inverse bijections. □

8.4. Trivialization of Stokes torsors. — As a consequence of the relationship 8.3.1 between Stokes filtered local systems and Stokes torsors, the variant 8.2.4 of Mochizuki's splitting criterion gives the following triviality criterion for torsors under the Stokes sheaf of a *good elementary local model.*

Lemma 8.4.1. — *Let (X, D) be a germ of regular pair at a point P. Let $D_1, \dots, D_m$ be the components of D. Let I be a subset in $[\![1, m]\!]$. Let $\mathcal{M}$ be a good elementary local* *model on X with poles along D. Let Q be a point in ∂P. Let $\mathcal{S}$ be a neighbourhood of Q in $\widetilde{X}$ of the form*

$$\prod_{i=1}^{m}([0,r[\times I_i)\times\Delta$$

where $r>0$, where $I_1,\dots,I_m$ are closed intervals in S^1, and where Δ is a ball in $\mathbb{C}^{n-m}$ centred at 0. We have

$$\mathcal{S}\cap\partial D_I^\circ=\prod_{i\in I}(\{0\}\times I_i)\times\prod_{i\notin I}(]0,r[\times I_i)\times\Delta$$

Then, at the cost of shrinking $\mathcal{S}$, Stokes torsors on $\mathcal{S}\cap\partial D_I^\circ$ are trivial.

Proof. — We use the notations from 8.3. Let $(L,L_\leqslant)$ be the $\xi_I(\mathcal{I})$-Stokes filtered local system on ∂D associated with $\mathcal{M}(I)$. From lemma 8.3.1, we are left to show that at the cost of shrinking $\mathcal{S}$, the $(L,L_\leqslant)$-marked Stokes filtered local systems on $\mathcal{S}\cap\partial D_I^\circ$ are trivial. To do this, it is enough to show that at the cost of shrinking $\mathcal{S}$, the $\xi_I(\mathcal{I})$-Stokes filtered local systems on $\mathcal{S}\cap\partial D_I^\circ$ come from $\xi_I(\mathcal{I})$-graded local systems on $\mathcal{S}\cap\partial D_I^\circ$. Then, lemma 8.4.1 is a consequence of proposition 8.2.4. □

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