

# Existence and uniqueness theorem for a 3-dimensional polytope in $\mathbb{R}^3$ with prescribed directions and perimeters of the facets

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## Abstract

We give a necessary and sufficient condition for the existence and uniqueness up to translations of a 3-dimensional polytope  $P$  in  $\mathbb{R}^3$  having  $N$  facets with given unit outward normal vectors  $n_1, \dots, n_N$  and corresponding facet perimeters  $L_1, \dots, L_N$ .

In 1897, Hermann Minkowski studied the problem of prescribing the areas and outer unit normals of the facets of a 3-dimensional polytope. The existence and uniqueness theorem that he obtained is one the most fundamental result in the theory of polytopes. This paper is devoted to the analogue problem of prescribing the perimeters and outer unit normals of the facets of a 3-dimensional polytope. Our main result (Theorem 5) gives a necessary and sufficient condition for the existence and uniqueness up to translations of a 3-dimensional polytope  $P$  in  $\mathbb{R}^3$  having  $N$  facets with given unit outward normals  $n_1, \dots, n_N$  and corresponding facet perimeters  $L_1, \dots, L_N$ .

## Introduction to the problem

In this paper, a polytope of  $\mathbb{R}^3$  is the convex hull of finitely many points in  $\mathbb{R}^3$ . The classical Minkowski problem for polytopes in  $\mathbb{R}^3$  concerns the following question:

*Given a collection  $n_1, \dots, n_N$  of  $N$  pairwise distinct unit vectors in  $\mathbb{R}^3$  and  $F_1, \dots, F_N$  a collection of  $N$  positive real numbers, is there a polytope  $P$  in  $\mathbb{R}^3$  having the  $n_i$  as its facet unit outward normals and the  $F_i$  as the corresponding facet areas ( $1 \leq i \leq N$ ), and, if so, is  $P$  unique up to translations?*

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2010 MSC: 52B10; 52A25

Key words and phrases: Polytopes, Minkowski Problem, existence and uniqueness theorem, perimeters of facets

H. Minkowski proved the following uniqueness theorem (see [1, Theorem 9, p. 107]):

**Theorem 1** (*H. Minkowski, 1897: [5] and [6, pp. 103-121]*)

*A polytope in  $\mathbb{R}^3$  is uniquely determined, up to translations, by the directions and the areas of its facets.*

A well-known necessary condition for the existence of a polytope having facet unit outward normals  $n_1, \dots, n_N$  and corresponding facet areas  $F_1, \dots, F_N$  is that:

$$\sum_{i=1}^N F_i n_i = 0.$$

An existence theorem of H. Minkowski ensures that this condition is both necessary and sufficient:

**Theorem 2** (*H. Minkowski, 1897: [5] and [6, pp. 103-121]*)

*Let  $n_1, \dots, n_N \in \mathbb{R}^3$  be  $N$  pairwise distinct unit vectors linearly spanning  $\mathbb{R}^3$  and let  $F_1, \dots, F_N$  be  $N$  positive real numbers. There exists a polytope  $P$  in  $\mathbb{R}^3$  having  $N$  facets with unit outward normals  $n_1, \dots, n_N$  and corresponding facet areas  $F_1, \dots, F_N$  if, and only if, we have*

$$\sum_{i=1}^N F_i n_i = 0.$$

Here, we have to mention that Theorem 2 is only the 3-dimensional version of the classical Minkowski existence and uniqueness theorem [7, p. 455], which is valid in  $\mathbb{R}^d$  for all  $d \geq 2$ . The proof of our main result (Theorem 5) will make use of the 2-dimensional version, which is almost trivial:

**The Minkowski theorem for convex polygons in  $\mathbb{R}^2$ .**

*Let  $n_1, \dots, n_N \in \mathbb{R}^2$  be  $N$  pairwise distinct unit vectors linearly spanning  $\mathbb{R}^2$  and let  $l_1, \dots, l_N$  be  $N$  positive real numbers. There exists a convex polygon  $P$  in  $\mathbb{R}^2$  having  $N$  edges with unit outward normals  $n_1, \dots, n_N$  and corresponding edge lengths  $l_1, \dots, l_N$  if, and only if,*

$$\sum_{i=1}^N l_i n_i = 0.$$

This paper is devoted to the analogue of the classical Minkowski problem obtained by replacing areas by perimeters. For this analogue, the following uniqueness result is known (see [1, p. 108]):

**Theorem 3** *A polytope in  $\mathbb{R}^3$  is uniquely determined, up to translations, by the directions and the perimeters of its facets.*

Theorems 1 and 3 are similar uniqueness theorems which are both corollaries of a same general result by A.D. Alexandrov (see [1, Theorem 8, p. 107]). Thus, we are led to the natural question of the existence of an analogue to Theorem 2 for the existence of a polytope with prescribed directions and perimeters of the facets.

For convenience, we will restrict ourselves to 3-dimensional polytopes in  $\mathbb{R}^3$ . Recall that the dimension of a convex body in  $\mathbb{R}^d$  is simply the dimension of its affine hull. Recall also that a facet of a 3-dimensional polytope  $P$  is a (convex) polygonal face of  $P$ , and that its perimeter is defined to be the sum of the lengths of all its sides (edges).

### Difficulty of the problem

This problem of prescribing the perimeters and outer unit normals of the facets of a 3-dimensional polytope has attracted the attention of geometers. Recently, a paper by V. Alexandrov highlighted its difficulty in explaining why a simple equation involving the prescribed perimeters cannot suffice to establish an analogue to Theorem 2 [2]. The main result of that paper reads as follows:

**Theorem 4** (*V. Alexandrov, 2018: [2]*)

*Let  $n_1, \dots, n_5$  in  $\mathbb{R}^3$  be defined by the formulas*

$$\begin{aligned} n_1 &:= (0, 0, 1), \quad n_2 := \left( \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \quad n_3 := \left( -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \\ n_4 &:= \left( 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right), \quad n_5 := \left( 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right). \end{aligned}$$

*Let  $\mathcal{L}(n_1, \dots, n_5) \subset \mathbb{R}^5$  be the set of all points  $(L_1, \dots, L_5) \in \mathbb{R}^5$  with the following property: there exists a polytope  $P \subset \mathbb{R}^3$  such that  $n_1, \dots, n_5$  (and no other vector) are the unit outward normals to the facets of  $P$ , and  $L_k$  is the perimeter of the face with the outward normal  $n_k$  for every  $k \in \{1, \dots, 5\}$ . Then the set  $\mathcal{L}(n_1, \dots, n_5) \subset \mathbb{R}^5$  is not locally-analytic.*

This result is of course interpreted by V. Alexandrov as an obstacle for finding an existence theorem for a polytope with prescribed directions and perimeters of the facets. This was the major source of inspiration for the work presented in this paper.

### Necessary conditions for the existence of a solution

Let  $n_1, \dots, n_N$  be a collection of  $N$  pairwise distinct unit vectors linearly spanning  $\mathbb{R}^3$  and let  $L_1, \dots, L_N$  be a collection of  $N$  positive

real numbers. The following set  $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$  of conditions is necessary for the existence of a 3-dimensional polytope  $P$  in  $\mathbb{R}^3$  having the  $n_i$  as its facet outward unit normals and the  $L_i$  as the corresponding facet perimeters.

(i) For each  $i \in \{1, \dots, N\}$ , there exists a decomposition of  $L_i$  into a sum of  $N$  non-negative real numbers,

$$L_i = \sum_{j=1}^N l_{ij},$$

in such a way that:

(ii) for all  $(i, j) \in (\{1, \dots, N\})^2$ ,  $l_{ij} = 0$  if  $n_i$  and  $n_j$  are collinear;

(iii) for all  $(i, j) \in (\{1, \dots, N\})^2$ ,  $l_{ji} = l_{ij}$ .

In other words, conditions (i) through (iii) require the existence of a symmetric matrix with nonnegative entries  $(l_{ij})_{1 \leq i, j \leq N}$ , such that row  $i$  sums to  $L_i$ , ( $1 \leq i \leq N$ ), and  $l_{ij} = 0$  for collinear  $n_i, n_j$ , ( $1 \leq i, j \leq N$ ).

Indeed, if such a polytope  $P$  exists, then denoting by  $f_1, \dots, f_N$  the  $N$  facets with respective unit outward normals  $n_1, \dots, n_N$ , the required relationships hold if we put:

$$l_{ij} := \begin{cases} 0 & \text{if } i = j \text{ or if } f_i \text{ and } f_j \text{ have no common edge} \\ \text{the length of the common edge} & \text{otherwise.} \end{cases}$$

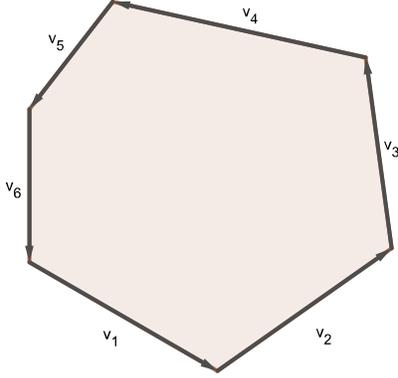
Our condition (iv) is a consequence of the fact that the edge vectors of a facet (which are perpendicular to the unit normals of both incident facets), oriented in positive direction with respect to the unit normal of the facet, concatenate into a (simple) closed circuit:

$$(\mathbf{iv}) \text{ For every } i \in \{1, \dots, N\}, \quad \sum_{j \in \{k | l_{ik} \neq 0\}} l_{ij} \left[ \frac{n_i \times n_j}{\sin(n_i, n_j)} \right] = 0,$$

where  $\times$  denotes the cross product (here, we of course assume that  $\mathbb{R}^3$  is oriented by its canonical basis), and  $(n_i, n_j)$  denotes the length of the shortest arc of great circle joining  $n_i$  to  $n_j$  on the unit sphere  $\mathbb{S}^2$  of  $\mathbb{R}^3$  (recall that  $n_i$  and  $n_j$  are non collinear by condition (ii) since  $l_{ij} \neq 0$ ). Indeed, each facet  $f_i$  of  $P$  is a convex polygon the boundary of which is a closed polygonal line. Here, it is worth noting that, for all  $j \in \{k | l_{ik} \neq 0\}$ ,

$$\overrightarrow{u_{i,j}} = (n_i \times n_j) / \sin(n_i, n_j)$$

is a unit vector that is such that the vector  $\vec{v}_{i,j} := l_{ij}\vec{u}_{i,j}$  is of the form  $\overrightarrow{MM'}$ , where  $M$  and  $M'$  are two consecutive vertices of the oriented boundary of the face  $f_i$  (see the figure 1 where  $v_1, \dots, v_m$  are the successive  $\vec{v}_{i,j} := l_{ij}\vec{u}_{i,j}$ , with  $j \in \{k \mid l_{ik} \neq 0\}$ ).



**Figure 1.** Illustration of the fact that

$$\sum_{j \in \{k \mid l_{ik} \neq 0\}} l_{ij}\vec{u}_{ij} = \sum_{l=1}^m v_l = 0$$

Our last necessary condition (v) will follow from Steinitz's theorem (e.g. see [4, Chapter 4, p. 103]), which characterizes in purely graph-theoretic terms those graphs that can be represented as the 1-skeleton of some 3-dimensional polytope:

**Theorem (Steinitz's theorem).** *A graph can be represented as the 1-skeleton of some 3-dimensional polytope if, and only if, it is simple, planar, and 3-connected.*

For the convenience of the reader, we shall summarize some basic definitions and facts on graphs and 1-skeletons just before the proof of Theorem 5. From Steinitz's theorem, the following last condition is also necessary in our case:

(v) The datum of the matrix  $(l_{ij})_{1 \leq i, j \leq N}$  determines as follows a simple 3-connected planar graph  $G$  drawn on the unit sphere  $\mathbb{S}^2$  (so that no two of the edges intersect at a point other than a vertex): the vertices of  $G$  are the unit vectors  $n_1, \dots, n_N$ , and any pair of non-collinear vertices  $\{n_i, n_j\}$  of  $G$  is connected by an edge that is given by the shortest arc of great circle joining the two vertices on  $\mathbb{S}^2$  if, and only if,  $l_{ij} \neq 0$ .

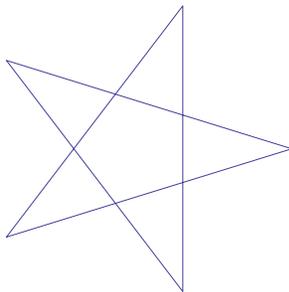
## Our main result

**Theorem 5** *Let  $n_1, \dots, n_N \in \mathbb{R}^3$  be  $N$  distinct unit vectors linearly spanning  $\mathbb{R}^3$  and let  $L_1, \dots, L_N$  be  $N$  positive real numbers. There exists a 3-dimensional polytope  $P$  in  $\mathbb{R}^3$  having facet unit outward normals  $n_1, \dots, n_N$  and corresponding facet perimeters  $L_1, \dots, L_N$  if, and only if, the set of conditions  $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$  holds.*

**Important remark.** The set of conditions  $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv})\}$  is far from being sufficient to ensure that there exists a 3-dimensional polytope in  $\mathbb{R}^3$  having facet unit outward normals  $n_1, \dots, n_N$  and corresponding facet perimeters  $L_1, \dots, L_N$ . Indeed, many problems can arise if we drop condition  $(\mathbf{v})$  from Theorem 5. If we retain only the first four conditions the data could correspond to a union of several polytopes or a non-convex polyhedron. The facets could themselves be non-convex. For instance, if we consider the unit vectors  $n_1, \dots, n_7 \in \mathbb{R}^3$  defined by  $n_1 := (0, 0, -1)$ ,  $n_2 := (\cos \frac{2\pi}{5}, \sin \frac{2\pi}{5}, 0)$ ,  $n_3 := (\cos \frac{6\pi}{5}, \sin \frac{6\pi}{5}, 0)$ ,  $n_4 := (1, 0, 0)$ ,  $n_5 := (\cos \frac{4\pi}{5}, \sin \frac{4\pi}{5}, 0)$ ,  $n_6 := (\cos \frac{8\pi}{5}, \sin \frac{8\pi}{5}, 0)$ ,  $n_7 := (0, 0, 1)$ , and the datum of the matrix

$$(l_{ij})_{1 \leq i, j \leq 7} := \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix},$$

for which the set of conditions  $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv})\}$  holds with  $L_1 = L_7 = 5$  and  $L_2 = L_3 = L_4 = L_5 = L_6 = 4$ , we obtain a non-convex right prism of which the bases are pentagrams (like the one shown in Figure 2), and there is no 3-dimensional polytope  $P$  in  $\mathbb{R}^3$  corresponding to these data.



**Figure 2.** A pentagram

In other words, given  $N$  distinct unit vectors  $n_1, \dots, n_N \in \mathbb{R}^3$  linearly spanning  $\mathbb{R}^3$  and  $N$  positive real numbers  $L_1, \dots, L_N$  such that the set of conditions  $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv})\}$  holds, we can of course associate a graph to the matrix  $(l_{ij})_{1 \leq i, j \leq N}$  but to be sure that this graph does correspond to the 1-skeleton of some 3-dimensional polytope, it is necessary and sufficient to assume that it satisfies the conditions of Steinitz's theorem. This is essentially what condition  $(\mathbf{v})$  requires.

### Basic definitions and facts on graphs and 1-skeletons

For the convenience of the reader, we summarize some basic definitions and facts on graphs and 1-skeletons:

- The *1-skeleton* of a polytope  $P$  is the graph whose vertices and edges are just the vertices and edges of  $P$  with the same incidence relation.
- A graph is said to be *polyhedral* if it can be represented as the 1-skeleton of some 3-dimensional polytope.
- A graph is said to be *simple* if it contains neither multiple edges nor loops.
- A graph is said to be *planar* if it can be drawn in the plane so that no two of the edges intersect at a point other than a vertex.
- A graph is called *3-connected* if it is connected, has at least 4 vertices, and remains connected whenever fewer than 3 vertices are removed.
- The dual of a polyhedral graph is also a polyhedral graph. More precisely, every polyhedral graph  $G$  has a well-defined dual graph  $G^*$  (independent of the plane embedding), corresponding to the 1-skeleton of the dual polytope.

**Proof of Theorem 5.** We have already seen that this set of conditions  $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$  is necessary for such a polytope  $P$  to exist in  $\mathbb{R}^3$ .

Conversely assume that the set of conditions  $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$  holds. Recall that two polytopes  $P$  and  $P'$  are said to be *combinatorially equivalent* if there is a bijection between their faces that preserves the inclusion relation. It is well-known that the combinatorial structure of a 3-dimensional polytope  $P$  is completely determined by its 1-skeleton [4, p. 105].

(a) *There exists a 3-dimensional polytope with the given combinatorial structure*

By Steinitz's theorem, the simple 3-connected planar graph  $G$  that is constructed on the sphere  $\mathbb{S}^2$  in accordance with condition  $(\mathbf{v})$  and its geometric dual graph  $G^*$  (which is also simple, planar, and 3-connected) are polyhedral: they can be represented in  $\mathbb{R}^3$  as the 1-skeletons of two dual 3-dimensional polytopes, say  $Q$  and  $Q^*$ , respectively.

(b) *The shape of one facet can be chosen*

Moreover, by the following refinement by Barnette and Grünbaum, we can preassign the shape of a face of one of these two polytopes [3]:

**Theorem (Barnette and Grünbaum).** *If one face of a 3-dimensional polytope  $Q$  is an  $n$ -gon, then there exists a polytope  $Q'$  combinatorially equivalent to  $Q$ , of which the corresponding face is any prescribed convex  $n$ -gon.*

Moreover, as noticed by Barnette and Grünbaum [3, p. 305]: "By an obvious application of duality, it follows from the theorem that the shape of one vertex-figure may be prescribed". Our desired polytope  $P$  will be combinatorially equivalent to  $Q^*$ .

(c) *Vocabulary convention*

In the remainder of the proof, the assembly of all the facets of a given 3-dimensional polytope that share a same vertex will be called a *corner* of the polytope. The *spherical representation* of such a corner, or of a 3-dimensional polytope, is defined as follows:

- A facet  $f$  corresponding to a unit normal  $n$  is represented on  $\mathbb{S}^2$  by  $n$ ;
- An edge is represented on  $\mathbb{S}^2$  by the arc of great circle joining the two points corresponding to the two adjacent facets of the edge;
- A vertex is represented on  $\mathbb{S}^2$  by the spherical polygon that is bounded by the spherical arcs corresponding to the edges that are adjacent at the vertex.

(d) *The polygons making up the desired facets are uniquely determined up to translations in space*

Our aim is to prove the existence of a 3-dimensional polytope  $P$  that satisfies the set of conditions  $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$ , and thus, the spherical representation of which is given by  $G$ .

Since condition  $(\mathbf{iv})$  is satisfied, the Minkowski existence and uniqueness theorem for convex polygons ensures that, for any  $i \in \{1, \dots, N\}$ , there exists in  $n_i^\perp$  (the linear plane with unit normal  $n_i$ , endowed with the induced orientation) a positively oriented convex polygon  $f_i$  whose edges  $e_{ij}$  are directed by the unit vectors

$$\overrightarrow{u_{i,j}} = (n_i \times n_j) / \sin(n_i, n_j), \quad (j \in \{k \mid l_{ik} \neq 0\}),$$

and have corresponding lengths  $l_{ij}$ ; and moreover, that this polygon  $f_i$ , of which the perimeter is  $L_i$  by condition  $(\mathbf{i})$ , is unique up to translations in  $n_i^\perp$ . Note that in the above expression of  $\overrightarrow{u_{i,j}}$ , the vectors  $n_i$  and  $n_j$  are non-collinear by condition  $(\mathbf{ii})$  since  $l_{ij} \neq 0$ .

Thus, for any  $i \in \{1, \dots, N\}$ , the desired oriented convex polygon  $f_i$ , with unit normal  $n_i$ , is well-defined and unique up to translations in  $\mathbb{R}^3$ .

(e) All the corners of the desired polytope are well-defined and unique up to translations in space

Now, let  $\mathcal{P}_n$  be any positively oriented  $n$ -gon on  $\mathbb{S}^2$  that is the oriented boundary of the closure of a connected component of the complementary of the graph  $G$ , which is drawn on  $\mathbb{S}^2$  according to condition (v), ( $n \geq 3$ ). Girard's theorem relates spherical angle excess and area of the spherical  $n$ -gon, which allows us to deduce that

$$\sum_{k=1}^n \alpha_k = (n-2)\pi + \text{Area}(\mathcal{P}_n),$$

where  $\alpha_1, \dots, \alpha_n$  denote the interior angles of  $\mathcal{P}_n$ . Because of condition (iv),  $\mathcal{P}_n$  has no reflex angle so that:

$$\sum_{k=1}^n \beta_k = 2\pi - \text{Area}(\mathcal{P}_n) < 2\pi,$$

where  $\beta_1, \dots, \beta_n$  denote the exterior angles of  $\mathcal{P}_n$ , (that is,  $\beta_k := \pi - \alpha_k$  for all  $k \in \{1, \dots, n\}$ ). Now, for any  $k \in \{1, \dots, n\}$ ,  $\beta_k > 0$  can be regarded as the interior angle of one of the desired convex polygons  $f_{i_k}$  (corresponding to a unit vector  $n_{i_k}$  that is a vertex of  $\mathcal{P}_n$ ) at the vertex of  $f_{i_k}$  that corresponds to  $\mathcal{P}_n$ . The above inequality, which says that the sum  $\sum_{i=1}^n \beta_k$  of these angles, is less than  $2\pi$  can be regarded as a *nonnegative curvature condition* that is satisfied from our conditions. Therefore, taking into account condition (iii), the convex polygons  $f_{i_k}$  that correspond to the unit vectors  $n_{i_k}$  that are the vertices of  $\mathcal{P}_n$  can be assembled (by gluing together their sides that correspond to a same edge of  $\mathcal{P}_n$ ) to form a corner (of a 3-dimensional polytope), the spherical representation of which corresponds to  $\mathcal{P}_n$  in  $\mathbb{S}^2$ . Here, the "convexity" at the corner is of course due to the nonnegative curvature condition.

Thus *all the corners of the desired polytope  $P$  are well-defined and unique up to translations in  $\mathbb{R}^3$ .*

(f) *They can be put together without contradiction*

Starting from any of these corners, we can construct by induction the desired polytope  $P$ , which is combinatorially equivalent to  $Q^*$  and satisfies the set of conditions  $\{(\mathbf{i}), (\mathbf{ii}), (\mathbf{iii}), (\mathbf{iv}), (\mathbf{v})\}$ , by assembling at each step, an adjacent corner to the part of  $P$ , say  $P_{art}(P)$ , that has already been constructed. Here, by "an adjacent corner" to  $P_{art}(P)$  we mean "a corner of  $P$  that is not included in  $P_{art}(P)$  but that shares

two facets with  $P_{art}(P)$ ". At each step, the spherical representation of the part of  $P$  that is constructed is controlled by condition (v), and the construction can continue until completion since we made sure that all the pieces had the required shape and dimensions. ■

**Remark.** It is worth noting that, as soon as the position of the first corner is fixed (with, for example, its vertex placed at the origin  $O$ ), the position of any other vertex  $S$  of  $P$  is deduced from that of  $O$  by a succession of translations from a vertex of  $P$  to another: consider any succession of adjacent regions of  $\mathbb{S}^2 \setminus G$  from that corresponding to  $O$  to that corresponding to  $S$  (two regions of  $\mathbb{S}^2 \setminus G$  are said to be adjacent if their boundaries share an edge of  $G$ ) and note that each crossing on  $\mathbb{S}^2$  from one region  $\mathcal{R}_i$  of  $\mathbb{S}^2 \setminus G$  to an adjacent one  $\mathcal{R}_j$  corresponds on  $P$  to the translation from a vertex to another by a translation by a vector  $\vec{v}_{ij}$  whose direction is determined by the arc of great circle  $\gamma_{ij}$  separating the two regions on  $\mathbb{S}^2$  ( $\vec{v}_{ij}$  is orthogonal to it and oriented in the sense of the crossing) and whose norm  $\|\vec{v}_{ij}\|$  is the length  $l_{ij}$  corresponding to  $\gamma_{ij}$ . Of course, thanks to condition (iv), the final position of the vertex (i.e. that of  $S$ ) does not depend on the succession of adjacent regions that has been considered.

### Acknowledgments

The author would like to thank Victor Alexandrov for carefully reading the manuscript and for giving detailed comments and useful suggestions. He also gratefully thanks the anonymous referees for their valuable suggestions that significantly helped to improve the manuscript.

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