# JONQUIÈRES MAPS AND SL(2; $\mathbb{C})$-COCYCLES 

by

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#### Abstract

We start the study of the family of birational maps $\left(f_{\alpha, \beta}\right)$ of $\mathbb{P}_{\mathbb{C}}^{2}$ in $[\mathbf{1 2}]$. For " $(\alpha, \beta)$ well chosen" of modulus 1 the centraliser of $f_{\alpha, \beta}$ is trivial, the topological entropy of $f_{\alpha, \beta}$ is 0 , there exist two domains of linearisation : in the first one the closure of the orbit of a point is a torus, in the other one the closure of the orbit of a point is the union of two circles. On $\mathbb{P}_{\mathbb{C}}^{1} \times \mathbb{P}_{\mathbb{C}}^{1}$ any $f_{\alpha, \beta}$ can be viewed as a cocyle ; using recent results about $\operatorname{SL}(2 ; \mathbb{C})$-cocycles ([1]) we determine the LyAPUNOV exponent of the cocyle associated to $f_{\alpha, \beta}$.


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## Introduction

In this article we deal with a family of birational maps $\left(f_{\alpha, \beta}\right)$ given by

$$
f_{\alpha, \beta}: \mathbb{P}_{\mathbb{C}}^{2} \longrightarrow \mathbb{P}_{\mathbb{C}}^{2} \quad(x: y: z) \rightarrow((\alpha x+y) z: \beta y(x+z): z(x+z))
$$

where $\alpha, \beta$ denote two complex numbers with modulus 1 , case where we know almost nothing about the dynamics. Let us consider the set $\Omega$ of pairs of complex numbers of modulus 1 that satisfy diophantine condition. The family $\left(f_{\alpha, \beta}\right)$ satisfies the following properties ( $\left.[\mathbf{1 2}]\right)$ :

- for $(\alpha, \beta) \in \Omega$ the centraliser of $f_{\alpha, \beta}$, that is the set of birational maps of $\mathbb{P}_{\mathbb{C}}^{2}$ that commutes with $f_{\alpha, \beta}$, is isomorphic to $\mathbb{Z}$;
- the topological entropy of $f_{\alpha, \beta}$ is 0 ;
- rotation domains of ranks 1 and 2 coexist: there is a domain of linearisation where the orbit of a generic point under $f_{\alpha, \beta}$ is a torus, and there is an other domain of linearisation where the orbit of a generic point under $f_{\alpha, \beta}^{2}$ is a circle.
We can also see $f_{\alpha, \beta}$ on $\mathbb{P}_{\mathbb{C}}^{1} \times \mathbb{P}_{\mathbb{C}}^{1}$ (since all the computations of [12] have been done in an affine chart they may all be carried on $\mathbb{P}_{\mathbb{C}}^{1} \times \mathbb{P}_{\mathbb{C}}^{1}$ ); the sets $\mathbb{P}_{\mathbb{C}}^{1} \times \mathbb{S}_{\rho}^{1}$, where $\mathbb{S}_{\rho}^{1}=\{y \in \mathbb{C}| | y \mid=\rho\}$, are invariant.

Let us define $A_{n}^{\alpha, \rho}: \mathbb{S}_{\rho}^{1} \rightarrow \mathbf{M}(2 ; \mathbb{C})$ given in terms of $A^{\alpha, \rho}(y)=\left[\begin{array}{cc}\alpha & y \\ 1 & 1\end{array}\right]$ by

$$
A_{n}^{\alpha, \rho}(\cdot)=A^{\alpha, \rho}\left(\beta^{n} \cdot\right) A^{\alpha, \rho}\left(\beta^{n-1} \cdot\right) \ldots A^{\alpha, \rho}(\beta \cdot) A^{\alpha, \rho}(\cdot)
$$

To compute $f_{\alpha, \beta}^{n}(x, y)$ is equivalent to compute $A_{n}^{\alpha, \rho}(y)$ as soon as $f_{\alpha, \beta}^{k}(x, y) \neq(-1, \alpha)$ for any $1 \leq k \leq n$.

Using [1] we are able to determine the LYAPUNOV exponent of the cocycle $\left(A^{\alpha, \rho}, \beta\right)$ :
Theorem A. - The LyAPUNOV exponent of $\left(A^{\alpha, \rho}, \beta\right)$ is

- positive as soon as $\rho>1$;
- zero as soon as $\rho \leq 1$.

More precisely $f_{\alpha, \beta}$ is semi conjugate to $\left(\frac{\alpha x+y^{2}}{x+1}, \beta^{1 / 2} y\right)$ and the LYAPUNOV exponent of the cocycle $\left(B^{\alpha, \rho}, \beta^{1 / 2}\right)$, where

$$
B^{\alpha, \rho}(y)=\left[\begin{array}{cc}
\alpha & y^{2} \\
1 & 1
\end{array}\right]
$$

is equal to $\max (0, \ln \rho)$.

In the next section we introduce the family $\left(f_{\alpha, \beta}\right)$ and its properties (§1). Then we deal with the recent works of AVILA on $\operatorname{SL}(2 ; \mathbb{C})$-cocyles. In the last section we give the proof of Theorem A (see §2). Let us explain the sketch of it. We associate to $\left(B^{\alpha, \rho}, \beta^{1 / 2}\right)$ a cocycle $\left(\widetilde{B}^{\alpha, \rho}, \beta^{1 / 2}\right)$ that belongs to $\operatorname{SL}(2 ; \mathbb{C})$. We first determine

$$
\lim _{\rho \rightarrow 0} L\left(\widetilde{B}^{\alpha, \rho}, \beta^{1 / 2}\right)
$$

and then

$$
\lim _{\rho \rightarrow+\infty} L\left(\widetilde{B}^{\alpha, \rho}, \beta^{1 / 2}\right)
$$

where $L(C, \gamma)$ denotes the LYAPUNOV exponent of the $\operatorname{SL}(2 ; \mathbb{C})$-cocyle $(C, \gamma)$. In both cases, we get 0 . Using [1, Theorem 5] we obtain that $L\left(\widetilde{B}^{\alpha, \rho}, \beta^{1 / 2}\right)$ vanishes everywhere; it allows us to determine $L\left(A^{\alpha, \rho}, \beta\right)$ since

$$
L\left(B^{\alpha, \rho}(y), \beta^{1 / 2}\right)=L\left(\widetilde{B}^{\alpha, \rho}(y), \beta^{1 / 2}\right)+\max (0, \ln \rho)
$$

and since $\left(A^{\alpha, \rho}, \beta\right)$ and $\left(\beta^{1 / 2}, B^{\alpha, \rho}\right)$ are conjugate.

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## 1. Some properties of the family $\left(f_{\alpha, \beta}\right)$

A rational map $\phi$ from $\mathbb{P}_{\mathbb{C}}^{2}$ into itself is a map of the form

$$
(x: y: z) \longrightarrow\left(\phi_{0}(x, y, z): \phi_{1}(x, y, z): \phi_{2}(x, y, z)\right)
$$

where the $\phi_{i}$ 's are some homogeneous polynomials of the same degree without common factor; $\phi$ is birational if it admits an inverse of the same type. We will denote by $\operatorname{Bir}\left(\mathbb{P}_{\mathbb{C}}^{2}\right)$ the group of birational maps of $\mathbb{P}_{\mathbb{C}}^{2}$, also called the Cremona group. The degree of $\phi$, denoted deg $\phi$, is the degree of the $\phi_{i}$ 's. The degree is not a birational invariant: $\operatorname{deg} \psi \phi \psi^{-1} \neq \operatorname{deg} \phi$ for generic birational maps $\phi$ and $\psi$. The first dynamical degree of $\phi$ given by

$$
\lambda(\phi)=\lim _{n \rightarrow+\infty}\left(\operatorname{deg} \phi^{n}\right)^{1 / n}
$$

is a birational invariant; it is strongly related to the topological entropy $h_{\text {top }}(\phi)$ of $\phi($ see $[\mathbf{1 7 , 2 0 ]})$

$$
\begin{equation*}
h_{\mathrm{top}}(\phi) \leq \log \lambda(\phi) \tag{1.1}
\end{equation*}
$$

Any birational map $\phi$ admits a resolution

where $\pi_{1}, \pi_{2}: S \rightarrow \mathbb{P}_{\mathbb{C}}^{2}$ are sequences of blow-ups (see $[\mathbf{3}]$ for example). The resolution is minimal if and only if no $(-1)$-curve of $S$ is contracted by both $\pi_{1}$ and $\pi_{2}$. The base-points of $\phi$ are the points blown-up in $\pi_{1}$, which can be points of $\mathbb{P}_{\mathbb{C}}^{2}$ or infinitely near points. We denote by $\mathfrak{b}(\phi)$ the number of such points, which is also equal to the difference of the ranks of $\operatorname{Pic}(S)$ and $\operatorname{Pic}\left(\mathbb{P}_{\mathbb{C}}^{2}\right)$, and thus equals to $\mathfrak{b}\left(\phi^{-1}\right)$. The dynamical number of base-points of $\phi$ introduced in [8] is by definition

$$
\mu(\phi)=\lim _{n \rightarrow+\infty} \frac{\mathfrak{b}\left(\phi^{n}\right)}{n} ;
$$

it is a real positive number that satisfies $\mu\left(\phi^{n}\right)=|n \mu(\phi)|$ for any $n \in \mathbb{Z}, \mu\left(\psi \phi \psi^{-1}\right)=\mu(\phi)$, and allows us to give a characterization of birational maps conjugate to automorphisms:

Theorem 1.1 ([8]). - Let S be a smooth projective surface; the birational map $\phi \in \operatorname{Bir}(\mathrm{S})$ is conjugate to an automorphism of a smooth projective surface if and only if $\mu(\phi)=0$.

The behavior of $\phi \in \operatorname{Bir}\left(\mathbb{P}_{\mathbb{C}}^{2}\right)$ is strongly related to the behavior of $\left(\operatorname{deg} \phi^{n}\right)_{n \in \mathbb{N}}($ see $[\mathbf{1 6}, \mathbf{1 5}, \mathbf{8}])$; up to birational conjugacy exactly one of the following holds:

1. the sequence $\left(\operatorname{deg} \phi^{n}\right)_{n \in \mathbb{N}}$ is bounded and either $\phi$ is of finite order, or $\phi$ is an automorphism of $\mathbb{P}_{\mathbb{C}}^{2}$;
2. there exists an integer $k$ such that

$$
\lim _{n \rightarrow+\infty} \frac{\operatorname{deg} \phi^{n}}{n}=k^{2} \frac{\mu(\phi)}{2}
$$

and $\phi$ is not an automorphism;
3. there exists an integer $k \geq 3$ such that

$$
\lim _{n \rightarrow+\infty} \frac{\operatorname{deg} \phi^{n}}{n^{2}}=k^{2} \frac{\kappa(\phi)}{9}
$$

where $\kappa(\phi) \in \mathbb{Q}$ is a birational invariant, and $\phi$ is an automorphism;
4. the sequence $\left(\operatorname{deg} \phi^{n}\right)_{n \in \mathbb{N}}$ grows exponentially (see [15] for more precise dynamical properties).
In the first three cases $\lambda(\phi)=1$, in the last one $\lambda(\phi)>1$. In case 2 . (resp. 3.) the map $\phi$ preserves a unique fibration which is rational (resp. elliptic).

In case 1. (resp. 2., resp. 3, resp. 4) we say that $\phi$ is elliptic (resp. a JonQuières twist, resp. an Halphen twist, resp. hyperbolic).

Let us give some examples. Let

$$
\phi(x, y)=\left(\frac{a(y) x+b(y)}{c(y) x+d(y)}, \frac{\alpha y+\beta}{\gamma y+\delta}\right)
$$

be an element of the JonQuières group $\operatorname{PGL}(2 ; \mathbb{C}(y)) \rtimes \operatorname{PGL}(2 ; \mathbb{C})$; either $\phi$ is elliptic (for instance $\phi:(x: y: z) \rightarrow(y z: x z: x y)$ ), or $\phi$ is a JonQuières twist (for example $\phi:(x: y: z) \rightarrow$ $\left(x z: x y: z^{2}\right)$ for which the unique invariant fibration is $y / z=$ constant $)$. The map

$$
\phi: \mathbb{P}_{\mathbb{C}}^{2} \longrightarrow \mathbb{P}_{\mathbb{C}}^{2} \quad(x: y: z) \rightarrow((2 y+z)(y+z): x(2 y-z): 2 z(y+z))
$$

is an Halphen twist ([15, Proposition 9.5]). HÉNON automorphisms give by homogeneization examples of hyperbolic maps.

Clearly elliptic birational maps have a poor dynamical behavior contrary to hyperbolic ones. The study of automorphisms of positive entropy is strongly related with birational maps of $\mathbb{P}_{\mathbb{C}}^{2}$ :
Theorem 1.2 ([9]). - Let S be a compact complex surface that carries an automorphism $\phi$ of positive topological entropy.

- Either the Kodaira dimension of S is zero and $\phi$ is conjugate to an automorphism on the unique minimal model of S that necessarily is a torus, or a K3 surface or an Enriques surface;
- or the surface S is a non-minimal rational one, isomorphic to $\mathbb{P}_{\mathbb{C}}^{2}$ blown up at $n$ points, $n \geq 10$, and $\phi$ is conjugate to a birational map of $\mathbb{P}_{\mathbb{C}}^{2}$.
This yields many examples of hyperbolic birational maps for which we can establish a lot of


Another way to measure chaos is to look at the size of centralisers. Let us give two examples. The polynomial automorphisms of $\mathbb{C}^{2}$ having rich dynamics are HÉNON maps; furthermore a polynomial automorphism of $\mathbb{C}^{2}$ is a HÉNON one if and only if its centraliser is countable. Let us now consider rational maps on $\mathbb{S}^{1}$; if the centraliser of such maps is not trivial ${ }^{(1)}$, then the JuLiA set is "special". The centraliser of an elliptic birational map of infinite order is uncountable ([8]). The centralisers of HALPHEN twists are described in [16]. The centraliser of an hyperbolic map is countable ([10]). In [11] we end the story by studying centralisers of JonQuières twists. If the

[^0]fibration is fiberwise invariant, then the centraliser is uncountable ; but if it isn't, then generically the centraliser is isomorphic to $\mathbb{Z}$. We don't know a lot about dynamics of these maps, in this article we will thus focus on a family of such maps. We consider the JONQUIÈRES maps
$$
f_{\alpha, \beta}: \mathbb{P}_{\mathbb{C}}^{2} \rightarrow \mathbb{P}_{\mathbb{C}}^{2} \quad(x: y: z) \rightarrow((\alpha x+y) z: \beta y(x+z): z(x+z))
$$
where $\alpha, \beta$ denote two complex numbers with modulus 1 . The base-points of $f_{\alpha, \beta}$ are
$$
(1: 0: 0), \quad(0: 1: 0), \quad(-1: \alpha: 1)
$$

Any $f_{\alpha, \beta}$ preserves a rational fibration (the fibration $y=$ constant in the affine chart $z=1$ ). Each element of the family $\left(f_{\alpha, \beta}\right)$ has first dynamical degree 1 hence topological entropy zero (1.1); more precisely one has ([8, Example 4.3])

$$
\mu\left(f_{\alpha, \beta}\right)=\frac{1}{2}
$$

so $f_{\alpha, \beta}$ is not conjugate to an automorphism (Theorem 1.1). The centralizer of $f_{\alpha, \beta}$ is isomorphic to $\mathbb{Z}$ (see [12, Theorem 1.6]). The idea of the proof is the following: the point $p=(1: \alpha: 1)$ is blown-up onto a fiber of the fibration $y=$ constant. Let $\psi$ be an element of

$$
\operatorname{Cent}\left(f_{\alpha, \beta}\right)=\left\{g \in \operatorname{Bir}\left(\mathbb{P}_{\mathbb{C}}^{2}\right) \mid g \circ f_{\alpha, \beta}=f_{\alpha, \beta} \circ g\right\} ;
$$

since $\psi$ blows down a finite number of curves there exists a positive integer $k$ (chosen minimal) such that $f_{\alpha, \beta}^{k}(p)$ is not blown down by $\psi$. Replacing $\psi$ by $\widetilde{\psi}=\psi f_{\alpha, \beta}^{k-1}$ one gets that $\widetilde{\psi}(p)$ is an indeterminacy point of $f_{\alpha, \beta}$. In other words $\widetilde{\psi}$ permutes the indeterminacy points of $f_{\alpha, \beta}$. A more precise study allows us to establish that $p$ is fixed by $\widetilde{\psi}$. The pair $(\alpha, \beta)$ being in $\Omega$, the closure of the negative orbit of $p$ under the action of $f_{\alpha, \beta}$ is ZARISKI dense; since $\widetilde{\psi}$ fixes any element of the orbit of $p$ one obtains $\widetilde{\psi}=\mathrm{id}$.

Let us recall that if $\psi$ is an automorphism on a compact complex manifold M , the FATOU set $\mathcal{F}(\psi)$ of $\psi$ is the set of points that have a neighborhood $\mathcal{V}$ such that $\left\{f_{\mid \mathcal{V}}^{n} \mid n \in \mathbb{N}\right\}$ is a normal family. Set

$$
\mathcal{G}(\mathcal{U})=\left\{\phi: \mathcal{U} \rightarrow \overline{\mathcal{U}} \mid \phi=\lim _{n_{j} \rightarrow+\infty} \psi^{n_{j}}\right\}
$$

we say that $\mathcal{U}$ is a rotation domain if $\mathcal{G}(\mathcal{U l})$ is a subgroup of $\operatorname{Aut}(\mathcal{U})$. An equivalent definition is the following: a component $\mathcal{U}$ of $\mathcal{F}(\psi)$ which is invariant by $\psi$ is a rotation domain if $\psi_{\mid \mathcal{U}}$ is conjugate to a linear rotation. If $\mathcal{U}$ is a rotation domain, $\mathcal{G}(\mathcal{U})$ is a compact LIE group, and the action of $\mathcal{G}(\mathcal{U l})$ on $\mathcal{U}$ is analytic real. Since $\mathcal{G}(\mathcal{U})$ is a compact, infinite, abelian LIE group, the connected component of the identity of $\mathcal{G}(\mathcal{U})$ is a torus of dimension $0 \leq d \leq \operatorname{dim}_{\mathbb{C}} \mathrm{M}$. The integer $d$ is the rank of the rotation domain. The rank coincides with the dimension of the closure of a generic orbit of a point in $\mathcal{U}$.

We can also see $f_{\alpha, \beta}$ on $\mathbb{P}_{\mathbb{C}}^{1} \times \mathbb{P}_{\mathbb{C}}^{1}$ and that is what we will do in the sequel (since all the computations of [12] have been done in an affine chart they may all be carried on $\left.\mathbb{P}_{\mathbb{C}}^{1} \times \mathbb{P}_{\mathbb{C}}^{1}\right)$; the sets $\mathbb{P}_{\mathbb{C}}^{1} \times \mathbb{S}_{\rho}^{1}$ are invariant. In [12] we show that there are two rotation domains for $f_{\alpha, \beta}^{2}$, one of
rank 1 , and the other one of rank $2^{(2)}$; in the first case we give here a more precise statement than in [12]:

Theorem 1.3. - Assume that $(\alpha, \beta)$ belongs to $\Omega$.
There exists a strictly positive real number $r$ such that $f_{\alpha, \beta}$ is conjugate to $(\alpha x, \beta y)$ on $\mathbb{P}_{\mathbb{C}}^{1} \times \mathbb{D}(0, r)$ where $\mathbb{D}(0, r)$ denotes the disk centered at the origin with radius $r$.

There exists a strictly positive real number $\widetilde{r}$ such that $f_{\alpha, \beta}^{2}$ is conjugate to $\left(\frac{x}{\beta}, \frac{z}{\beta^{2}}\right)$ on $\mathbb{P}_{\mathbb{C}}^{1} \times$ $\mathbb{D}(0, \widetilde{r})$.
Remark 1.4. - The point $(\alpha-1,0)$ is also a fixed point of $f_{\alpha, \beta}$ where the behavior of $f_{\alpha, \beta}$ is the same as near $(0,0)$.

Proof. - The first assertion is proved in [12].
Let us consider the map $\psi(x, z)=\left(\frac{a(z) x+b(z)}{c(z) x+1}, z\right)$. The equation

$$
\psi^{-1} f_{\alpha, \beta}^{2} \psi=\left(\frac{x}{\beta}, \frac{z}{\beta^{2}}\right)
$$

yields

$$
\begin{align*}
& \quad \beta a\left(\beta^{-2} z\right) c(z)+\beta a\left(\beta^{-2} z\right) a(z)-c\left(\beta^{-2} z\right) a(z)+\alpha a\left(\beta^{-2} z\right) a(z) \\
& \quad+z\left(\alpha^{2} a\left(\beta^{-2} z\right) c(z)-\alpha c\left(\beta^{-2} z\right) c(z)-c\left(\beta^{-2} z\right) c(z)-c\left(\beta^{-2} z\right) a(z)\right)=0  \tag{1.2}\\
& \beta a\left(\beta^{-2} z\right)-\beta a(z)+z\left(\alpha^{2} a\left(\beta^{-2} z\right)-\alpha \beta c(z)-\beta c(z)-\beta a(z)-\alpha c\left(\beta^{-2} z\right)-c\left(\beta^{-2} z\right)\right) \\
& +\beta(\alpha+\beta) a(z) b\left(\beta^{-2} z\right)+(\alpha+\beta) b(z) a\left(\beta^{-2} z\right)+\beta^{2} b\left(\beta^{-2} z\right) c(z)-b(z) c\left(\beta^{-2} z\right) \\
& +z\left(\alpha^{2} \beta b\left(\beta^{-2} z\right) c(z)-b(z) c\left(\beta^{-2} z\right)\right)=0 \tag{1.3}
\end{align*}
$$

and

$$
\begin{equation*}
(\alpha+1) z+b(z)-\beta b\left(\beta^{-2} z\right)-\alpha^{2} z b\left(\beta^{-2} z\right)+z b(z)-(\alpha+\beta) b\left(\beta^{-2} z\right) b(z)=0 \tag{1.4}
\end{equation*}
$$

Let us set

$$
a(z)=\sum_{i \geq 0} a_{i} z^{i}, \quad b(z)=\sum_{i \geq 0} b_{i} z^{i}, \quad c(z)=\sum_{i \geq 0} c_{i} z^{i}
$$

We easily get $a_{0}=1-\beta, b_{0}=0$ and $c_{0}=\alpha+\beta$.
Relation (1.4) implies that

$$
b_{1}=\frac{\beta(1+\alpha)}{1-\beta} \quad \& \quad \beta b_{v}\left(1-\beta^{1-2 v}\right)+F_{i}\left(b_{i} \mid i<v\right)=0 \quad \forall v>1
$$

(1.3) yields

$$
a_{v}\left(\beta^{1-2 v}-\beta\right)+b_{v}\left((\alpha+\beta) a_{0}\left(1+\beta^{1-2 v}\right)+c_{0}\left(\beta^{2-2 v}-1\right)\right)+G_{i}\left(a_{i}, b_{i}, c_{i} \mid i<v\right)=0
$$

and (1.2) to

$$
c_{v} a_{0}\left(\beta-\beta^{-2 v}\right)+a_{v}\left((\alpha+\beta) a_{0}\left(1+\beta^{-2 v}\right)+c_{0}\left(\beta^{1-2 v}-1\right)\right)+H_{i}\left(a_{i}, b_{i}, c_{i} \mid i<v\right)=0
$$

[^1]where the $F_{i}$ 's, $G_{i}$ 's and $H_{i}$ 's denote universal polynomials; this allows to compute $b_{v}, a_{v}$ and $c_{V}$. Thus we get a formal conjugacy of $f_{\alpha, \beta}^{2}$ to its linear part. Since this linear part satisfies a Rüssmann condition (see [19, Theorem 2.1] condition (2)), according to [19, Theorem 2.1] any formal linearizing map conjugating $f_{\alpha, \beta}^{2}$ to its linear part is convergent on a polydisc.

## 2. About $\operatorname{SL}(2 ; \mathbb{C})$-cocycles

A (one-frequency, analytic) quasiperiodic $\operatorname{SL}(2 ; \mathbb{C})$-cocycle is a pair $(A, \beta)$, where $\beta \in \mathbb{R}$ and

$$
A: \mathbb{S}_{1}^{1} \rightarrow \mathrm{SL}(2 ; \mathbb{C})
$$

is analytic, and defines a linear skew product acting on $\mathbb{C}^{2} \times \mathbb{S}_{1}^{1}$ by

$$
(x, y) \mapsto(A(y) \cdot x, \beta y)
$$

The iterates of the cocyle are given by $\left(A_{n}, n \beta\right)$ where $A_{n}$ is given by

$$
A_{n}(y)=A\left(\beta^{n-1} y\right) \ldots A(y) \quad n \geq 1, \quad A_{0}(y)=\mathrm{id}, \quad A_{-n}(y)=A_{n}\left(\beta^{-n} y\right)^{-1}
$$

The LYAPUNOV exponent $L(A, \beta)$ of a quasiperiodic $\operatorname{SL}(2 ; \mathbb{C})$-cocycle $(A, \beta)$ is given by

$$
\lim _{n \rightarrow+\infty} \frac{1}{n} \int_{\mathbb{S}_{1}^{1}} \ln \left\|A_{n}(y)\right\| \mathrm{d} y
$$

A quasiperiodic $\operatorname{SL}(2 ; \mathbb{C})$-cocycle $(A, \beta)$ is uniformly hyperbolic if there exist analytic functions

$$
u, s: \mathbb{S}_{1}^{1} \rightarrow \mathbb{P}_{\mathbb{C}}^{2}
$$

called the unstable and stable directions, and $n \geq 1$ such that for any $y \in \mathbb{S}_{1}^{1}$,

$$
A(y) \cdot u(y)=u(\beta y) \quad A(y) \cdot s(y)=s(\beta y)
$$

and for any unit vector $x \in s(y)$ (resp. $x \in u(y)$ ) we have $\left\|A_{n}(y) \cdot x\right\|<1$ (resp. $\left.\left\|A_{n}(y) \cdot x\right\|>1\right)$. The unstable and stable directions are uniquely characterized by those properties, and clearly $u(y) \neq s(y)$ for any $y \in \mathbb{S}_{1}^{1}$. If $(A, \beta)$ is uniformly hyperbolic, then $L(A, \beta)>0$. Let us denote by

$$
\mathcal{U} \mathcal{H} \subset C^{\omega}\left(\mathrm{SL}(2 ; \mathbb{C}), \mathbb{S}_{1}^{1}\right)
$$

the set of $A$ such that $(A, \beta)$ is uniformly hyperbolic. Uniform hyperbolicity is a stable property: $\mathcal{U} \mathcal{H}$ is open, and $A \mapsto L(A, \beta)$ is analytic over $\mathcal{U} \mathcal{H}$ (regularity properties of the LyAPUNOV exponent are consequence of the regularity of the unstable and stable directions which depend smoothly on both variables).

Definition. - Let $(A, \beta)$ be a quasiperiodic $\operatorname{SL}(2 ; \mathbb{C})$-cocycle. If $L(A, \beta)>0$ but $(A, \beta) \notin \mathcal{U} \mathcal{H}$, then $(A, \beta)$ is nonuniformly hyperbolic.

If $A \in C^{\omega}\left(\operatorname{SL}(2 ; \mathbb{C}), \mathbb{S}_{1}^{1}\right)$ admits a holomorphic extension to $|\operatorname{Im} y|<\delta$ then for $|\varepsilon|<\delta$ we can define $A_{\varepsilon} \in C^{\omega}\left(\operatorname{SL}(2 ; \mathbb{C}), \mathbb{S}_{1}^{1}\right)$ by

$$
A_{\varepsilon}(y)=A(y+\mathbf{i} \varepsilon)
$$

The LYAPUNOV exponent $L\left(A_{\varepsilon}, \beta\right)$ is a convex function of $\varepsilon$. We can thus introduce the following notion. The acceleration of a quasiperiodic $\operatorname{SL}(2 ; \mathbb{C})$-cocyle $(A, \beta)$ is given by

$$
\omega(A, \beta)=\lim _{\varepsilon \rightarrow 0^{+}} \frac{1}{2 \pi \varepsilon}\left(L\left(A_{\varepsilon}, \beta\right)-L(A, \beta)\right) .
$$

Remark 2.1. - The convexity of the LYAPUNOV exponent in function of $\varepsilon$ implies that the acceleration is decreasing.

Since the LYAPUNOV exponent is a convex and continuous function the acceleration is an upper semi-continuous function in $\mathbb{R} \backslash \mathbb{Q} \times C^{\omega}\left(\operatorname{SL}(2 ; \mathbb{C}), \mathbb{S}_{1}^{1}\right)$. The acceleration is quantized:

Theorem $2.2([1]) .-\operatorname{If}(A, \beta)$ is a $\operatorname{SL}(2 ; \mathbb{C})$-cocycle with $\beta \in \mathbb{R} \backslash \mathbb{Q}$, then $\omega(A, \beta)$ is always an integer.

A direct consequence is the following:
Corollary 2.3. - The function $\varepsilon \mapsto L\left(A_{\varepsilon}, \beta\right)$ is a piecewise affine function of $\varepsilon$.
It is thus natural to introduce the notion of regularity. A cocycle

$$
(A, \beta) \in C^{\omega}\left(\mathrm{SL}(2 ; \mathbb{C}), \mathbb{S}_{1}^{1}\right) \times \mathbb{R} \backslash \mathbb{Q}
$$

is regular if $L\left(A_{\varepsilon}, \beta\right)$ is affine for $\varepsilon$ in a neighborhood of 0 . In other words $(A, \beta)$ is regular if the equality

$$
L\left(A_{\varepsilon}, \beta\right)-L(A, \beta)=2 \pi \varepsilon \omega(A, \beta)
$$

holds for all $\varepsilon$ small, and not only for the positive ones. Regularity is equivalent to the acceleration being locally constant near $(A, \beta)$. It is an open condition in $C^{\omega}\left(\operatorname{SL}(2 ; \mathbb{C}), \mathbb{S}_{1}^{1}\right) \times \mathbb{R} \backslash \mathbb{Q}$. The following statement gives a characterization of the dynamics of regular cocycles with positive LYAPUNOV exponent:

Theorem $2.4([1])$. - Let $(A, \beta)$ be a $\operatorname{SL}(2 ; \mathbb{C})$-cocycle with $\beta \in \mathbb{R} \backslash \mathbb{Q}$. Assume that $L(A, \beta)>0$; then $(A, \beta)$ is regular if and only if $(A, \beta)$ is $\mathcal{U H}$.

One striking consequence is the following:
Corollary $2.5([1])$. - For any $(A, \beta)$ in $C^{\omega}\left(\operatorname{SL}(2 ; \mathbb{C}), \mathbb{S}_{1}^{1}\right) \times \mathbb{R} \backslash \mathbb{Q}$ there exists $\varepsilon_{0}$ such that

- $L\left(A_{\varepsilon}, \beta\right)=0$ (and $\omega(A, \beta)=0$ ) for every $0<\varepsilon<\varepsilon_{0}$,
- or $\left(A_{\varepsilon}, \beta\right) \in \mathcal{U} \mathcal{H}$ for every $0<\varepsilon<\varepsilon_{0}$.

Remark 2.6. - Let us mention that there is a link between $\operatorname{SL}(2 ; \mathbb{C})$-cocycles and SCHRÖDINGER operators (see [1] for more details).

## 3. Proof of Theorem $\mathbf{A}$

Suppose that $\rho \neq 1$, and let us consider the cocycle $\left(B^{\alpha, \rho}, \beta^{1 / 2}\right)$ where

$$
B^{\alpha, \rho}(y)=\left[\begin{array}{cc}
\alpha & y^{2} \\
1 & 1
\end{array}\right]
$$

Since

$$
\left(\frac{\alpha x+y}{x+1}, \beta y\right)\left(x, y^{2}\right)=\left(x, y^{2}\right)\left(\frac{\alpha x+y^{2}}{x+1}, \beta^{1 / 2} y\right)
$$

the cocycles $\left(A^{\alpha, \rho}, \beta\right)$ and $\left(B^{\alpha, \rho}, \beta^{1 / 2}\right)$ have the same behavior. Using two different arguments of monodromy (one for $\rho<1$, and the other one for $\rho>1$ ) we see that there is a continuous determination for the square root of $\operatorname{det} B^{\alpha, \rho}(y)=\alpha-y^{2}$. Let us set

$$
\widetilde{B}^{\alpha, \rho}(y)=\frac{1}{\sqrt{\alpha-y^{2}}} B^{\alpha, \rho}(y) \in \operatorname{SL}(2 ; \mathbb{C})
$$

that is thus defined on two different domains of analyticity.
According to Theorem 1.3 one has $L\left(\widetilde{B}^{\alpha, \rho}, \beta^{1 / 2}\right)=0$ when $\rho$ is close to both 0 and $\infty$.
Assume that $L\left(\widetilde{B}^{\alpha, \rho}, \beta^{1 / 2}\right)$ is non constant. When $\widetilde{B}^{\alpha, \rho}$ is holomorphic, so in particular when $\rho<1$ and $\rho>1$, the acceleration is decreasing (Remark 2.1); furthermore the acceleration is positive for $\rho<1$ and negative for $\rho>1$ (because $L$ is continuous). Theorem 2.2 thus implies

$$
\omega\left(\widetilde{B}^{\alpha, 1^{+}}, \beta^{1 / 2}\right)-\omega\left(\widetilde{B}^{\alpha, 1^{-}}, \beta^{1 / 2}\right) \leq-2
$$

By definition of $\widetilde{B}^{\alpha, \rho}$ we have

$$
\begin{aligned}
L\left(\widetilde{B}^{\alpha, \rho}(y), \beta^{1 / 2}\right) & =L\left(B^{\alpha, \rho}(y), \beta^{1 / 2}\right)-\int_{\mathbb{S}_{\rho}^{1}} \ln \sqrt{\alpha-y^{2}} \mathrm{~d} y \\
& =L\left(B^{\alpha, \rho}(y), \beta^{1 / 2}\right)-\max (0, \ln \rho)
\end{aligned}
$$

Even though $\left(B^{\alpha, \rho}(y), \beta^{1 / 2}\right)$ is not a $\operatorname{SL}(2 ; \mathbb{C})$-cocycle, the LYAPUNOV exponent is still a convex function of $\log \rho$ (see for example [2]). The jump of $\omega\left(B^{\alpha, \rho}(y), \beta^{1 / 2}\right)$ is thus $\geq 0$, and the jump for the second term of the right member is -1 . Therefore the jump of $L\left(\widetilde{B}^{\alpha, \rho}(y), \beta^{1 / 2}\right)$ is $\geq-1$ : contradiction.

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[^2]
[^0]:    1. The centraliser of a map $\phi$ is trivial if it coincides with the iterates of $\phi$.
[^1]:    2. There already exists an example of automorphism of positive entropy with rotation domains of rank 1 and 2 (see [5]), but $f_{\alpha, \beta}$ is not conjugate to an automorphism on a rational surface.
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