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The k -parent spatial Lambda-Fleming-Viot process as a stochastic measure-valued model for an expanding population

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Abstract

We model spatially expanding populations by means of a spatial Λ -Fleming Viot process (SLFV) with selection : the k -parent SLFV. We fill empty areas with type 0 "ghost" individuals, which have a strong selective disadvantage against "real" type 1 individuals. This model is a special case of the SLFV with selection introduced in [19, 22] : natural selection acts during all reproduction events, and the fraction of individuals replaced during a reproduction event is constant equal to 1. Letting the selective advantage k of type 1 individuals over type 0 individuals grow to $+\infty$, and without rescaling time nor space, we obtain a new model for expanding populations, the ∞ -parent SLFV. This model is reminiscent of the Eden growth model [13], but with an associated dual process of potential ancestors, making it possible to investigate the genetic diversity in a population sample. In order to obtain the limit $k \rightarrow +\infty$ of the k -parent SLFV, we introduce an alternative construction of the k -parent SLFV adapted from [38], which allows us to couple SLFVs with different selection strengths.

Running headline: A spatial Lambda-Fleming-Viot process for expanding populations

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Contents

1	Introduction	2
1.1	The k -parent SLFV and its dual	4
1.1.1	The k -parent SLFV	4

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1.1.2	The k -parent ancestral process	9
1.2	Construction of the ∞ -parent SLFV	12
1.3	Dual of the ∞ -parent SLFV	14
1.4	Structure of the paper	15
2	The ∞-parent SLFV	16
2.1	Alternative construction of the k -parent SLFV	16
2.2	Definition of the ∞ -parent SLFV	21
2.3	Characterization via a martingale problem	23
3	The ∞-parent ancestral process : definition and characterization	30
3.1	Definition and first properties	30
3.2	Characterization via a martingale problem	33
4	Uniqueness of the solution to the martingale problem characterizing the ∞-parent SLFV	35
4.1	Extended martingale problem for the ∞ -parent SLFV	35
4.2	Extended martingale problem for the ∞ -parent ancestral process	41
4.3	Uniqueness of the solution to the martingale problem characterizing the ∞ -parent SLFV	44
5	Technical lemmas	47
5.1	Properties of the operators \mathcal{L}_μ^k and \mathcal{L}_μ^∞	47
5.2	Properties of the operator \mathcal{G}_μ^∞	51
5.3	Properties of the densities of coupled k -parent SLFVs	53
	References	55

1 Introduction

Population expansions are common events occurring at all biological scales. The growth of a population in a new environment generates interfaces with distinctive features [27, 30] and specific patterns of genetic variation [23, 24, 26], both being a consequence of the stochasticity of reproduction at the front, where local population sizes are small. The models which are used to study expanding populations can be divided in two main categories : growth models, mostly used to investigate the front features, and models coming from population genetics, which are more suited to study genetic diversity patterns.

Experimental approaches suggest that the dynamics of fronts of real expanding populations belongs to the universality class of the Kardar-Parisi-Zhang (KPZ) equation introduced in [30] (see e.g [27]). It has been conjectured (and demonstrated in the case of the solid-on-solid (SOS) growth model [6]) that many growth models generate similar interfaces. One of these models is the Eden growth model, initially introduced on a lattice in [13]. Under this model, each node of the lattice is either occupied or empty. At each time step, an empty node with at least one occupied neighbour is chosen, and becomes occupied. There exist alternative update rules for this model [28], as well as off-lattice variants (see e.g [39]). While this model can be used to study the growth of an expanding population, it is less suited to study genetic diversity patterns.

Conversely, models used in population genetics are generally associated with tools allowing one to investigate these patterns. The analysis of the genetic diversity of a population often goes through modelling the ancestral lineages of a subset of individuals, and studying how these lineages coalesce into common ancestors [15]. However, most classical population genetics models assume that populations have constant sizes and that individuals are uniformly distributed over the area of interest. Therefore, they appear at first ill-suited to model a population during an expansion event. One way to overcome this consists in filling empty areas with "ghost" individuals, which can reproduce but have a very strong selective disadvantage against "real" individuals [12, 25]. Under this framework, the reproduction of "ghost" individuals can be interpreted as a local extinction of real individuals.

Using this idea, it is possible to model a population expansion as the spread of a genetic type favoured by natural selection. Such a question was already studied by means of different models including a stochastic component, mostly in one dimension (see e.g [3, 18, 33]). The most classical one is based on the Fisher-KPP equation [21, 32], in which stochasticity is introduced through a Wright-Fisher noise term. If $0 \leq p(t, x) \leq 1$ represents the proportion of individuals of the favoured type at location $x \in \mathbb{R}$ at time $t \geq 0$, then $p(t, x)$ solves the stochastic Fisher-KPP equation if for $x \in \mathbb{R}$ and $t > 0$,

$$\frac{\partial p}{\partial t}(t, x) = \frac{m}{2} \Delta p(t, x) dt + s_0 p(t, x)(1 - p(t, x)) + \sqrt{\frac{1}{p_e} p(t, x)(1 - p(t, x))} W(dt, dx) \quad (1)$$

where W is a space-time white noise and p_e an effective population density. In one dimension, the stochastic Fisher-KPP equation exhibits travelling wave solutions [35], which describe how does the advantageous type spreads through space. However, Eq. (1) has no solution in higher dimensions. Many variants of the deterministic version of the Fisher-KPP equation have been studied, including versions with individuals having different motilities [8, 9], different growth rates (see e.g [11]), other diffusion kernels, or other choices for the nonlinearity [5].

Other models are more individual-based, and are adaptations of classical population genetics models, such as the Moran model [12, 18, 25] or the stepping-stone model [2, 3, 37]. They generally require to divide the space into subunits called *demes*, to which reproduction events are limited and which are connected by migration.

In this article, we will focus instead on a "reproduction event-based" model allowing us to keep the spatial continuum : the spatial Λ -Fleming Viot process, or SLFV [4, 14]. Its main feature is that it models reproduction events affecting whole areas rather than reproduction individual by individual, by means of a Poisson point process of reproduction events.

The original version of the SLFV does not account for the presence of a selectively favoured genetic type, but it can be modified in order to incorporate selection : see [22] for different forms of fixed selection mechanisms, and [7, 10, 31] for ways to introduce fluctuating selection. Our approach will be based on a version of the SLFV with selection introduced in [22] and rigorously constructed in [19]. Most of the work on the SLFV with selection involved investigating scaling limits under different forms of *weak* selection (see also [16, 17]). However, in our case, since the selectively disadvantaged individuals do not actually exist, the selection can be considered as very strong. Therefore, we shall consider a different limit, when selection goes stronger and stronger, and neither time nor space are rescaled. The limiting model we shall obtain will be close to the off-lattice Eden growth model, hence we can expect it to generate interfaces similar to the ones observed in real expanding populations. Moreover, and contrary to the off-lattice Eden growth model, a dual process of potential ancestors is also associated to it, giving us tools to investigate the genetic diversity patterns observed in an expanding population. This model therefore constitutes a new model for expanding populations, which naturally appears as the limit of other well-known processes, and which seems promising in order to investigate both the front features and the genetic diversity patterns of an expanding population. In this work, we will focus on the area occupied by the population, but future works will include genetic diversity inside the expanding population, using for instance tracer dynamics [12, 25].

1.1 The k -parent SLFV and its dual

1.1.1 The k -parent SLFV

All the random objects introduced in this section will be defined over some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Before presenting the processes we will consider, we need to introduce some notation.

Let $d \geq 1$. Let $C_c(\mathbb{R}^d)$ be the space of all continuous and compactly supported functions $\mathbb{R}^d \rightarrow \mathbb{R}$, let $C^1(\mathbb{R})$ be the space of all continuously differentiable functions on \mathbb{R} , let $C_b^1(\mathbb{R})$ be the space of all bounded functions $\mathbb{R} \rightarrow \mathbb{R}$ that are C^1 and whose first derivative is also bounded, and let $\mathcal{B}(\mathbb{R}^d)$ be

the space of all measurable functions $\mathbb{R}^d \rightarrow \mathbb{R}$.

We start by introducing the state space over which the variant of the SLFV with selection we consider is defined. Let $\widetilde{\mathcal{M}}_\lambda$ be the space of all measures M on $\mathbb{R}^d \times \{0, 1\}$ such that for all $f \in C_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d \times \{0,1\}} f(x)M(dx, d\kappa) = \int_{\mathbb{R}^d} f(x)dx.$$

In other words, $\widetilde{\mathcal{M}}_\lambda$ is the space of all measures on $\mathbb{R}^d \times \{0, 1\}$ whose marginal over \mathbb{R}^d is Lebesgue measure. By a standard decomposition theorem (see e.g [29], p.561), for all $M \in \widetilde{\mathcal{M}}_\lambda$, there exists $\omega : \mathbb{R}^d \rightarrow [0, 1]$ measurable such that

$$M(dx, d\kappa) = ((\omega(x)\delta_0(d\kappa) + (1 - \omega(x))\delta_1(d\kappa))dx). \quad (2)$$

Such a ω is not unique, but defined up to a Lebesgue null set. The state space we consider is the set \mathcal{M}_λ of all measures $M \in \widetilde{\mathcal{M}}_\lambda$ such that there exists a measurable function $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$ (instead of $\omega : \mathbb{R}^d \rightarrow [0, 1]$) satisfying (2).

We endow $\widetilde{\mathcal{M}}_\lambda$ and \mathcal{M}_λ with the topology of vague convergence. Moreover, let $D_{\mathcal{M}_\lambda}[0, +\infty)$ (resp. $D_{\widetilde{\mathcal{M}}_\lambda}[0, +\infty)$) denote the space of all càdlàg \mathcal{M}_λ -valued paths (resp. $\widetilde{\mathcal{M}}_\lambda$ -valued paths), endowed with the standard Skorokhod topology.

Let $M \in \mathcal{M}_\lambda$, and let $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$ be a measurable function satisfying Eq. (2). The function ω can be interpreted as the indicator function of a measurable set $E \subset \mathbb{R}^d$ corresponding to the area occupied by what will be called "type 0" individuals, while $\mathbb{R}^d \setminus E$ corresponds to the area occupied by "type 1" individuals. We will consider that type 0 individuals correspond to the "ghost" individuals mentioned in the introduction, and type 1 individuals to the "real" individuals. Therefore, type 0 individuals have a strong selective disadvantage against type 1 individuals, and E corresponds to the area not yet invaded by the real population (up to a Lebesgue null set). In all that follows, any $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$ such that (2) is true will be called a *density* of M , and the notation ω_M will be used to denote an arbitrarily chosen density of M .

For all $f \in C_c(\mathbb{R}^d)$, $F \in C^1(\mathbb{R})$ and $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$ measurable, we set :

$$\langle \omega, f \rangle := \int_{\mathbb{R}^d} f(x)\omega(x)dx$$

and we define the function $\Psi_{F,f} \in C_b(\mathcal{M}_\lambda)$ as :

$$\forall M \in \mathcal{M}_\lambda, \Psi_{F,f}(M) := F\left(\int_{\mathbb{R}^d \times \{0,1\}} f(x)\mathbb{1}_{\{\kappa=0\}}M(dx, d\kappa)\right) \quad (3)$$

$$\begin{aligned}
&= F\left(\int_{\mathbb{R}^d} f(x)\omega_M(x)dx\right) \\
&= F(\langle \omega_M, f \rangle).
\end{aligned}$$

For all $f \in C_c(\mathbb{R}^d)$, we denote the support of f by $Supp(f)$, and for all $\mathcal{R} \in \mathbb{R}_+^*$, we set :

$$Supp^{\mathcal{R}}(f) := \{y \in \mathbb{R}^d : \exists x \in Supp(f), \|y - x\| \leq \mathcal{R}\}$$

$$\text{and } V_{\mathcal{R}} := \text{Vol}(\mathcal{B}(0, \mathcal{R})).$$

In other words, $V_{\mathcal{R}}$ is the volume of a ball of radius \mathcal{R} , and $Supp^{\mathcal{R}}(f)$ is the set of all points which are at a distance of at most \mathcal{R} of a point in the support of f .

For all $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$, $\mathcal{R} \in \mathbb{R}_+^*$ and $x \in \mathbb{R}^d$, we define the functions $\Theta_{x, \mathcal{R}}^+(\omega) : \mathbb{R}^d \rightarrow \{0, 1\}$ and $\Theta_{x, \mathcal{R}}^-(\omega) : \mathbb{R}^d \rightarrow \{0, 1\}$ by :

$$\begin{aligned}
\Theta_{x, \mathcal{R}}^+(\omega) &:= \mathbb{1}_{B(x, \mathcal{R})^c} \times \omega + \mathbb{1}_{B(x, \mathcal{R})}, \\
\Theta_{x, \mathcal{R}}^-(\omega) &:= \mathbb{1}_{B(x, \mathcal{R})^c} \times \omega.
\end{aligned}$$

$\Theta_{x, \mathcal{R}}^+(\omega)$ corresponds to filling the ball $B(x, \mathcal{R})$ with type 0 individuals (or equivalently, emptying the ball $B(x, \mathcal{R})$ of all real individuals), while $\Theta_{x, \mathcal{R}}^-(\omega)$ can be interpreted as filling the ball $B(x, \mathcal{R})$ with type 1 individuals. Notice that if $M \in \mathcal{M}_\lambda$, then $\Theta_{x, \mathcal{R}}^+(\omega_M) \in \mathcal{M}_\lambda$ and $\Theta_{x, \mathcal{R}}^-(\omega_M) \in \mathcal{M}_\lambda$.

We now introduce the operator which will be used to define the specific SLFV with selection we will consider as the solution to a well-posed martingale problem. Let $k \in \mathbb{N} \setminus \{0, 1\}$, and let μ be a σ -finite measure on \mathbb{R}_+^* such that

$$\int_0^\infty \mathcal{R}^d \mu(d\mathcal{R}) < +\infty. \quad (4)$$

Let \mathcal{L}_μ^k be the operator acting on functions of the form $\Psi_{F, f}$ with $f \in C_c(\mathbb{R}^d)$ and $F \in C^1(\mathbb{R})$, defined the following way. Let $f \in C_c(\mathbb{R}^d)$ and $F \in C^1(\mathbb{R})$. Then, for all $M \in \mathcal{M}_\lambda$,

$$\begin{aligned}
\mathcal{L}_\mu^k \Psi_{F, f}(M) &:= \int_{\mathbb{R}^d} \int_0^\infty \int_{B(x, \mathcal{R})^k} \frac{1}{V_{\mathcal{R}}^k} \times \left[\left(\prod_{j=1}^k \omega_M(y_j) \right) \times F(\langle \Theta_{x, \mathcal{R}}^+(\omega_M), f \rangle) \right. \\
&\quad + \left(1 - \prod_{j=1}^k \omega_M(y_j) \right) \times F(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f \rangle) \\
&\quad \left. - F(\langle \omega_M, f \rangle) \right] dy_1 \dots dy_k \mu(d\mathcal{R}) dx.
\end{aligned}$$

In Section 5, it is shown that this operator is well-defined, and that it can be rewritten as

$$\begin{aligned} \mathcal{L}_\mu^k \Psi_{F,f}(M) = & \int_0^\infty \int_{\text{Supp}^{\mathcal{R}}(f)} \int_{B(x,\mathcal{R})^k} \frac{1}{V_{\mathcal{R}}^k} \times \left[\prod_{j=1}^k \omega_M(y_j) \times F(\langle \Theta_{x,\mathcal{R}}^+(\omega_M), f \rangle) \right. \\ & + (1 - \prod_{j=1}^k \omega_M(y_j)) \times F(\langle \Theta_{x,\mathcal{R}}^-(\omega_M), f \rangle) \\ & \left. - F(\langle \omega_M, f \rangle) \right] dy_1 \dots dy_k dx \mu(d\mathcal{R}). \end{aligned}$$

Intuitively, an interpretation of this operator in terms of reproduction events is the following. Whenever a reproduction event affects the ball $B(x, \mathcal{R})$, k positions y_1, \dots, y_k are sampled inside the ball, and we take k individuals occupying each one of these positions. Since the density of type 0 individuals ω_M is $\{0, 1\}$ -valued, we can consider that all the individuals occupying the position y_1 (resp. y_2, \dots, y_k) are of type $1 - \omega_M(y_1)$ (resp. $1 - \omega_M(y_2), \dots, 1 - \omega_M(y_k)$). If $\prod_{j=1}^k \omega_M(y_j) = 1$, then all the individuals are of type 0, and we fill the ball $B(x, \mathcal{R})$ with type 0 individuals. Conversely, if $1 - \prod_{j=1}^k \omega_M(y_j) = 1$, then at least one individual is of type 1, and this time we fill the ball $B(x, \mathcal{R})$ with type 1 individuals. Since type 0 individuals model "ghost" individuals, they are supposed to have a selective disadvantage against "real" type 1 individuals, hence the exclusion of the case $k = 1$ which would not give any advantage to type 1 individuals. Moreover, k can be interpreted as measuring the strength of the selective advantage of "real" individuals against "ghost" individuals, or in other words, the capacity of "real" individuals to invade an empty environment.

If $k = 2$, \mathcal{L}_μ^2 is the operator introduced in [19] to define and characterize the "selection part" of the SLFV with selection, in the special case for which there are no neutral events and all reproduction events have an impact of $u = 1$. Their proof of the existence and uniqueness of the $D_{\widetilde{\mathcal{M}}_\lambda}[0, +\infty)$ -valued solution to the martingale problem associated to \mathcal{L}_μ^k can easily be extended to the case $k \geq 2$, by restricting the martingale problem to an increasing sequence of compact subsets of \mathbb{R}^d converging to \mathbb{R}^d . In Section 2, we will show that this unique solution is in fact $D_{\mathcal{M}_\lambda}[0, +\infty)$ -valued if the initial value belongs to \mathcal{M}_λ .

Theorem 1. *Let $k \geq 2$, and let μ be a σ -finite measure on $(0, +\infty)$ satisfying condition (4). For all $M^0 \in \mathcal{M}_\lambda$, there exists a unique $D_{\mathcal{M}_\lambda}[0, +\infty)$ -valued process $(M_t)_{t \geq 0}$ such that $M_0 = M^0$ and, for all $F \in C^1(\mathbb{R})$ and $f \in C_c(\mathbb{R}^d)$,*

$$\left(\Psi_{F,f}(M_t) - \Psi_{F,f}(M_0) - \int_0^t \mathcal{L}_\mu^k \Psi_{F,f}(M_s) ds \right)_{t \geq 0}$$

is a martingale. Moreover, the process $(M_t)_{t \geq 0}$ is Markovian, and the corresponding semigroup is Feller.

Definition 2 (Definition of the k-SLFV). *Let $k \geq 2$, let μ be a σ -finite measure on $(0, \infty)$ satisfying (4), and let $M^0 \in \mathcal{M}_\lambda$. Then, the k -parent spatial Λ -Fleming-Viot process (or k -SLFV process) with initial condition M^0 associated to μ is the unique solution to the martingale problem (\mathcal{L}_μ^k, M^0) stated in Theorem 1. In particular, the k -SLFV is a strong Markov process with càdlàg paths a.s.*

By extension, if $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$ is measurable, we will define the k -SLFV process with initial density ω associated to μ to be the k -SLFV process with initial condition M^0 associated to μ , with $M^0 \in \mathcal{M}_\lambda$ of density ω .

Intuitively, the k -SLFV process can be constructed in the following way. Let $M^0 \in \mathcal{M}_\lambda$, and let μ be a σ -finite measure on $(0, \infty)$ satisfying (4). Moreover, let Π be a Poisson point process on $\mathbb{R} \times \mathbb{R}^d \times (0, +\infty)$ with intensity $dt \otimes dx \otimes \mu(dr)$. Initially, the k -SLFV is equal to M^0 . The dynamics of the k -SLFV process $(M_t)_{t \geq 0}$ is then as follows. If $(t, x, \mathcal{R}) \in \Pi$, a reproduction event happens at time t in the ball $\mathcal{B}(x, \mathcal{R})$. We sample k types according to the type distribution in the ball $\mathcal{B}(x, \mathcal{R})$ at the time $t-$. We interpret these types as the types of k potential "parents". With probability

$$\frac{1}{V_{\mathcal{R}}^k} \int_{B(x, \mathcal{R})^k} \left[\prod_{j=1}^k \omega_{M_{t-}}(y_j) \right] dy_1 \dots dy_k,$$

the k types sampled are 0, so the k potential parents are of type 0. In this case, all the individuals in the ball $B(x, \mathcal{R})$ die, the k -th potential parent (of type 0) fills the ball $B(x, \mathcal{R})$ with its descendants, which means that we set :

$$\forall z \in B(x, \mathcal{R}), \omega_{M_t}(z) = 1.$$

Conversely, with probability

$$1 - \frac{1}{V_{\mathcal{R}}^k} \int_{B(x, \mathcal{R})^k} \left[\prod_{j=1}^k \omega_{M_{t-}}(y_j) \right] dy_1 \dots dy_k,$$

at least one of the k types sampled is 1. As in the other case, all the individuals in the ball $B(x, \mathcal{R})$ die, but this time the first potential parent to be of type 1 fills the ball $B(x, \mathcal{R})$ with its descendants, which amounts to setting

$$\forall z \in B(x, \mathcal{R}), \omega_{M_t}(z) = 0.$$

Note that the position of the parent which actually reproduces is then uniformly distributed over the closure of the region $\{y \in B(x, \mathcal{R}) : \omega_{M_{t-}}(y) = 0\}$. The value taken by the density out of the ball $B(x, \mathcal{R})$ at time t is not affected by this reproduction event. We repeat this for each $(t, x, \mathcal{R}) \in \Pi$.

This construction can be made rigorous using arguments adapted from [38], and will be used in

Section 2 to complete the proof of Theorem 1.

Remark 3. The k -SLFV process is a special case of the general definition of an SLFV with selection in [22], with impact parameter $u = 1$, selection parameter $s = 1$, and selection function $F : x \rightarrow x - x^k$.

Remark 4. The condition (4) on μ matches the standard condition for the existence of the SLFV [4].

It comes from the fact that a point $x \in \mathbb{R}^d$ is affected by a reproduction event at rate :

$$\int_{\mathbb{R}^d} \int_0^{+\infty} \mathbb{1}_{y \in B(x, \mathcal{R})} \mu(d\mathcal{R}) dy = \int_0^{+\infty} V_{\mathcal{R}} \mu(d\mathcal{R}) \propto \int_0^{+\infty} \mathcal{R}^d \mu(d\mathcal{R}).$$

Remark 5. Since the density ω_M is only defined up to a Lebesgue null set, the type of individuals present in a given position $y \in \mathbb{R}^d$ cannot be uniquely defined. Therefore, even though intuitively we can first sample parental positions, and deduce parental types from ω_M , we cannot formally sample positions in order to sample parental types.

A particularly interesting feature of this model is that there exists a dual process of potential ancestors associated to it, which follows the locations of the potential ancestors of a set of individuals. In other words, the genetic diversity in a sample of the population can be determined by going *backwards in time*, and reconstructing the genealogical tree of the sample. For $k = 2$, the dual process is analogous to the Ancestral Selection Graph (ASG) [34, 36], but with a spatial structure.

1.1.2 The k -parent ancestral process

All the new objects introduced in relation with the dual process will be defined on a new probability space (Ω, \mathcal{F}, P) . As before, we let μ be a σ -finite measure on $(0, +\infty)$ satisfying condition (4), and we let $\overleftarrow{\Pi}$ be a Poisson point process on $\mathbb{R} \times \mathbb{R}^d \times (0, +\infty)$ with intensity $dt \otimes dx \otimes \mu(d\mathcal{R})$.

Let $\mathcal{M}_p(\mathbb{R}^d)$ denote the set of all finite point measures on \mathbb{R}^d , equipped of the topology of weak convergence. For all $\Xi = \sum_{i=1}^l \delta_{\xi_i} \in \mathcal{M}_p(\mathbb{R}^d)$, for all $x \in \mathbb{R}^d$ and $\mathcal{R} > 0$, we define

$$I_{x, \mathcal{R}}(\Xi) = \{i \in \llbracket 1, l \rrbracket : \|x - \xi_i\| \leq \mathcal{R}\}$$

and $S^{\mathcal{R}}(\Xi) = \{x \in \mathbb{R}^d : \exists i \in \llbracket 1, l \rrbracket : \|x - \xi_i\| \leq \mathcal{R}\}.$

In other words, $I_{x, \mathcal{R}}(\Xi)$ is the set of all the indices of the points in Ξ which are at distance at most \mathcal{R} of x , while $S^{\mathcal{R}}(\Xi)$ is the set of all the points in \mathbb{R}^d which are at distance at most \mathcal{R} of a point of Ξ .

Definition 6. Let $\Xi^0 \in \mathcal{M}_p(\mathbb{R}^d)$. The k -parent ancestral process $(\Xi_t)_{t \geq 0}$ associated to μ (or equivalently to $\overleftarrow{\Pi}$) and with initial condition Ξ^0 is the $\mathcal{M}_p(\mathbb{R}^d)$ -valued Markov jump process defined as follows.

- First, we set $\Xi_0 = \Xi^0$.
- Then, for all $(t, x, \mathcal{R}) \in \overleftarrow{\Pi}$, if $I_{x, \mathcal{R}}(\Xi_{t-}) \neq \emptyset$ and if we write

$$\Xi_{t-} = \sum_{i=1}^{N_{t-}} \delta_{\xi_{t-}^i},$$

we sample k points y_1, \dots, y_k independently and uniformly at random in $B(x, \mathcal{R})$, and we set

$$\Xi_t := \sum_{i=1}^{N_{t-}} \delta_{\xi_{t-}^i} - \sum_{i \in I_{x, \mathcal{R}}(\Xi_{t-})} \delta_{\xi_{t-}^i} + \sum_{j=1}^k \delta_{y_j}.$$

In other words, we remove all the atoms of Ξ_{t-} sitting in $B(x, \mathcal{R})$, and we add k atoms at locations that are i.i.d and uniformly distributed over the ball $B(x, \mathcal{R})$.

This process is well-defined, since N_t is stochastically bounded by the number $(Y_t^k)_{t \geq 0}$ of particles in a Yule process with k children and with individual branching rate $\int_0^\infty V_{\mathcal{R}} \mu(d\mathcal{R}) < +\infty$ (see [19] for a proof in the case $k = 2$, which can be generalized to the case $k \geq 2$).

The k -parent ancestral process solves a martingale problem that we now introduce. For all $F \in C_b^1(\mathbb{R})$ and $f \in \mathcal{B}(\mathbb{R}^d)$, we define the function $\Phi_{F, f} : \mathcal{M}_p(\mathbb{R}^d) \rightarrow \mathbb{R}$ by :

$$\forall \Xi \in \mathcal{M}_p(\mathbb{R}^d), \Phi_{F, f}(\Xi) = F\left(\int_{\mathbb{R}^d} f(x) \Xi(dx)\right) = F(\langle \Xi, f \rangle).$$

We now define the operator \mathcal{G}_μ^k on the set of functions of the form $\Phi_{F, f}$, which will be at the basis of the martingale problem satisfied by $(\Xi_t)_{t \geq 0}$. Let $F \in C_b^1(\mathbb{R})$ and $f \in \mathcal{B}(\mathbb{R}^d)$, then for all $\Xi = \sum_{i=1}^l \delta_{\xi_i} \in \mathcal{M}_p(\mathbb{R}^d)$, we set :

$$\begin{aligned} \mathcal{G}_\mu^k \Phi_{F, f}(\Xi) := & \int_{\mathbb{R}^d} \int_0^{+\infty} \int_{B(x, \mathcal{R})^k} \left[\mathbf{1}_{x \in S^{\mathcal{R}}(\Xi)} \times \frac{1}{V_{\mathcal{R}}^k} \times F\left(\langle \Xi, f \rangle - \sum_{i \in I_{x, \mathcal{R}}(\Xi)} f(x_i) + \sum_{j=1}^k f(y_j)\right) \right. \\ & \left. - \mathbf{1}_{x \in S^{\mathcal{R}}(\Xi)} \times \frac{1}{V_{\mathcal{R}}^k} F(\langle \Xi, f \rangle) \right] dy_1 \dots dy_k \mu(d\mathcal{R}) dx \end{aligned}$$

This operator is well defined. Indeed, for all $\Xi \in \mathcal{M}_p(\mathbb{R}^d)$, by Fubini's theorem,

$$\begin{aligned} |\mathcal{G}_\mu^k \Phi_{F, f}(\Xi)| & \leq \int_0^{+\infty} \int_{S^{\mathcal{R}}(\Xi)} \int_{B(x, \mathcal{R})^k} 2 \times \frac{1}{V_{\mathcal{R}}^k} \times \|F\|_\infty dy_1 \dots dy_k dx \mu(d\mathcal{R}) \\ & \leq 2 \|F\|_\infty \times \int_0^{+\infty} \text{Vol}(S^{\mathcal{R}}(\Xi)) \mu(d\mathcal{R}) \\ & \leq 2 \|F\|_\infty \times \Xi(\mathbb{R}^d) \times \int_0^{+\infty} V_{\mathcal{R}} \mu(d\mathcal{R}) \\ & < +\infty \end{aligned}$$

by Condition (4).

Proposition 7. *Let $\Xi^0 \in \mathcal{M}_p(\mathbb{R}^d)$, and let $(\Xi_t)_{t \geq 0}$ be the k -parent ancestral process of initial condition Ξ^0 associated to μ . Then, for all $F \in C_b^1(\mathbb{R})$ and for all $f \in \mathcal{B}(\mathbb{R}^d)$, the process*

$$\left(\Phi_{F,f}(\Xi_t) - \Phi_{F,f}(\Xi_0) - \int_0^t \mathcal{G}_\mu^k \Phi_{F,f}(\Xi_s) ds \right)_{t \geq 0}$$

is a martingale.

Intuitively, the k -parent ancestral process records the locations of the potential ancestors of a given sample of individuals. However, because densities are only defined up to a Lebesgue null set, it is not possible to assign uniquely a type to an individual located at $x \in \mathbb{R}^d$ looking at the value of the density at this point. Therefore, as in [19], in order to give a duality relation between the k -parent SLFV and the k -parent ancestral process, we will need to consider a distribution of sampling locations, rather than fixed locations.

More specifically, for all $l \geq 1$ and $x_1, \dots, x_l \in (\mathbb{R}^d)^l$, we define :

$$\Xi[x_1, \dots, x_l] := \sum_{i=1}^l \delta_{x_i} \in \mathcal{M}_p(\mathbb{R}^d).$$

If Ψ is a density function on $(\mathbb{R}^d)^l$, let μ_Ψ be the law of the random point measure $\sum_{i=1}^l \delta_{X_i}$, where (X_1, \dots, X_l) is sampled according to Ψ . If $M \in \mathcal{M}_\lambda$ and $\Xi = \sum_{i=1}^l \delta_{x_i} \in \mathcal{M}_p(\mathbb{R}^d)$, we set :

$$D(M, \Xi) := \prod_{i=1}^l \omega_M(x_i).$$

Notice that for all $l \in \mathbb{N}^*$ and for all density function Ψ on $(\mathbb{R}^d)^l$,

$$\begin{aligned} \int_{\mathcal{M}_p(\mathbb{R}^d)} D(M, \Xi) \mu_\Psi(d\Xi) &= \int_{(\mathbb{R}^d)^l} \Psi(x_1, \dots, x_l) \left\{ \prod_{j=1}^l \omega_M(x_j) \right\} dx_1 \dots dx_l \\ &= \int_{(\mathbb{R}^d \times \{0,1\})^l} \Psi(x_1, \dots, x_l) \left\{ \prod_{j=1}^l \mathbb{1}_0(\kappa_j) \right\} M(dx_1, d\kappa_1) \dots M(dx_l, d\kappa_l) \end{aligned}$$

does not depend on the choice of a density ω_M of M .

A straightforward adaptation of the proof of Proposition 1.7 in [19] to the case $k \geq 2$ leads to the following proposition.

Proposition 8. *Let $k \geq 2$. Let $M^0 \in \mathcal{M}_\lambda$, let $l \in \mathbb{N}^*$, and let Ψ be a density function on $(\mathbb{R}^d)^l$. Let μ be a σ -finite measure on $(0, +\infty)$ satisfying (4). Let $(M_t)_{t \geq 0}$ be the k -SLFV with initial condition*

M^0 associated to μ and $(\Xi_t)_{t \geq 0}$ be the k -parent ancestral process associated to μ . Then, for all $t \geq 0$,

$$\int_{\mathcal{M}_p(\mathbb{R}^d)} \mathbb{E}[D(M_t, \xi) | M_0 = M^0] \mu_\Psi(d\xi) = \mathbf{E}[D(M^0, \Xi_t) | \Xi_0 \sim \mu_\Psi].$$

Equivalently, for all $t \geq 0$,

$$\mathbb{E}_{M^0} \left[\int_{(\mathbb{R}^d)^l} \Psi(x_1, \dots, x_l) \left\{ \prod_{j=1}^l \omega_{M_t}(x_j) \right\} dx_1 \dots dx_l \right] = \int_{(\mathbb{R}^d)^l} \Psi(x_1, \dots, x_l) \mathbf{E}_{\Xi[x_1, \dots, x_l]} \left[\prod_{j=1}^{N_t} \omega_{M^0}(\xi_t^j) \right] dx_1 \dots dx_l.$$

1.2 Construction of the ∞ -parent SLFV

Let us now introduce the limit of the k -parent SLFV when $k \rightarrow \infty$ somewhat informally. We will call it the ∞ -parent spatial Λ -Fleming Viot process, or ∞ -parent SLFV. It will be constructed rigourously in Section 2 using an alternative construction of the k -parent SLFV inspired by [38], but we now give an intuitive idea of its definition.

Let μ be a σ -finite measure on $(0, +\infty)$ satisfying Condition (4), and let Π be a Poisson point process on $\mathbb{R} \times \mathbb{R}^d \times (0, +\infty)$ with intensity $dt \otimes dx \otimes \mu(d\mathcal{R})$. Let also $M^0 \in \mathcal{M}_\lambda$. We start the ∞ -parent spatial Λ -Fleming Viot process $(M_t^\infty)_{t \geq 0}$, or ∞ -parent SLFV, at $M_0^\infty = M^0$. Then, if $(t, x, \mathcal{R}) \in \Pi$, as before, we consider that a reproduction event occurs in the ball $B(x, \mathcal{R})$ at time t . However, this time we do not sample a finite number of potential parents. Instead, we look at the value of the integral

$$\int_{B(x, \mathcal{R})} \left(1 - \omega_{M_t^\infty}(z) \right) dz,$$

which amounts to sampling an infinite number of potential parents over the ball $B(x, \mathcal{R})$ and looking at the proportion of them which are of the "existing" type (i.e, type 1).

If $\int_{B(x, \mathcal{R})} \left(1 - \omega_{M_t^\infty}(z) \right) dz = 0$, we consider that the parent which reproduces is of type 0, and we set :

$$\forall z \in B(x, \mathcal{R}), \omega_{M_t^\infty}(z) = 1.$$

Note that in this case, the "parent" which reproduces was "sampled" at a location which is uniformly distributed over the ball $B(x, \mathcal{R})$.

Conversely, if $\int_{B(x, \mathcal{R})} \left(1 - \omega_{M_t^\infty}(z) \right) dz \neq 0$, there is a non negligible number of individuals of type 1 in $B(x, \mathcal{R})$. We impose that it is one of them which reproduces, and in such a way that its offspring invades the whole region. That is, we set :

$$\forall z \in B(x, \mathcal{R}), \omega_{M_t^\infty}(z) = 0.$$

Again, note that in our interpretation, the location of the parent which actually reproduces is uniformly distributed over the closure of the region $\{y \in B(x, \mathcal{R}) : \omega_{M_{t^-}}^\infty(y) = 1\}$.

As for the k -parent SLFV, the ∞ -parent SLFV is solution to a martingale problem. However, and in contrast with the case of the k -parent SLFV, the condition (4) on μ will not be sufficient to ensure that this solution is unique. Instead, we will need the following stronger condition.

Definition 9. Let $a_d > 0$ such that the minimal number of d -dimensional balls of radius 1 needed to cover the border of an hypersphere of radius n in d dimensions is bounded from above by $a_d \times n^{d-1}$ for every $n \geq 1$. A σ -finite measure μ on \mathbb{R}_+^* is said to satisfy Condition (5) if it satisfies Condition (4), and if there exists $\mathcal{R} > 0$ such that

$$\sum_{n=1}^{+\infty} \left(\int_{(n-1)\mathcal{R}}^{n\mathcal{R}} (\mathcal{R} + r)^d \mu(dr) \right) (a_d \times n^{d-1} + 1) < +\infty. \quad (5)$$

Examples of σ -finite measures μ on \mathbb{R}_+^* satisfying Condition (5) are the following :

1. Measures μ on \mathbb{R}_+^* having a bounded support.
2. Measures μ on \mathbb{R}_+^* of the form $\alpha \times (1 + r)^{-3d-1} dr$, with $\alpha > 0$.

We define the operator \mathcal{L}^∞ on functions of the form $\Psi_{F,f}$ where $F \in C^1(\mathbb{R})$ and $f \in C_c(\mathbb{R}^d)$ in the following way. For all $M \in \mathcal{M}_\lambda$, we set :

$$\begin{aligned} \mathcal{L}_\mu^\infty \Psi_{F,f}(M) := & \int_0^{+\infty} \int_{Supp^{\mathcal{R}}(f)} \left[\delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega_M(z)) dz \right) \times F(\langle \Theta_{x,\mathcal{R}}^+(\omega_M), f \rangle) \right. \\ & + \left(1 - \delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega_M(z)) dz \right) \right) \times F(\langle \Theta_{x,\mathcal{R}}^-(\omega_M), f \rangle) \\ & \left. - F(\langle \omega_M, f \rangle) \right] dx \mu(d\mathcal{R}). \end{aligned}$$

Note that if $\delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega_M(z)) dz \right) = 1$, then for all $y \in B(x, \mathcal{R})$ except possibly on a Lebesgue null set,

$$\Theta_{x,\mathcal{R}}^+(\omega_M)(y) = \omega_M(y).$$

In other words, the ball $B(x, \mathcal{R})$ is already completely void of "existing" individuals, and filling it with "ghost" individuals does not change anything. Therefore, we also have

$$\begin{aligned} \mathcal{L}_\mu^\infty \Psi_{F,f}(M) = & \int_0^{+\infty} \int_{Supp^{\mathcal{R}}(f)} \left(1 - \delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega_M(z)) dz \right) \right) \\ & \times \left[F(\langle \Theta_{x,\mathcal{R}}^-(\omega_M), f \rangle) - F(\langle \omega_M, f \rangle) \right] dx \mu(d\mathcal{R}). \end{aligned}$$

In Section 5, we show that this operator is well-defined, even if μ satisfies Condition (4) rather than Condition (5). If μ satisfies Condition (5), then the associated martingale problem can be used to define and fully characterize the ∞ -parent SLFV. If μ satisfies only Condition (4), then the ∞ -parent SLFV is still solution to the martingale problem, but we no longer know whether this solution is unique, as stated in the following theorem. Therefore, we will provide in Section 2 a construction of the ∞ -parent SLFV which does not rely on the martingale problem, and works even if μ only satisfies Condition (4).

Theorem 10. *Let $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$, let $M^0 \in \mathcal{M}_\lambda$, and let μ be a σ -finite measure on $(0, +\infty)$ satisfying Condition (4). Then, the ∞ -parent SLFV with initial condition M^0 associated to μ defined in Section 2.2 is a solution to the martingale problem for $(\mathcal{L}_\mu^\infty, M^0)$.*

Moreover, if μ satisfies Condition (5), the martingale problem associated to $(\mathcal{L}_\mu^\infty, M^0)$ is well-posed, and the ∞ -parent SLFV with initial condition M^0 associated to μ is the unique solution to it in $D_{\mathcal{M}_\lambda}[0, +\infty)$.

1.3 Dual of the ∞ -parent SLFV

As for the k -parent SLFV, the ∞ -parent SLFV also has a dual process of potential ancestors.

Let \mathcal{E}^c be the set of Lebesgue measurable and connected subsets of \mathbb{R}^d whose Lebesgue measure is finite. Let \mathcal{E}^{cf} be the set of all finite unions of elements of \mathcal{E}^c . If $E \in \mathcal{E}^{cf}$ can be written as $E = \cup_{i=1}^l E^i$ where for all $1 \leq i \leq l$, $E^i \in \mathcal{E}^c$, we let $m(E) = m(E^1, \dots, E^l)$ be the measure on \mathbb{R}^d defined by $m(E)(dx) := \mathbf{1}_{x \in E} dx$, and we set :

$$\mathcal{M}^{cf} := \{m(E) : E \in \mathcal{E}^{cf}\}.$$

Definition 11 (∞ -parent ancestral process). *Let μ be a σ -finite measure on $(0, +\infty)$ satisfying Condition (5). Let $\overleftarrow{\Pi}$ be a Poisson point process on $\mathbb{R}_+ \times \mathbb{R}^d \times (0, +\infty)$ with intensity $dt \otimes dx \otimes \mu(d\mathcal{R})$, defined on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$.*

Let $\Xi^0 = m(E_0^1, \dots, E_0^l) \in \mathcal{M}^{cf}$. Then, the \mathcal{M}^{cf} -valued ∞ -parent ancestral process $(\Xi_t^\infty)_{t \geq 0}$ with initial condition Ξ^0 associated to μ (or equivalently to $\overleftarrow{\Pi}$) is defined in the following way.

First, we set $\Xi_0^\infty = \Xi^0$. Then, if for all $t \geq 0$, we write Ξ_t^∞ as

$$\Xi_t^\infty = m(E_t),$$

then for all $(t, x, \mathcal{R}) \in \overleftarrow{\Pi}$, if $E_{t-} \cap B(x, \mathcal{R})$ has a non zero Lebesgue measure,

$$\Xi_t^\infty = m(E_{t-} \cup B(x, \mathcal{R})).$$

Moreover, this process is Markovian.

We will show that this process is well-defined in Section 3.

Remark 12. Note that the case $E_{t-} \cap B(x, \mathcal{R}) = B(x, \mathcal{R})$ is equivalent to $E_{t-} \cup B(x, \mathcal{R}) = E_{t-}$, and hence does not correspond to a visible jump of $(\Xi_t^\infty)_{t \geq 0}$.

For all $M \in \mathcal{M}_\lambda$ with density ω and for all $\Xi = m(E) \in \mathcal{M}^{cf}$, we set :

$$\tilde{D}(M, \Xi) := \delta_0 \left(\int_E (1 - \omega(x)) dx \right).$$

If we know the value of $\tilde{D}(M, \Xi)$ for all $\Xi \in \mathcal{M}^{cf}$, since ω is $\{0, 1\}$ -valued, we know the value of ω everywhere up to a Lebesgue null set, and so we have completely characterized M . Therefore, the following duality result shows that the solution to the martingale problem associated to \mathcal{L}_μ^∞ is unique.

Proposition 13. *Let μ be a σ -finite measure on $(0, +\infty)$ satisfying Condition (5). Let $M^0 \in \mathcal{M}_\lambda$, and let $(M_t^\infty)_{t \geq 0}$ be a solution to the martingale problem associated to $(\mathcal{L}_\mu^\infty, \delta_{M^0})$. Then, for all $t \geq 0$ and for all $E^0 \in \mathcal{M}^{cf}$,*

$$\mathbb{E}_{M^0} \left[\tilde{D}(M_t^\infty, m(E^0)) \right] = \mathbf{E}_{m(E^0)} \left[\tilde{D}(M^0, \Xi_t^\infty) \right],$$

where (Ξ_t^∞) is the ∞ -parent ancestral process of initial condition $m(E^0)$ associated to μ . Equivalently, for every $t \geq 0$, if ω_t et ω_0 are $\{0, 1\}$ -valued densities of M_t^∞ and M^0 ,

$$\mathbb{E} \left[\delta_0 \left(\int_{E^0} (1 - \omega_t(x)) dx \right) \right] = \mathbf{E}_{m(E^0)} \left[\delta_0 \left(\int_{E_t} (1 - \omega_0(x)) dx \right) \right].$$

1.4 Structure of the paper

In Section 2, we construct the ∞ -parent SLFV rigorously, by introducing a coupling between a sequence of k -parent SLFV processes with the same initial conditions. We also show the first part of Theorem 10, i.e, that the ∞ -parent SLFV is a solution to the martingale problem associated to \mathcal{L}_μ^∞ . In Section 3, we first demonstrate that the ∞ -parent ancestral process is well defined, and we then show that it can be characterized as the unique solution to a specific martingale problem. Section 4 is devoted to the proof of the duality relation between the ∞ -parent SLFV and the ∞ -parent ancestral process stated

in Proposition 13. The second part of Theorem 10 is then a direct consequence of Proposition 13. Section 5 contains technical lemmas used throughout the paper.

2 The ∞ -parent SLFV

2.1 Alternative construction of the k -parent SLFV

In order to construct the ∞ -parent SLFV rigorously, we start by introducing an alternative construction of the k -parent SLFV, based on a variant of its dual. It relies on the sampling of parental locations *along with* reproduction events, and is an adaptation of the concept of *parental skeleton* presented in Section 2.3.1 of [38].

In all that follows, let μ be a σ -finite measure on $(0, +\infty)$ satisfying Condition (4). Let $U = \mathcal{B}(0, 1)^{\mathbb{N}}$, and let \tilde{u} be the law of a sequence of i.i.d random variables $(\mathcal{P}_n)_{n \geq 1}$ uniformly distributed over $\mathcal{B}(0, 1)$. We will call an element of U a *sequence of potential parents*. Let us now extend the Poisson point process Π considered earlier by adding to each event a countable sequence of locations of potential parents. Indeed, let Π^c be a Poisson point process on $\mathbb{R} \times \mathbb{R}^d \times (0, +\infty) \times U$ with intensity

$$dt \otimes dx \otimes \mu(d\mathcal{R}) \otimes \tilde{u}(d(p_n)_{n \geq 1}).$$

Then for all $(t, x, \mathcal{R}, (p_n)_{n \geq 1}) \in \Pi^c$,

- as before, t can be interpreted as the time at which the reproduction event occurs, and we can see $B(x, \mathcal{R})$ as being the area affected by the reproduction event.
- For all $n \geq 1$, $x + \mathcal{R} \times p_n$ is uniformly distributed over the ball $B(x, \mathcal{R})$, and can be interpreted as the location of the n -th potential parent sampled, if at least n potential parents have to be sampled.

We start by defining the variant of the k -parent ancestral process, on which the alternative construction of the k -parent SLFV is based.

Definition 14 (Quenched k -parent ancestral process). *Let $k \geq 2$, let $\Xi^0 \in M_p(\mathbb{R}^d)$, and let $\tilde{t} \geq 0$. The k -parent ancestral process $(\Xi_{k,t}^{\Pi^c, \tilde{t}, \Xi^0})_{t \geq 0}$ associated to Π^c , started at time \tilde{t} and with initial condition Ξ^0 is the $\mathcal{M}_p(\mathbb{R}^d)$ -valued Markov jump process defined as follows.*

- First, we set $\Xi_{k,0}^{\Pi^c, \tilde{t}, \Xi^0} = \Xi^0$.
- Then, for all $(t, x, \mathcal{R}, (p_n)_{n \geq 1}) \in \Pi^c$ such that $t \leq \tilde{t}$, recalling that for $\Xi = \sum_{i=1}^l \delta_{\xi_i} \in \mathcal{M}_p(\mathbb{R}^d)$,

$I_{x,\mathcal{R}}(\Xi) = \{i \in \llbracket 1, l \rrbracket : \|x - \xi_i\| \leq \mathcal{R}\}$, if

$$I_{x,\mathcal{R}}(\Xi_{k,(\tilde{t}-t)_-}^{\Pi^c, \tilde{t}, \Xi^0}) \neq \emptyset,$$

then for all $1 \leq l \leq k$, we set

$$y_l := x + \mathcal{R} \times p_l$$

and

$$\Xi_{k, \tilde{t}-t}^{\Pi^c, \tilde{t}, \Xi^0} := \Xi_{k, (\tilde{t}-t)_-}^{\Pi^c, \tilde{t}, \Xi^0} - \sum_{x' \in I_{x, \mathcal{R}}(\Xi_{k, (\tilde{t}-t)_-}^{\Pi^c, \tilde{t}, \Xi^0})} \delta_{x'} + \sum_{l=1}^k \delta_{y_l}.$$

It is straightforward to check that this process has the same distribution as the k -parent ancestral process associated to μ and with initial condition Ξ^0 . Its interest is twofold. First, conditionally on Π^c , $(\Xi_{k,t}^{\Pi^c, \tilde{t}, \Xi^0})_{t \geq 0}$ is completely deterministic. Moreover, if for all $\Xi = \sum_{i=1}^l \delta_{x_i} \in \mathcal{M}_p(\mathbb{R}^d)$, we denote the set of atoms of Ξ by

$$A(\Xi) := \{x_i : i \in \llbracket 1, l \rrbracket\},$$

then the process satisfies the following property, which will be useful in the coupling that we will introduce later.

Lemma 15. *Let $2 \leq k \leq k'$, let $\Xi^0 \in \mathcal{M}_p(\mathbb{R}^d)$, let $\tilde{t} \geq 0$, and let Π^c be a Poisson point process on $\mathbb{R} \times \mathbb{R}^d \times (0, +\infty) \times U$ with intensity $dt \otimes dx \otimes \mu(d\mathcal{R}) \otimes \tilde{u}(d(p_n)_{n \geq 1})$.*

Then, for all $t \geq 0$,

$$A(\Xi_{k,t}^{\Pi^c, \tilde{t}, \Xi^0}) \subseteq A(\Xi_{k',t}^{\Pi^c, \tilde{t}, \Xi^0}).$$

In particular, for all $t \geq 0$ and $x \in \mathbb{R}^d$,

$$A(\Xi_{k,t}^{\Pi^c, \tilde{t}, \delta_x}) \subseteq A(\Xi_{k',t}^{\Pi^c, \tilde{t}, \delta_x}).$$

Remark 16. Since $A(\Xi)$ is a set, if there exists $i \neq j$ such that $x_i = x_j$, then x_i appears only once in $A(\Xi)$.

Intuitively, the idea behind this lemma is the following. Since the coupled k -parent and k' -parent ancestral processes are based on the same extended Poisson point process of reproduction events, their evolutions are determined by the same reproduction events. Moreover, since $k' \geq k$, all the potential parents which are involved in the dynamics of the k -parent ancestral process are also potential parents for the k' -ancestral process. Therefore, we can consider that the k -parent ancestral process is embedded

in the k' -parent ancestral process.

We now introduce an alternative way of constructing the k -parent SLFV, by associating it to the extended Poisson point process Π^c .

Definition 17 (Quenched k -parents SLFV). *Let $k \geq 2$, and let $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$ be a measurable function. The k -parent SLFV $(M_{k,t}^{\Pi^c, \omega})_{t \geq 0}$ associated to Π^c and of initial density ω is the \mathcal{M}_λ -valued Markov process defined as follows.*

- First, we set $\omega_{k,0}^{\Pi^c, \omega} = \omega$.
- Then, for all $t \geq 0$ and for all $x \in \mathbb{R}^d$, we set

$$\omega_{k,t}^{\Pi^c, \omega}(x) := \prod_{y \in A(\Xi_{k,t}^{\Pi^c, \omega}, \delta x)} \omega(y). \quad (6)$$

- We conclude by setting for all $t \geq 0$,

$$M_{k,t}^{\Pi^c, \omega} := ((\omega_{k,t}^{\Pi^c, \omega}(x) \delta_0(d\kappa) + (1 - \omega_{k,t}^{\Pi^c, \omega}(x)) \delta_1(d\kappa)) dx).$$

$(\omega_{k,t}^{\Pi^c, \omega})_{t \geq 0}$ will be called the density of the k -parent SLFV associated to Π^c and of initial condition ω .

Note that $\omega_{k,t}^{\Pi^c, \omega}(x)$ in Eq. 6 is thus equal to 1 if and only if all potential ancestors at time 0 of the individuals at x at time t are of type 0, i.e are all ghosts.

We show below that this process corresponds to another way of constructing the k -parent SLFV using the parental skeleton, and in particular, that $(M_{k,t}^{\Pi^c, \omega})_{t \geq 0} \in D_{\mathcal{M}_\lambda}[0, +\infty)$. This alternative construction will allow us to couple SLFV processes with different numbers of potential parents, using the same Poisson process. However, even though it is possible to define the k -parent SLFV for an initial condition $M \in \mathcal{M}_\lambda$ instead of an initial density ω of M , this coupling can only be used if all processes are constructed using the same initial density.

Proof. In order for the process to have a chance to correspond to the k -parent SLFV, we first need to check that

$$(M_{k,t}^{\Pi^c, \omega})_{t \geq 0} \in D_{\mathcal{M}_\lambda}[0, +\infty).$$

Let $t \geq 0$. Since ω is $\{0, 1\}$ -valued, by definition $\omega_{k,t}^{\Pi^c, \omega}$ is $\{0, 1\}$ -valued. Moreover, the values taken by ω are changed over balls of the form $B(x, \mathcal{R})$ in order to compute $\omega_{k,t}^{\Pi^c, \omega}$. Therefore, as ω is measurable, $\omega_{k,t}^{\Pi^c, \omega}$ is measurable as well, and we obtain that for all $t \geq 0$, $M_{k,t}^{\Pi^c, \omega} \in \mathcal{M}_\lambda$.

We now show that the process is càdlàg. Let $f \in C_c(\mathbb{R}^d)$, and let

$$t_f := \max\{t' < t : \exists \mathcal{R} > 0, \exists x \in \text{Supp}^{\mathcal{R}}(f), \exists (p_n)_{n \geq 1} \in U, (t', x, \mathcal{R}, (p_n)_{n \geq 1}) \in \Pi^c\}.$$

Since there exists $C > 0$ such that for all $\mathcal{R} > 0$,

$$\text{Vol}(\text{Supp}^{\mathcal{R}}(f)) \leq C \times (\mathcal{R}^d \vee 1),$$

the support of f is affected by reproduction events at rate

$$\begin{aligned} \int_0^\infty \text{Vol}(\text{Supp}^{\mathcal{R}}(f)) \mu(d\mathcal{R}) &\leq C \times \int_0^\infty (\mathcal{R}^d \vee 1) \mu(d\mathcal{R}) \\ &< +\infty \end{aligned}$$

as μ satisfies Condition (4), and thus we obtain that $t_f < t$. Therefore, $s \rightarrow \langle \omega_{k,s}^{\Pi^c, \omega}, f \rangle$ is constant over $[t_f, t)$, and we can conclude. \square

Lemma 18. *Under the notation of Definition 17, $(M_{k,t}^{\Pi^c, \omega})_{t \geq 0}$ has the same distribution as the k -parent SLFV associated to μ and with initial density ω .*

Proof. We set $M^0 = M_{k,0}^{\Pi^c, \omega}$, and we use the characterization of the k -SLFV by the duality relation in Proposition 8.

Let $l \in \mathbb{N}^*$, let Ψ be a density function on $(\mathbb{R}^d)^l$, and let $t \geq 0$. Then,

$$\begin{aligned} &\mathbb{E}_{M^0} \left[\int_{(\mathbb{R}^d)^l} \Psi(x_1, \dots, x_l) \left(\prod_{j=1}^l \omega_{k,t}^{\Pi^c, \omega}(x_j) \right) dx_1 \dots dx_l \right] \\ &= \int_{(\mathbb{R}^d)^l} \Psi(x_1, \dots, x_l) \mathbb{E}_{M^0} \left[\prod_{j=1}^l \omega_{k,t}^{\Pi^c, \omega}(x_j) \right] dx_1 \dots dx_l \\ &= \int_{(\mathbb{R}^d)^l} \Psi(x_1, \dots, x_l) \mathbb{E}_{M^0} \left[\prod_{j=1}^l \prod_{y \in A(\Xi_{k,t}^{\Pi^c, t, \delta x_j})} \omega(y) \right] dx_1 \dots dx_l \\ &= \int_{(\mathbb{R}^d)^l} \Psi(x_1, \dots, x_l) \mathbb{E}_{M^0} \left[\prod_{y \in A(\Xi_{k,t}^{\Pi^c, t, \sum_{j=1}^l \delta x_j})} \omega(y) \right] dx_1 \dots dx_l \\ &= \int_{(\mathbb{R}^d)^l} \Psi(x_1, \dots, x_l) \mathbf{E}_{\Xi[x_1, \dots, x_l]} \left[\prod_{y \in A(\Xi_t)} \omega(y) \right] dx_1 \dots dx_l, \end{aligned}$$

with $(\Xi_t)_{t \geq 0}$ the k -parent ancestral process associated to μ with initial condition $\Xi[x_1, \dots, x_l]$. We used the definition of the quenched k -parent SLFV to pass from line 2 to line 3, and the fact that ω is

$\{0, 1\}$ -valued to pass from line 3 to line 4.

Writing $\Xi_t = \sum_{j=1}^{N_t} \xi_t^j$, we obtain

$$\begin{aligned} & \mathbb{E}_{M^0} \left[\int_{(\mathbb{R}^d)^l} \Psi(x_1, \dots, x_l) \left(\prod_{j=1}^l \omega_{k,t}^{\Pi^c, \omega}(x_j) \right) dx_1 \dots dx_l \right] \\ &= \int_{(\mathbb{R}^d)^l} \Psi(x_1, \dots, x_l) \mathbf{E}_{\Xi[x_1, \dots, x_l]} \left[\prod_{j=1}^{N_t} \omega(\xi_t^j) \right] dx_1 \dots dx_l. \end{aligned}$$

This concludes the proof. \square

This lemma has two direct consequences. First, $(M_{k,t}^{\Pi^c, \omega})_{t \geq 0}$ is Markovian. Moreover, since this process is \mathcal{M}_λ -valued, we have proved the second part of Theorem 1, that is, that the unique solution to the martingale problem characterizing the k -parent SLFV is \mathcal{M}_λ -valued.

The interest of the coupling lies in the fact that given a sequence of coupled k -parent SLFV constructed using the same extended Poisson point process Π^c , their corresponding densities, as constructed in Definition 17, satisfy the following property.

Lemma 19. *Let $2 \leq k < k'$, and let $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$ be a measurable function. Let Π^c be a Poisson point process on $\mathbb{R} \times \mathbb{R}^d \times (0, +\infty) \times U$ with intensity $dt \otimes dx \otimes \mu(d\mathcal{R}) \otimes \tilde{u}((p_n)_{n \geq 1})$.*

Then, for all $t \geq 0$ and $x \in \mathbb{R}^d$,

$$\omega_{k',t}^{\Pi^c, \omega}(x) \leq \omega_{k,t}^{\Pi^c, \omega}(x).$$

In particular, for all $t \geq 0$ and $x \in \mathbb{R}^d$, $(\omega_{k,t}^{\Pi^c, \omega}(x))_{k \geq 2}$ converges to some $\omega_t^\infty(x) \in \{0, 1\}$ as $k \rightarrow +\infty$.

Proof. Let $t \geq 0$ and $x \in \mathbb{R}^d$. By Lemma 15,

$$A(\Xi_{k,t}^{\Pi^c, t, \delta_x}) \subseteq A(\Xi_{k',t}^{\Pi^c, t, \delta_x}).$$

Therefore, as ω is $\{0, 1\}$ -valued,

$$\begin{aligned} \omega_{k',t}^{\Pi^c, \omega}(x) &= \prod_{y \in A(\Xi_{k',t}^{\Pi^c, t, \delta_x})} \omega(y) \\ &\leq \prod_{y \in A(\Xi_{k,t}^{\Pi^c, t, \delta_x})} \omega(y) \\ &\leq \omega_{k,t}^{\Pi^c, \omega}(x). \end{aligned}$$

The second part of the lemma is a consequence of the fact that $(\omega_{k,t}^{\Pi^c, \omega}(x))_{k \geq 2}$ is a non-increasing $\{0, 1\}$ -valued sequence. \square

2.2 Definition of the ∞ -parent SLFV

We can now define the ∞ -parent SLFV.

Definition 20. *Let $M^0 \in \mathcal{M}_\lambda$ with density $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$. The ∞ -parent spatial Λ -Fleming Viot process, or ∞ -parent SLFV, with initial density ω associated to the extended Poisson point process Π^c is the \mathcal{M}_λ -valued process $(M_t^\infty)_{t \geq 0}$ defined the following way.*

First, we set $M_0^\infty = M^0$. Then, for all $t \geq 0$ and $x \in \mathbb{R}^d$, we set

$$\omega_t^\infty(x) := \lim_{k \rightarrow +\infty} \omega_{k,t}^{\Pi^c, \omega}(x)$$

and we set

$$M_t^\infty(dx, d\kappa) := (\omega_t^\infty(x)\delta_0(d\kappa) + (1 - \omega_t^\infty(x))\delta_1(d\kappa))dx.$$

Π^c will be called the associated extended Poisson point process, and $(\omega_t^\infty)_{t \geq 0}$ will be called the density of the ∞ -parent SLFV associated to Π^c and of initial density ω .

In its more general form, the ∞ -parent SLFV is defined for an initial condition $M^0 \in \mathcal{M}_\lambda$ and a σ -finite measure μ . However, we construct it using a density ω of M^0 , and an extended Poisson point process Π^c , and in the following, we will need both the initial density and the extended Poisson process used in order to prove some properties satisfied by the ∞ -parent SLFV. Therefore, we considered two complementary definitions of the process, one based on the initial condition and the measure μ , and the other one based on the initial density and the extended Poisson point process, both definitions corresponding to the same process. In the following, we will use one or the other of the two definitions, depending on whether the initial density and extended Poisson point process used to construct the process are needed or not.

As in the proof of Definition 17, we can show that $(M_t^\infty)_{t \geq 0} \in D_{\mathcal{M}_\lambda}[0, +\infty)$.

Lemma 21. *Under the notation of Definition 20, $(M_t^\infty)_{t \geq 0}$ is Markovian.*

Proof. First, notice that the definition of $(M_t^\infty)_{t \geq 0}$ implies that we only need to demonstrate that $(\omega_t^\infty)_{t \geq 0}$ is Markovian.

Let $0 \leq s \leq t$ and let $x \in \mathbb{R}^d$. Our goal is to show that

$$\omega_t^\infty(x) = \lim_{k \rightarrow +\infty} \prod_{x' \in A\left(\Xi_{k,t-s}^{\Pi^c, t, \delta x}\right)} \omega_s^\infty(x').$$

Indeed, if this result is true, since $A\left(\Xi_{\tilde{k}, t-s}^{\Pi^c, t, \delta_x}\right)$ depends on events occurring during the interval $[s, t]$, it is independent from $(\omega_{s'}^\infty)_{0 \leq s' \leq s}$ and we can conclude.

By definition of the ∞ -parent SLFV,

$$\omega_t^\infty(x) = \lim_{k \rightarrow +\infty} \omega_{k,t}^{\Pi^c, \omega}(x).$$

Using Lemma 46 from Section 5, we obtain

$$\omega_t^\infty(x) = \lim_{k \rightarrow +\infty} \prod_{x' \in A\left(\Xi_{k, t-s}^{\Pi^c, t, \delta_x}\right)} \omega_{k,s}^{\Pi^c, \omega}(x').$$

Let $\tilde{k} \geq 2$. By Lemma 19 and since for all $k \geq 2$, $\omega_{k,s}^{\Pi^c, \omega}$ is $\{0, 1\}$ -valued,

$$\begin{aligned} \omega_t^\infty(x) &\leq \lim_{k \rightarrow +\infty} \prod_{x' \in A\left(\Xi_{\tilde{k}, t-s}^{\Pi^c, t, \delta_x}\right)} \omega_{k,s}^{\Pi^c, \omega}(x') \\ &\leq \prod_{x' \in A\left(\Xi_{\tilde{k}, t-s}^{\Pi^c, t, \delta_x}\right)} \lim_{k \rightarrow +\infty} \omega_{k,s}^{\Pi^c, \omega}(x') \\ &\leq \prod_{x' \in A\left(\Xi_{\tilde{k}, t-s}^{\Pi^c, t, \delta_x}\right)} \omega_s^\infty(x'). \end{aligned}$$

Here we used Lemma 47 to pass from the first to the second line, and the definition of the ∞ -parent SLFV to pass from the second to the third line.

Since this is true for all $\tilde{k} \geq 2$,

$$\omega_t^\infty(x) \leq \lim_{\tilde{k} \rightarrow +\infty} \prod_{x' \in A\left(\Xi_{\tilde{k}, t-s}^{\Pi^c, t, \delta_x}\right)} \omega_s^\infty(x').$$

Then, starting back from the equation

$$\omega_t^\infty(x) = \lim_{k \rightarrow +\infty} \prod_{x' \in A\left(\Xi_{k, t-s}^{\Pi^c, t, \delta_x}\right)} \omega_{k,s}^{\Pi^c, \omega}(x'),$$

as for all $x \in \mathbb{R}^d$, $(\omega_{k,s}^{\Pi^c, \omega}(x'))_{k \geq 2}$ is decreasing, we obtain that

$$\omega_t^\infty(x) \geq \lim_{k \rightarrow +\infty} \prod_{x' \in A\left(\Xi_{k, t-s}^{\Pi^c, t, \delta_x}\right)} \omega_s^\infty(x')$$

and we can conclude. \square

2.3 Characterization via a martingale problem

Let $M^0 \in \mathcal{M}_\lambda$ with density $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$. We recall that the operator \mathcal{L}_μ^∞ is defined by

$$\begin{aligned} \mathcal{L}_\mu^\infty \Psi_{F,f}(M) &= \int_0^{+\infty} \int_{\text{Supp}^{\mathcal{R}}(f)} \left(1 - \delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega_M(z)) dz \right) \right) \\ &\quad \times \left[F(\langle \Theta_{x,\mathcal{R}}^-(\omega_M), f \rangle) - F(\langle \omega_M, f \rangle) \right] dx \mu(d\mathcal{R}). \end{aligned}$$

The goal of this section is to demonstrate the following result, which is also the first part of Theorem 10.

Proposition 22. *Let $(M_t^\infty)_{t \geq 0}$ be the ∞ -parent SLFV with initial density ω , associated to Π^c . Then, for all $F \in C^1(\mathbb{R})$ and $f \in C_c(\mathbb{R}^d)$,*

$$\left(\Psi_{F,f}(M_t) - \Psi_{F,f}(M_0) - \int_0^t \mathcal{L}_\mu^\infty \Psi_{F,f}(M_s) ds \right)_{t \geq 0}$$

is a martingale.

In other words, $(M_t^\infty)_{t \geq 0}$ is a solution of the martingale problem $(\mathcal{L}_\mu^\infty, M_0^\infty)$, but this solution is not necessarily unique. In fact, we will show in Section 4 that this solution is unique when μ satisfies the stronger Condition (5), but the question of uniqueness when μ does not satisfy Condition (5) remains open.

We start by justifying why the operator \mathcal{L}_μ^∞ is a suitable candidate for an operator characterizing the limit $k \rightarrow +\infty$ of the k -parent SLFV.

Lemma 23. *Let $\omega : \mathbb{R}^d \rightarrow [0, 1]$, and let $x \in \mathbb{R}$. Then, for all $\mathcal{R} > 0$,*

$$\delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega(z)) dz \right) = \lim_{k \rightarrow +\infty} \frac{1}{V_{\mathcal{R}}^k} \int_{B(x,\mathcal{R})^k} \left(\prod_{j=1}^k \omega(y_j) \right) dy_1 \dots dy_k.$$

Proof. For all $k \geq 2$,

$$\frac{1}{V_{\mathcal{R}}^k} \int_{B(x,\mathcal{R})^k} \left[\prod_{j=1}^k \omega(y_j) \right] dy_1 \dots dy_k = \left(\frac{1}{V_{\mathcal{R}}} \int_{B(x,\mathcal{R})} \omega(y) dy \right)^k.$$

As $V_{\mathcal{R}}^{-1} \int_{B(x,\mathcal{R})} \omega(y) dy \in [0, 1]$,

$$\begin{aligned} \lim_{k \rightarrow +\infty} \frac{1}{V_{\mathcal{R}}^k} \int_{B(x,\mathcal{R})^k} \left[\prod_{j=1}^k \omega(y_j) \right] dy_1 \dots dy_k &= 1 \\ \iff \frac{1}{V_{\mathcal{R}}} \int_{B(x,\mathcal{R})} \omega(y) dy &= 1 \\ \iff \int_{B(x,\mathcal{R})} (1 - \omega(z)) dz &= 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \lim_{k \rightarrow +\infty} \frac{1}{V_{\mathcal{R}}^k} \int_{B(x, \mathcal{R})^k} \left[\prod_{j=1}^k \omega(y_j) \right] dy_1 \dots dy_k &= 0 \\ \iff \frac{1}{V_{\mathcal{R}}} \int_{B(x, \mathcal{R})} \omega(y) dy &< 1 \\ \iff \int_{B(x, \mathcal{R})} (1 - \omega(z)) dz &> 0, \end{aligned}$$

and we can conclude. \square

Let $F \in C^1(\mathbb{R})$ and $f \in C_c(\mathbb{R}^d)$. For all $M \in \mathcal{M}_\lambda$,

$$|F(\langle \omega_M, f \rangle)| \leq \max\{F(x) : x \in [-\text{Vol}(\text{Supp}(f)), \text{Vol}(\text{Supp}(f))]\}, \quad (7)$$

which means in particular that for all $x \in \mathbb{R}^d$ and for all $\mathcal{R} > 0$,

$$\begin{aligned} |F(\langle \Theta_{x, \mathcal{R}}^+(\omega_M), f \rangle)| &\leq \max\{F(x) : x \in [-\text{Vol}(\text{Supp}(f)), \text{Vol}(\text{Supp}(f))]\} \\ \text{and } |F(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f \rangle)| &\leq \max\{F(x) : x \in [-\text{Vol}(\text{Supp}(f)), \text{Vol}(\text{Supp}(f))]\}. \end{aligned}$$

Therefore, a direct consequence of the dominated convergence theorem is the following lemma.

Lemma 24. *Let $M \in \mathcal{M}_\lambda$, and let $(M_n)_{n \in \mathbb{N}} \in \mathcal{M}_\lambda$ such that M_n converges vaguely to M . Then, for all $x \in \mathbb{R}^d$ and for all $\mathcal{R} > 0$,*

$$\begin{aligned} F(\langle \omega_{M_n}, f \rangle) &\xrightarrow{n \rightarrow +\infty} F(\langle \omega_M, f \rangle) \\ F(\langle \Theta_{x, \mathcal{R}}^+(\omega_{M_n}), f \rangle) &\xrightarrow{n \rightarrow +\infty} F(\langle \Theta_{x, \mathcal{R}}^+(\omega_M), f \rangle) \\ F(\langle \Theta_{x, \mathcal{R}}^-(\omega_{M_n}), f \rangle) &\xrightarrow{n \rightarrow +\infty} F(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f \rangle). \end{aligned}$$

In contrast with $\mathcal{L}_\mu^k \Psi_{F, f}$, the function $\mathcal{L}_\mu^\infty \Psi_{F, f}$ is not continuous. However, we have the following result.

Lemma 25. *Let $M \in \mathcal{M}_\lambda$, and $(M_n)_{n \in \mathbb{N}} \in \mathcal{M}_\lambda$ such that M_n converges to M in the topology of vague convergence. Assume that there exists a density ω of M and densities ω_n of M_n for all $n \in \mathbb{N}$ such that :*

$$\forall n \in \mathbb{N}, \forall z \in \mathbb{R}^d, \omega(z) \leq \omega_n(z).$$

Then,

$$\lim_{n \rightarrow +\infty} \mathcal{L}_\mu^\infty \Psi_{F, f}(M_n) = \mathcal{L}_\mu^\infty \Psi_{F, f}(M).$$

Proof. First, since $(M_n)_{n \in \mathbb{N}}$ converges vaguely to M , by Lemma 24, for all $\mathcal{R} > 0$ and for all $x \in \text{Supp}^{\mathcal{R}}(f)$,

$$F(\langle \Theta_{x, \mathcal{R}}^-(\omega_n), f \rangle) \xrightarrow{n \rightarrow +\infty} F(\langle \Theta_{x, \mathcal{R}}^-(\omega), f \rangle).$$

Then, let $\mathcal{R} > 0$ and $n \in \mathbb{N}$. Since for all $z \in \mathcal{B}(x, \mathcal{R})$, $\omega(z) \leq \omega_n(z)$,

$$\int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega(z)) dz \geq \int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega_n(z)) dz.$$

Moreover, since

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega_n(z)) dz &= \int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega(z)) dz \\ &\geq \int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega_n(z)) dz, \end{aligned}$$

if $\lim_{n \rightarrow +\infty} \int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega_n(z)) dz = 0$, then for all $n \in \mathbb{N}$, $\int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega_n(z)) dz = 0$, and thus :

$$\lim_{n \rightarrow +\infty} \delta_0 \left(\int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega_n(z)) dz \right) = \delta_0 \left(\int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega(z)) dz \right).$$

Conversely, if $\lim_{n \rightarrow +\infty} \int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega_n(z)) dz \neq 0$, since $\delta_0(\bullet)$ is continuously equal to 0 over \mathbb{R}_+^* ,

$$\lim_{n \rightarrow +\infty} \delta_0 \left(\int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega_n(z)) dz \right) = \delta_0 \left(\int_{\mathcal{B}(x, \mathcal{R})} (1 - \omega(z)) dz \right).$$

We conclude by using the dominated convergence theorem. □

In order to prove Proposition 22, we will need the following result, which illustrates in which sense the ∞ -parent SLFV can be considered as the limit $k \rightarrow +\infty$ of the k -parent SLFV.

Lemma 26. *Let $(M_t^\infty)_{t \geq 0}$ be the ∞ -parent SLFV with initial density ω associated to Π^c . Then, for all $t \geq 0$, $(M_{k,t}^{\Pi^c, \omega})_{k \geq 2}$ converges vaguely to M_t^∞ as $k \rightarrow +\infty$.*

Proof. Let $t \geq 0$, and let ω_t^∞ be the density of the ∞ -SLFV with initial density ω associated to Π^c , considered at time t . Let $f \in C_c(\mathbb{R}^d)$. Then f is integrable and

$$\begin{aligned} \forall x \in \mathbb{R}^d, f(x) \omega_{k,t}^{\Pi^c, \omega}(x) &\xrightarrow{k \rightarrow +\infty} f(x) \omega_t^\infty(x) \\ \forall x \in \mathbb{R}^d, |f(x) \omega_{k,t}^{\Pi^c, \omega}(x)| &\leq |f(x)|. \end{aligned}$$

Therefore, by the dominated convergence theorem,

$$\lim_{k \rightarrow +\infty} \int_{\mathbb{R}^d} f(x) \omega_{k,t}^{\Pi^c, \omega}(x) dx = \int_{\mathbb{R}^d} f(x) \omega_t^\infty(x) dx$$

and

$$\lim_{k \rightarrow +\infty} \int_{B(x, \mathcal{R})} f(y) \omega_{k,t}^{\Pi^c, \omega}(y) dx = \int_{B(x, \mathcal{R})} f(y) \omega_t^\infty(y) dy.$$

We now consider $\tilde{f} \in C_c(\mathbb{R}^d \times \{0, 1\})$. Then, there exists $f_0, f_1 \in C_c(\mathbb{R}^d)$ such that

$$\forall (x, \kappa) \in \mathbb{R}^d \times \{0, 1\}, \tilde{f}(x, \kappa) = f_0(x) \mathbb{1}_{\{0\}}(\kappa) + f_1(x) \mathbb{1}_{\{1\}}(\kappa).$$

Therefore, for all $k \geq 2$,

$$\begin{aligned} \int_{\mathbb{R}^d \times \{0, 1\}} \tilde{f}(x, \kappa) M_{k,t}^{\Pi^c, k}(dx, d\kappa) &= \int_{\mathbb{R}^d} f_0(x) \omega_{k,t}^{\Pi^c, k}(x) dx + \int_{\mathbb{R}^d} f_1(x) (1 - \omega_{k,t}^{\Pi^c, k}(x)) dx \\ &\xrightarrow{k \rightarrow +\infty} \int_{\mathbb{R}^d} f_0(x) \omega_t^\infty(x) dx + \int_{\mathbb{R}^d} f_1(x) (1 - \omega_t^\infty(x)) dx \\ &= \int_{\mathbb{R}^d \times \{0, 1\}} \tilde{f}(x, \kappa) M_t^\infty(dx, d\kappa) \end{aligned}$$

and we conclude that $(M_{k,t}^{\Pi^c, \omega})_{k \geq 2}$ converges vaguely to M_t^∞ as $k \rightarrow +\infty$. \square

Lemma 27. *Let $(M_t^\infty)_{t \geq 0}$ be the ∞ -parent SLFV of initial condition M^0 , constructed using the initial density ω and Π^c . Then, for all $F \in C^1(\mathbb{R})$ and $f \in C_c(\mathbb{R}^d)$, for all $l \geq 1$, for all $0 \leq t_1 < \dots < t_l \leq t < t+s$, for all $h_1, \dots, h_l \in C_b(\mathcal{M}_\lambda)$,*

$$\lim_{k \rightarrow +\infty} \mathbb{E} \left[\left(\Psi_{F,f}(M_{t+s}^k) - \Psi_{F,f}(M_t^k) - \int_t^{t+s} \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^k) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}^k) \right) \right] = 0.$$

Proof. For all $k \geq 2$, we set $(M_u^k)_{u \geq 0} = (M_{k,u}^{\Pi^c, \omega})_{u \geq 0}$ the k -parent SLFV associated to Π^c and with initial condition ω . Moreover, for all $u \geq 0$, let ω_u^k be a density of M_u^k .

Let $l \geq 1$, $0 \leq t_1 < \dots < t_l \leq t < t+s$ and $h_1, \dots, h_l \in C_b(\mathcal{M}_\lambda)$. Then, for all $k \geq 2$,

$$\begin{aligned} &\mathbb{E} \left[\left(\Psi_{F,f}(M_{t+s}^k) - \Psi_{F,f}(M_t^k) - \int_t^{t+s} \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^k) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}^k) \right) \right] \\ &= \mathbb{E} \left[\left(\Psi_{F,f}(M_{t+s}^k) - \Psi_{F,f}(M_t^k) - \int_t^{t+s} \mathcal{L}_\mu^k \Psi_{F,f}(M_u^k) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}^k) \right) \right] \\ &\quad + \mathbb{E} \left[\left(\int_t^{t+s} \mathcal{L}_\mu^k \Psi_{F,f}(M_u^k) - \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^k) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}^k) \right) \right]. \end{aligned}$$

Since $(M_u^k)_{u \geq 0}$ is solution to the martingale problem associated to $(\mathcal{L}^k, \delta_{M_0^k})$, the above is equal to

$$0 + \mathbb{E} \left[\left(\int_t^{t+s} \mathcal{L}_\mu^k \Psi_{F,f}(M_u^k) - \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^k) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}^k) \right) \right].$$

From Lemmas 40 and 41 in Section 5, we can apply the dominated convergence theorem to

$$\mathbb{E} \left[\left(\int_t^{t+s} |\mathcal{L}_\mu^k \Psi_{F,f}(M_u^k) - \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^k)| du \right) \times \left(\prod_{i=1}^l |h_i(M_{t_i}^k)| \right) \right],$$

and we obtain

$$\begin{aligned} & \lim_{k \rightarrow +\infty} \mathbb{E} \left[\left(\Psi_{F,f}(M_{t+s}^k) - \Psi_{F,f}(M_t^k) - \int_t^{t+s} \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^k) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}^k) \right) \right] \\ &= \mathbb{E} \left[\left(\int_t^{t+s} \lim_{k \rightarrow +\infty} \left(\mathcal{L}_\mu^k \Psi_{F,f}(M_u^k) - \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^k) \right) du \right) \times \left(\lim_{k \rightarrow +\infty} \prod_{i=1}^l h_i(M_{t_i}^k) \right) \right], \end{aligned}$$

assuming that the different limits exist.

Now, let $k \geq 2$ and $u \in [t, t+s]$. We have

$$\begin{aligned} & \mathcal{L}_\mu^k \Psi_{F,f}(M_u^k) - \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^k) \\ &= \int_0^\infty \int_{\text{Supp}^{\mathcal{R}}(f)} \left(F(\langle \Theta_{x,\mathcal{R}}^+(\omega_u^k), f \rangle) - F(\langle \omega_u^k, f \rangle) \right) \\ & \quad \times \left[\int_{B(x,\mathcal{R})^k} \prod_{j=1}^k \left(\frac{\omega_u^k(y_j)}{V_{\mathcal{R}}} \right) dy_1 \dots dy_k - \delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega_u^k(y)) dy \right) \right] \\ & \quad + \left(F(\langle \Theta_{x,\mathcal{R}}^-(\omega_u^k), f \rangle) - F(\langle \omega_u^k, f \rangle) \right) \\ & \quad \times \left[\delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega_u^k(y)) dy \right) - \int_{B(x,\mathcal{R})^k} \prod_{j=1}^k \left(\frac{\omega_u^k(y_j)}{V_{\mathcal{R}}} \right) dy_1 \dots dy_k \right] dx \mu(d\mathcal{R}) \\ &= \int_0^\infty \int_{\text{Supp}^{\mathcal{R}}(f)} \left(F(\langle \Theta_{x,\mathcal{R}}^+(\omega_u^k), f \rangle) - F(\langle \Theta_{x,\mathcal{R}}^-(\omega_u^k), f \rangle) \right) \\ & \quad \times \left[\int_{B(x,\mathcal{R})^k} \prod_{j=1}^k \left(\frac{\omega_u^k(y_j)}{V_{\mathcal{R}}} \right) dy_1 \dots dy_k - \delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega_u^k(y)) dy \right) \right] dx \mu(d\mathcal{R}). \end{aligned}$$

The term inside the integral is bounded in absolute value, by Lemma 37 in Section 5. Moreover, as

$(M_u^k)_{k \geq 2}$ converges vaguely to M_u^∞ by Lemma 26, we can apply Lemma 24 and we obtain

$$\lim_{k \rightarrow +\infty} F(\langle \Theta_{x,\mathcal{R}}^+(\omega_u^k), f \rangle) - F(\langle \Theta_{x,\mathcal{R}}^-(\omega_u^k), f \rangle) = F(\langle \Theta_{x,\mathcal{R}}^+(\omega_u^\infty), f \rangle) - F(\langle \Theta_{x,\mathcal{R}}^-(\omega_u^\infty), f \rangle).$$

Therefore, we have to show that

$$\lim_{k \rightarrow +\infty} \int_{B(x, \mathcal{R})^k} \prod_{j=1}^k \left(\frac{\omega_u^k(y_j)}{V_{\mathcal{R}}} \right) dy_1 \dots dy_k - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_u^k(y)) dy \right) = 0.$$

We cannot apply directly Lemma 23, because the density also depends on k . However,

$$\begin{aligned} & \left| \int_{B(x, \mathcal{R})^k} \prod_{j=1}^k \left(\frac{\omega_u^k(y_j)}{V_{\mathcal{R}}} \right) dy_1 \dots dy_k - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_u^k(y)) dy \right) \right| \\ & \leq \left| \int_{B(x, \mathcal{R})^k} \left(\prod_{j=1}^k \frac{\omega_u^k(y_j)}{V_{\mathcal{R}}} - \prod_{j=1}^k \frac{\omega_u^\infty(y_j)}{V_{\mathcal{R}}} \right) dy_1 \dots dy_k \right| \\ & + \left| \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_u^k(y)) dy \right) - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_u^\infty(y)) dy \right) \right| \\ & + \left| \int_{B(x, \mathcal{R})^k} \prod_{j=1}^k \left(\frac{\omega_u^\infty(y_j)}{V_{\mathcal{R}}} \right) dy_1 \dots dy_k - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_u^\infty(y)) dy \right) \right|. \end{aligned}$$

We can apply Lemma 23 to the third term. Since for all $y \in \mathbb{R}^d$, $\omega_u^\infty(y) \leq \omega_u^k(y)$, we showed in the proof of Lemma 25 that

$$\lim_{k \rightarrow +\infty} \left| \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_u^k(y)) dy \right) - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_u^\infty(y)) dy \right) \right| = 0.$$

Regarding the first term, we distinguish two cases. If $V_{\mathcal{R}}^{-1} \int_{B(x, \mathcal{R})} \omega_u^\infty(y) dy = 1$, since

$$\int_{B(x, \mathcal{R})} \frac{\omega_u^\infty(y)}{V_{\mathcal{R}}} dy \leq \int_{B(x, \mathcal{R})} \frac{\omega_u^k(y)}{V_{\mathcal{R}}} dy \leq 1,$$

we obtain that in fact for every $k \geq 2$

$$\int_{B(x, \mathcal{R})^k} \prod_{j=1}^k \left(\frac{\omega_u^k(y_j)}{V_{\mathcal{R}}} - \prod_{j=1}^k \frac{\omega_u^\infty(y_j)}{V_{\mathcal{R}}} \right) dy_1 \dots dy_k = 0.$$

Conversely, assume $V_{\mathcal{R}}^{-1} \int_{B(x, \mathcal{R})} \omega_u^\infty(y) dy < 1$. Since,

$$\int_{B(x, \mathcal{R})} \frac{\omega_u^k(y)}{V_{\mathcal{R}}} dy \xrightarrow{k \rightarrow +\infty} \int_{B(x, \mathcal{R})} \frac{\omega_u^\infty(y)}{V_{\mathcal{R}}} dy,$$

there exist $0 < M < 1$ and $k' \geq 2$ such that :

$$\forall k \geq k', \int_{B(x, \mathcal{R})} \frac{\omega_u^k(y)}{V_{\mathcal{R}}} dy \leq M.$$

Therefore,

$$\left| \int_{B(x, \mathcal{R})^k} \prod_{j=1}^k \left(\frac{\omega_u^k(y_j)}{V_{\mathcal{R}}} \right) - \prod_{j=1}^k \left(\frac{\omega_u^\infty(y_j)}{V_{\mathcal{R}}} \right) dy_1 \dots dy_k \right| = \left| \left(\int_{B(x, \mathcal{R})} \frac{\omega_u^k(y)}{V_{\mathcal{R}}} dy \right)^k - \left(\int_{B(x, \mathcal{R})} \frac{\omega_u^\infty(y)}{V_{\mathcal{R}}} dy \right)^k \right|$$

$$\xrightarrow[k \rightarrow +\infty]{} 0,$$

and we can conclude. \square

We can now show that the ∞ -parent SLFV is solution of the martingale problem introduced in the Proposition 22.

Proof. (Proposition 22) Let $F \in C^1(\mathbb{R})$ and $f \in C_c(\mathbb{R}^d)$. For all $k \geq 2$, we set $(M_t^\infty)_{t \geq 0} = (M_{k,t}^{\Pi^c, \omega})_{t \geq 0}$. Let $l \geq 1$, let $0 \leq t_1 < \dots < t_l \leq t < t + s$, and let $h_1, \dots, h_l \in C_b(\mathcal{M}_\lambda)$. By Lemma 27,

$$\lim_{k \rightarrow +\infty} \mathbb{E} \left[\left(\Psi_{F,f}(M_{t+s}^k) - \Psi_{F,f}(M_t^k) - \int_t^{t+s} \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^k) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}^k) \right) \right] = 0.$$

Since $(M_{t+s}^k)_{k \geq 2}$ (resp. $(M_t^k)_{k \geq 2}$) converges vaguely to M_{t+s}^∞ (resp. M_t^∞) by Lemma 26, we can apply Lemma 24 and we obtain

$$\lim_{k \rightarrow +\infty} \Psi_{F,f}(M_{t+s}^k) = \Psi_{F,f}(M_{t+s}^\infty)$$

$$\text{and } \lim_{k \rightarrow +\infty} \Psi_{F,f}(M_t^k) = \Psi_{F,f}(M_t^\infty).$$

Moreover, by Lemma 25, for all $u \in [t, t + s]$,

$$\lim_{k \rightarrow +\infty} \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^k) = \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^\infty),$$

which is uniformly bounded in $M \in \mathcal{M}_\lambda$ by Lemma 41 in Section 5. Since for all $i \in \llbracket 1, l \rrbracket$, $h_i \in C_b(\mathcal{M}_\lambda)$,

$$\forall i \in \llbracket 1, l \rrbracket, \lim_{k \rightarrow +\infty} h_i(M_{t_i}^k) = h_i(M_{t_i}^\infty).$$

Therefore, by Eq.(7) and by Lemmas 40, 41 in Section 5, we can apply the dominated convergence theorem and obtain

$$\mathbb{E} \left[\left(\Psi_{F,f}(M_{t+s}^\infty) - \Psi_{F,f}(M_t^\infty) - \int_t^{t+s} \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^\infty) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}^\infty) \right) \right] = 0.$$

We conclude that

$$\left(\Psi_{F,f}(M_t^\infty) - \Psi_{F,f}(M_0^\infty) - \int_0^t \mathcal{L}_\mu^\infty \Psi_{F,f}(M_u^\infty) du \right)_{t \geq 0}$$

is indeed a martingale. □

3 The ∞ -parent ancestral process : definition and characterization

3.1 Definition and first properties

In order to show that the ∞ -parent ancestral process $(\Xi_t^\infty)_{t \geq 0}$ introduced in Definition 11 is well-defined, we start by observing that the only reproduction events affecting Ξ_t^∞ are the ones intersecting its border $\overline{\Xi_t^\infty} \setminus \overset{\circ}{\Xi_t^\infty}$. Therefore, it is sufficient to consider only the reproduction events affecting its border, or the ones affecting a well-chosen space containing it.

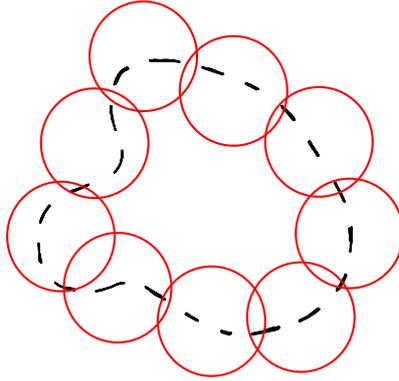
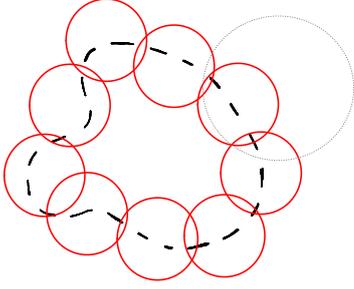


Figure 1: Initial state of the ∞ -parent ancestral process (dashed line), and a covering of its border by balls of radius $\tilde{\mathcal{R}}$.

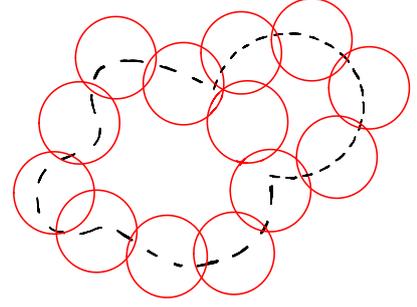
In order to control the rate at which the ∞ -parent ancestral process jumps, we start by taking $\tilde{\mathcal{R}} > 0$ satisfying some condition which will be introduced later, and we cover the border $\overline{\Xi_0^\infty} \setminus \overset{\circ}{\Xi_0^\infty}$ of Ξ_0^∞ with balls of radius $\tilde{\mathcal{R}}$ (see Figure 1). Then, informally, whenever a reproduction event overlaps what we will call the $\tilde{\mathcal{R}}$ -covering :

- if this reproduction event has a radius of at most $\tilde{\mathcal{R}}$, it is included in the ball of same center but of radius $\tilde{\mathcal{R}}$. We add this ball of radius $\tilde{\mathcal{R}}$ to the covering.
- Otherwise, we cover the border of the area of the reproduction event by balls of radius $\tilde{\mathcal{R}}$, and we add these balls to the covering.

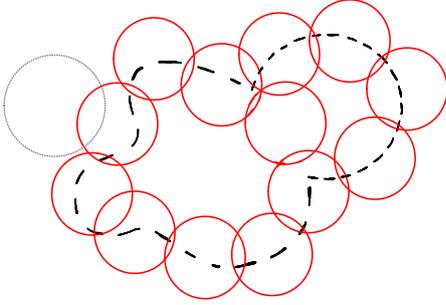
See Figure 2 for an illustration of this dynamics.



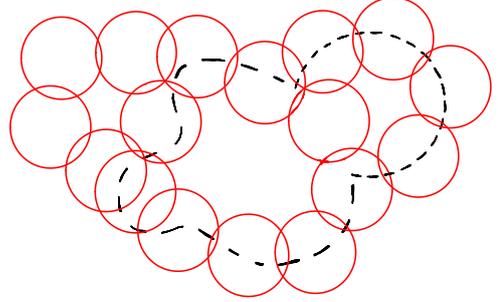
(a) Reproduction event (grey line) affecting the ∞ -parent ancestral process at time $t > 0$.



(b) The ∞ -parent ancestral process is updated, and a covering of the border of the reproduction event by balls of radius $\tilde{\mathcal{R}}$ is added to the $\tilde{\mathcal{R}}$ -covering process.



(c) Since the $\tilde{\mathcal{R}}$ -covering process is bigger than the border of the ∞ -parent ancestral process, it can be affected by reproduction events (grey line) which do not intersect the ∞ -parent ancestral process.



(d) Updated $\tilde{\mathcal{R}}$ -covering process after a reproduction event affecting it while not intersecting the ∞ -parent ancestral process.

Figure 2: Illustration of the dynamics of the ∞ -parent ancestral process (dashed line) and its associated $\tilde{\mathcal{R}}$ -covering process.

Note that since the covering contains the border $\overline{\Xi_t^\infty} \setminus \overset{\circ}{\Xi}_t^\infty$ of Ξ_t^∞ but is not equal to it, there are more reproduction events affecting the $\tilde{\mathcal{R}}$ -covering than reproduction events affecting Ξ_t^∞ .

Constructed this way, the $\tilde{\mathcal{R}}$ -covering contains only balls of radius $\tilde{\mathcal{R}}$, each one being overlapped by a reproduction event at rate

$$\int_0^\infty V_1(\tilde{\mathcal{R}} + r)^d \mu(dr).$$

Moreover, since the covering is constructed using the same Poisson point process as for $(\Xi_t^\infty)_{t \geq 0}$, at any time t the current state of the covering contains the border $\overline{\Xi_t^\infty} \setminus \overset{\circ}{\Xi}_t^\infty$ of Ξ_t^∞ . Since the rate at which $(\Xi_t^\infty)_{t \geq 0}$ jumps is bounded by the rate at which the covering we just constructed is updated, we can show that $(\Xi_t^\infty)_{t \geq 0}$ is well-defined by controlling the rate at which new balls are added to the $\tilde{\mathcal{R}}$ -covering.

Let us now define the border covering process we just introduced rigorously.

Definition 28 (Border covering process). *In the notation of Definition 11, let $\tilde{\mathcal{R}} > 0$ be such that μ satisfies Condition (5). Let $x_1, \dots, x_N \in \mathbb{R}^d$, $N \geq 1$ such that initially the border of Ξ^0 is entirely covered by the N balls of radius $\tilde{\mathcal{R}}$ $(B(x_i, \tilde{\mathcal{R}}))_{1 \leq i \leq N}$. Then, the $\tilde{\mathcal{R}}$ -covering process $(C_t)_{t \geq 0}$ associated to $(\Xi_t^\infty)_{t \geq 0}$ is constructed in the following way.*

First, we set $C_0 = \{x_1, \dots, x_N : 1 \leq i \leq N\}$. Then, for all $(t, x, \mathcal{R}) \in \tilde{\Pi}$, if $C_{t-} \cap B(x, \mathcal{R}) \neq \emptyset$, let $n \in \mathbb{N}^$ such that $(n-1)\tilde{\mathcal{R}} \leq \mathcal{R} \leq n\tilde{\mathcal{R}}$. We construct a covering of the border of $B(x, \mathcal{R})$ by at most $a_d \times n^{d-1}$ balls of radius $\tilde{\mathcal{R}}$, and C_t is obtained by adding the center of these balls to C_{t-} .*

The interest of the border covering process lies in the fact that, as we argued earlier, for all $t \geq 0$,

$$\overline{\Xi_t^\infty} \setminus \overset{\circ}{\Xi}_t^\infty \subseteq C_t.$$

Therefore, the jump rate of Ξ_t^∞ is bounded above by

$$\text{Card}(C_t) \times \int_0^\infty V_1(\tilde{\mathcal{R}} + r)^d \mu(dr).$$

Lemma 29. *In the notation of Definitions 11 and 28, $(\text{Card}(C_t))_{t \geq 0}$ is bounded from above by $(Y_t)_{t \geq 0}$ the number of particles in a branching process in which each particle branches independently of the others at rate*

$$\int_0^\infty V_1(\tilde{\mathcal{R}} + r)^d \mu(dr),$$

and in which at each branching event, the number of descendants is equal to $a_d \times n^{d-1} + 1$, $n \geq 1$ with probability

$$\frac{\int_{(n-1)\tilde{\mathcal{R}}}^{n\tilde{\mathcal{R}}} (\tilde{\mathcal{R}} + r)^d \mu(dr)}{\int_0^\infty (\tilde{\mathcal{R}} + r)^d \mu(dr)}.$$

Moreover, for all $t \geq 0$, $Y_t < +\infty$ p.s, and $\mathbb{E}[Y_t] < +\infty$.

Proof. How to construct the branching process $(Y_t)_{t \geq 0}$ from $(C_t)_{t \geq 0}$ is clear. The jump rates and transition probabilities come from the fact that for any point $x \in C_t$ and for all $n \geq 1$, the ball $\mathcal{B}(x, \tilde{\mathcal{R}})$ is affected by a reproduction event of radius $(n-1)\tilde{\mathcal{R}} \leq \mathcal{R} \leq n\tilde{\mathcal{R}}$ at rate

$$\int_{(n-1)\tilde{\mathcal{R}}}^{n\tilde{\mathcal{R}}} V_1(\tilde{\mathcal{R}} + \mathcal{R})^d \mu(d\mathcal{R}),$$

and such a reproduction event generates $a_d \times n^{d-1}$ new balls in the border covering process.

Then, if Φ is the probability generating function of the number of descendants,

$$\Phi'(1) = \sum_{n=1}^{+\infty} \left(\int_{(n-1)\tilde{\mathcal{R}}}^{n\tilde{\mathcal{R}}} V_1(\tilde{\mathcal{R}} + r)^d \mu(dr) \right) \times (a_d \times n^{d-1} + 1) < +\infty$$

since μ satisfies Condition (5). Therefore, by Theorem III.2.1 in [1], Y_t is finite for all $t \geq 0$ a.s, and $\mathbb{E}[Y_t] < +\infty$ for all $t \geq 0$. \square

We can then conclude that $(\Xi_t^\infty)_{t \geq 0}$ is well-defined using the fact that the jump rate of Ξ_t^∞ is bounded from above by

$$Y_t \times \int_0^\infty V_1 \times (\tilde{\mathcal{R}} + \mathcal{R})^d \mu(d\mathcal{R}) < +\infty \text{ p.s.}$$

3.2 Characterization via a martingale problem

The goal of this section is to introduce how to characterize the ∞ -parent ancestral process as the unique solution to a martingale problem.

In all that follows, let $F \in C_b^1(\mathbb{R})$ and $f \in \mathbb{B}(\mathbb{R}^d)$. We extend the definition of the function $\Phi_{F,f}$ to the space of measures $m(E) \in \mathcal{M}^{cf}$, setting

$$\begin{aligned} \Phi_{F,f}(m(E)) &:= F \left(\int_{\mathbb{R}^d} f(x) m(E) dx \right) \\ &= F \left(\int_E f(x) dx \right). \end{aligned}$$

Moreover, for all $E \in \mathcal{E}^{cf}$ and $\mathcal{R} > 0$, we set

$$S^{\mathcal{R}}(E) := \{x \in \mathbb{R}^d : \exists y \in E, \|x - y\| \leq \mathcal{R}\}.$$

Note that this definition is reminiscent of the definition of $S^{\mathcal{R}}(\Xi)$ with $\Xi \in \mathcal{M}_p(\mathbb{R}^d)$.

Let μ be a σ -finite measure on \mathbb{R}_+^* satisfying Condition (5). We define the operator \mathcal{G}_μ^∞ on functions of the form $\Phi_{F,f}$ the following way. For all $m(E) \in \mathcal{M}^{cf}$, we set

$$\mathcal{G}_\mu^\infty \Phi_{F,f}(m(E)) := \int_0^\infty \int_{S^{\mathcal{R}}(E)} F(\langle m(E \cup B(x, \mathcal{R})), f \rangle) - F(\langle m(E), f \rangle) dx \mu(d\mathcal{R}).$$

We show in Section 5 that this operator is well-defined, and give some properties that it satisfies. The ∞ -parent ancestral process is then solution to the following martingale problem.

Proposition 30. *Let μ be a σ -finite measure on $(0, +\infty)$ satisfying Condition (5). Let $\Xi^0 \in \mathcal{M}^{cf}$, and let $(\Xi_t^\infty)_{t \geq 0}$ be the ∞ -parent ancestral process associated to μ with initial condition Ξ^0 .*

Then, for all $F \in C_b^1(\mathbb{R})$ and for all measurable function $f : \mathbb{R}^d \rightarrow \{0, 1\}$, the process

$$\left(\Phi_{F,f}(\Xi_t^\infty) - \Phi_{F,f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_s^\infty) ds \right)_{t \geq 0}$$

is a martingale.

Proof. Let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by $(\Xi_t)_{t \geq 0}$, and let $0 \leq s \leq t$.

By Lemma 43 in Section 5,

$$\mathbf{E} \left[\Phi_{F,f}(\Xi_t^\infty) - \Phi_{F,f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right]$$

is well-defined, and

$$\begin{aligned} & \mathbf{E} \left[\Phi_{F,f}(\Xi_t^\infty) - \Phi_{F,f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right] \\ = & \mathbf{E} \left[\Phi_{F,f}(\Xi_t^\infty) - \Phi_{F,f}(\Xi_s^\infty) - \int_s^t \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right] \\ & + \mathbf{E} \left[\Phi_{F,f}(\Xi_s^\infty) - \Phi_{F,f}(\Xi_0^\infty) - \int_0^s \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right] \\ = & \mathbf{E} \left[\Phi_{F,f}(\Xi_t^\infty) \middle| \Xi_s^\infty \right] - \Phi_{F,f}(\Xi_s^\infty) - \mathbf{E} \left[\int_s^t \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_u^\infty) du \middle| \Xi_s^\infty \right] + \Phi_{F,f}(\Xi_s^\infty) - \Phi_{F,f}(\Xi_0^\infty) \\ & - \int_0^s \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_u^\infty) du \end{aligned}$$

since $(\Xi_u^\infty)_{u \geq 0}$ is Markovian. Let $(\tilde{\Xi}_u)_{u \geq 0}$ be another ∞ -parent ancestral process associated to μ , this time with initial condition Ξ_s^∞ . Then,

$$\begin{aligned} & \mathbf{E} \left[\Phi_{F,f}(\Xi_t^\infty) - \Phi_{F,f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right] \\ = & \mathbf{E} \left[\Phi_{F,f}(\Xi_t^\infty) \middle| \Xi_s^\infty \right] - \Phi_{F,f}(\Xi_s^\infty) - \mathbf{E} \left[\int_0^{t-s} \mathcal{G}_\mu^\infty \Phi_{F,f}(\tilde{\Xi}_u) du \middle| \Xi_s^\infty \right] + \Phi_{F,f}(\Xi_s^\infty) - \Phi_{F,f}(\Xi_0^\infty) \\ & - \int_0^s \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_u^\infty) du. \end{aligned}$$

By Lemmas 43 and 44 in Section 5,

$$\begin{aligned} \mathbf{E} \left[\int_0^{t-s} \mathcal{G}_\mu^\infty \Phi_{F,f}(\tilde{\Xi}_u) du \middle| \Xi_s^\infty \right] &= \int_0^{t-s} \mathbf{E} \left[\mathcal{G}_\mu^\infty \Phi_{F,f}(\tilde{\Xi}_u) \middle| \Xi_s^\infty \right] du \\ &= \int_0^{t-s} \frac{d}{dv} \mathbf{E} \left[\Phi_{F,f}(\tilde{\Xi}_v) \middle| \Xi_s^\infty \right] \Big|_{v=u} du \\ &= \mathbf{E} \left[\Phi_{F,f}(\tilde{\Xi}_{t-s}) \middle| \Xi_s^\infty \right] - \mathbf{E} \left[\Phi_{F,f}(\tilde{\Xi}_0) \middle| \Xi_s^\infty \right] \\ &= \mathbf{E} \left[\Phi_{F,f}(\Xi_t^\infty) \middle| \Xi_s^\infty \right] - \Phi_{F,f}(\Xi_s^\infty). \end{aligned}$$

Therefore, we obtain

$$\mathbf{E} \left[\Phi_{F,f}(\Xi_t^\infty) - \Phi_{F,f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right] = \Phi_{F,f}(\Xi_s^\infty) - \Phi_{F,f}(\Xi_0^\infty) - \int_0^s \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_u^\infty) du$$

and we can conclude. \square

4 Uniqueness of the solution to the martingale problem characterizing the ∞ -parent SLFV

In order to show the uniqueness of the solution to the martingale problem characterizing the ∞ -parent SLFV, we first need to extend the set of functions over which the operators \mathcal{L}_μ^∞ and \mathcal{G}_μ^∞ are defined.

4.1 Extended martingale problem for the ∞ -parent SLFV

For all $\alpha \in \mathbb{R}$, we set $F^\alpha : x \rightarrow \delta_\alpha(x)$, and for all $E \in \mathcal{E}^{cf}$, we set $f^E : x \rightarrow \mathbf{1}_{x \in E}$. Let μ a σ -finite measure on $(0, +\infty)$ satisfying Condition (4), and let $M^0 \in \mathcal{M}_\lambda$. The goal of this section is to prove the following result.

Lemma 31. *Let M be a solution to the martingale problem associated to $(\mathcal{L}_\mu^\infty, \delta_{M^0})$. Then, for all $E \in \mathcal{E}^{cf}$,*

$$\left(\Psi_{F^{\text{Vol}(E)}, f^E}(M_t) - \Psi_{F^{\text{Vol}(E)}, f^E}(M_0) - \int_0^t \mathcal{L}^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M_s) ds \right)_{t \geq 0}$$

is a martingale, where $\Psi_{F^{\text{Vol}(E)}, f^E} : M \in \mathcal{M}_\lambda \rightarrow \Psi_{F^{\text{Vol}(E)}, f^E}(M)$ is the function defined by

$$\begin{aligned} \forall M \in \mathcal{M}_\lambda, \Psi_{F^{\text{Vol}(E)}, f^E}(M) &:= F^{\text{Vol}(E)}(\langle \omega_M, f \rangle) \\ &= \delta_{\text{Vol}(E)}(\langle \omega_M, f \rangle) \\ &= \delta_0(\text{Vol}(E) - \langle \omega_M, f \rangle). \end{aligned}$$

This lemma is a direct consequence of the following lemma.

Lemma 32. *Let M be a solution to the martingale problem associated to $(\mathcal{L}_\mu^\infty, \delta_{M^0})$. Then, for all $E \in \mathcal{E}^{cf}$, for all $l \geq 1$, for all $0 \leq t_1 < \dots < t_l \leq t < t + s$, for all $h_1, \dots, h_l \in C_b(\mathcal{M}_\lambda)$,*

$$\mathbf{E} \left[\left(\Psi_{F^{\text{Vol}(E)}, f^E}(M_{t+s}) - \Psi_{F^{\text{Vol}(E)}, f^E}(M_t) - \int_t^{t+s} \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M_u) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}) \right) \right] = 0$$

Let $E \in \mathcal{E}^{cf}$. Let $(F_n^{\text{Vol}(E)})_{n \in \mathbb{N}} \in C^1(\mathbb{R})$ and $(f_n^E)_{n \in \mathbb{N}} \in C_c(\mathbb{R}^d)$ be two sequences satisfying the

following conditions.

- (A) $F_n^{\text{Vol}(E)} \xrightarrow[n \rightarrow +\infty]{} F^{\text{Vol}(E)}$ pointwise and in L^1 ,
- (B) $f_n^E \xrightarrow[n \rightarrow +\infty]{} f^E$ pointwise and in L^1 ,
- (C) $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}, 0 \leq F_n^{\text{Vol}(E)}(x) \leq 1$ and $F_n^{\text{Vol}(E)}(\text{Vol}(E)) = 1$,
- (D) $\forall n \in \mathbb{N}, \forall x \in \mathbb{R}^d, 0 \leq f_n^E(x) \leq 1$ and $\forall z \in E, f_n^E(z) = 1$,
- (E) $\forall n \in \mathbb{N}, F_n^{\text{Vol}(E)}$ is increasing over $(-\infty, \text{Vol}(E)]$ and decreasing over $[\text{Vol}(E), +\infty)$,
- (F) $\forall n \in \mathbb{N}, \text{Vol}(\text{Supp}(f_n^E) \setminus E) \leq n^{-1}$, and $\text{Supp}(f_{n+1}^E) \subseteq \text{Supp}(f_n^E)$
- (G) $\forall n \in \mathbb{N}, F_n^{\text{Vol}(E)}(\text{Vol}(E) + n^{-1}) \geq 1 - n^{-1}$ and $F_n^{\text{Vol}(E)}(\text{Vol}(E) - n^{-1}) \geq 1 - n^{-1}$.

First, we observe that since $F^{\text{Vol}(E)}$ and $(F_n^{\text{Vol}(E)})_{n \in \mathbb{N}^*}$ are bounded by one (by Hypothesis (C)), for all $M \in \mathcal{M}_\lambda$ and $n \in \mathbb{N}^*$

$$\left| \Psi_{F^{\text{Vol}(E)}, f^E}(M) \right| \leq 1 \quad (8)$$

$$\left| \Psi_{F_n^{\text{Vol}(E)}, f^E}(M) \right| \leq 1 \quad (9)$$

$$\left| \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M) \right| \leq 1 \quad (10)$$

Moreover, there exists $C_E > 0$ such that for all $\mathcal{R} > 0$,

$$\text{Vol}(S^\mathcal{R}(E)) \leq C^E \times (\mathcal{R}^d \vee 1), \quad (11)$$

where we recall that $S^\mathcal{R}(E)$ is defined by

$$S^\mathcal{R}(E) := \{x \in \mathbb{R}^d : \exists y \in E, \|x - y\| \leq \mathcal{R}\}.$$

Therefore, we have the following lemma.

Lemma 33. *There exists $C_2^E > 0$ such that for all $M \in \mathcal{M}_\lambda$ and $n \in \mathbb{N}^*$,*

$$\left| \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M) \right| \leq C_2^E$$

$$\left| \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M) \right| \leq C_2^E$$

$$\left| \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M) \right| \leq C_2^E.$$

Proof. Let $M \in \mathcal{M}_\lambda$.

$$\begin{aligned}
\left| \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M) \right| &\leq \int_0^\infty \int_{S^{\mathcal{R}(E)}} \left| 1 - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_M(z)) dz \right) \right| \\
&\quad \times \left| F^{\text{Vol}(E)} \left(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f \rangle \right) - F^{\text{Vol}(E)} \left(\langle \omega_M, f \rangle \right) \right| dx \mu(d\mathcal{R}) \\
&\leq \int_0^\infty \int_{S^{\mathcal{R}(E)}} 2 dx \mu(d\mathcal{R}) \\
&\leq 2 \times \int_0^\infty C^E \times (\mathcal{R}^d \vee 1) \mu(d\mathcal{R}) \\
&< +\infty
\end{aligned}$$

since μ satisfies Condition (4). Here we passed from line 1 to line 2 using the fact that $F^{\text{Vol}(E)}$ is bounded by 1, and from line 2 to line 3 using Eq. (11).

Setting $C_2^E = 2C^E \times \int_0^\infty (\mathcal{R}^d \vee 1) \mu(d\mathcal{R})$, we obtain

$$\left| \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M) \right| \leq C_2^E.$$

Similarly, we can show that for all $n \in \mathbb{N}^*$,

$$\begin{aligned}
\left| \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M) \right| &\leq C_2^E \\
\text{and } \left| \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M) \right| &\leq C_2^E.
\end{aligned}$$

□

This lemma, along with Eqs. (8, 9, 10, 11), will allow us to use the dominated convergence theorem in the proof of Lemma 32.

Since by Hypothesis (A) the sequence $(F_n^{\text{Vol}(E)})_{n \in \mathbb{N}^*}$ converges pointwise to $F^{\text{Vol}(E)}$, we obtain that

$$\forall M \in \mathcal{M}_\lambda, \Psi_{F_n^{\text{Vol}(E)}, f^E}(M) \xrightarrow{n \rightarrow +\infty} \Psi_{F^{\text{Vol}(E)}, f^E}(M). \quad (12)$$

We want to show a similar result regarding $(\Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M))_{n \in \mathbb{N}^*}$.

Lemma 34. *For all $M \in \mathcal{M}_\lambda$,*

$$\Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M) - \Psi_{F_n^{\text{Vol}(E)}, f^E}(M) \xrightarrow{n \rightarrow +\infty} 0.$$

Proof. Let $M \in \mathcal{M}_\lambda$. We distinguish two cases.

Case 1 : $\int_E \omega_M(z) dz = \text{Vol}(E)$.

Let $n \in \mathbb{N}^*$. Then, since by Hypothesis (D) we have $E \subseteq \text{Supp}(f_n^E)$,

$$\begin{aligned} \text{Vol}(E) &\leq \langle \omega_M, f_n^E \rangle \leq \text{Vol}(E) + \int_{\text{Supp}(f_n^E) \setminus E} f_n^E(z) \omega_M(z) dz \\ &\leq \text{Vol}(E) + \text{Vol}(\text{Supp}(f_n^E) \setminus E) \\ &\leq \text{Vol}(E) + \frac{1}{n} \end{aligned}$$

using Hypotheses (D) and (F). Therefore, since $F_n^{\text{Vol}(E)}$ is decreasing over $[\text{Vol}(E), +\infty)$ by Hypothesis (E),

$$F_n^{\text{Vol}(E)}(\text{Vol}(E)) \geq \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M) \geq F_n^{\text{Vol}(E)}(\text{Vol}(E) + \frac{1}{n})$$

or, in other words,

$$1 \geq \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M) \geq 1 - \frac{1}{n}$$

by Hypothesis (C) and (G). Moreover,

$$\begin{aligned} \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M) &= F_n^{\text{Vol}(E)} \left(\int_E \omega_M(z) dz \right) \\ &= F_n^{\text{Vol}(E)}(\text{Vol}(E)) \\ &= 1 \end{aligned}$$

by Hypothesis (C), and we can conclude.

Case 2: $\int_E \omega_M(z) dz < \text{Vol}(E)$.

Let $N \in \mathbb{N}^*$ such that $N^{-1} \leq 2^{-1} \times (\text{Vol}(E) - \int_E \omega_M(z) dz)$. Then, for all $n \geq N$, using Hypotheses (D) and (F),

$$\begin{aligned} 0 \leq \langle \omega_M, f_n^E \rangle &\leq \int_E \omega_M(z) dz + \int_{\text{Supp}(f_n^E) \setminus E} \omega_M(z) dz \\ &\leq \int_E \omega_M(z) dz + \text{Vol}(\text{Supp}(f_n^E) \setminus E) \\ &\leq \int_E \omega_M(z) dz + \frac{1}{n} \\ &\leq \int_E \omega_M(z) dz + \frac{1}{N} \\ &\leq \frac{1}{2} \times \int_E \omega_M(z) dz + \frac{1}{2} \times \text{Vol}(E) \\ &< \text{Vol}(E), \end{aligned}$$

so by Hypothesis (E),

$$\Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M) \xrightarrow{n \rightarrow +\infty} 0.$$

Moreover, since $\langle \omega_M, f^E \rangle < \text{Vol}(E)$, again by Hypothesis (E),

$$\Psi_{F_n^{\text{Vol}(E)}, f^E}(M) \xrightarrow{n \rightarrow +\infty} 0,$$

and we can conclude. □

We now prove a similar result involving \mathcal{L}_μ^∞ .

Lemma 35. *For all $M \in \mathcal{M}_\lambda$,*

$$\begin{aligned} \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M) - \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M) &\xrightarrow{n \rightarrow +\infty} 0, \\ \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M) - \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M) &\xrightarrow{n \rightarrow +\infty} 0. \end{aligned}$$

Proof. Let $M \in \mathcal{M}_\lambda$, and let $n \in \mathbb{N}^*$. We have

$$\begin{aligned} &\mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M) - \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M) \\ &= \int_0^\infty \int_{S^{\mathcal{R}(E)}} \left(1 - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_M(z)) dz \right) \right) \\ &\quad \times \left(F_n^{\text{Vol}(E)} \left(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f^E \rangle \right) - F_n^{\text{Vol}(E)} \left(\langle \omega_M, f^E \rangle \right) \right) dx \mu(d\mathcal{R}) \\ &- \int_0^\infty \int_{S^{\mathcal{R}(E)}} \left(1 - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_M(z)) dz \right) \right) \\ &\quad \times \left(F^{\text{Vol}(E)} \left(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f^E \rangle \right) - F^{\text{Vol}(E)} \left(\langle \omega_M, f^E \rangle \right) \right) dx \mu(d\mathcal{R}) \\ &= \int_0^\infty \int_{S^{\mathcal{R}(E)}} \left(1 - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_M(z)) dz \right) \right) \\ &\quad \times \left(F_n^{\text{Vol}(E)} \left(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f^E \rangle \right) - F^{\text{Vol}(E)} \left(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f^E \rangle \right) \right) dx \mu(d\mathcal{R}) \\ &+ \int_0^\infty \int_{S^{\mathcal{R}(E)}} \left(1 - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_M(z)) dz \right) \right) \\ &\quad \times \left(F^{\text{Vol}(E)} \left(\langle \omega_M, f^E \rangle \right) - F_n^{\text{Vol}(E)} \left(\langle \omega_M, f^E \rangle \right) \right) dx \mu(d\mathcal{R}). \end{aligned}$$

By Eq. (12), for all $x \in \mathbb{R}^d$ and $\mathcal{R} > 0$,

$$F_n^{\text{Vol}(E)} \left(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f^E \rangle \right) - F^{\text{Vol}(E)} \left(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f^E \rangle \right) \xrightarrow{n \rightarrow +\infty} 0$$

and

$$F^{\text{Vol}(E)} \left(\langle \omega_M, f^E \rangle \right) - F_n^{\text{Vol}(E)} \left(\langle \omega_M, f^E \rangle \right) \xrightarrow{n \rightarrow +\infty} 0.$$

Therefore, using the bounds from the proof of Lemma 33, we can apply the dominated convergence theorem and obtain

$$\mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M) - \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M) \xrightarrow{n \rightarrow +\infty} 0.$$

We can similarly show that

$$\mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M) - \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M) \xrightarrow{n \rightarrow +\infty} 0$$

using Lemma 34 instead of Eq. (12). \square

We can now prove Lemma 32, from which we will directly deduce Lemma 31.

Proof. (Lemma 32) Let $l \geq 1$, let $0 \leq t_1 < \dots < t_l \leq t < t + s$ and let $h_1, \dots, h_l \in C_b(\mathcal{M}_\lambda)$. Let $n \in \mathbb{N}^*$.

Then,

$$\begin{aligned} \Psi_{F^{\text{Vol}(E)}, f^E}(M_{t+s}) &= \Psi_{F^{\text{Vol}(E)}, f^E}(M_{t+s}) - \Psi_{F_n^{\text{Vol}(E)}, f^E}(M_{t+s}) \\ &\quad + \Psi_{F_n^{\text{Vol}(E)}, f^E}(M_{t+s}) - \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_{t+s}) + \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_{t+s}) \\ \Psi_{F^{\text{Vol}(E)}, f^E}(M_t) &= \Psi_{F^{\text{Vol}(E)}, f^E}(M_t) - \Psi_{F_n^{\text{Vol}(E)}, f^E}(M_t) \\ &\quad + \Psi_{F_n^{\text{Vol}(E)}, f^E}(M_t) - \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_t) + \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_t), \end{aligned}$$

and for all $u \in [t, t + s]$,

$$\begin{aligned} \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M_u) &= \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M_u) - \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M_u) \\ &\quad + \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M_u) - \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_u) + \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_u). \end{aligned}$$

Since M is a solution of the martingale problem associated to $(\mathcal{L}_\mu^\infty, \delta_{M^0})$, for all $n \in \mathbb{N}^*$,

$$\mathbb{E} \left[\left(\Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_{t+s}) - \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_t) - \int_t^{t+s} \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_u) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}) \right) \right] = 0.$$

Therefore, since all the equations written above are true for all $n \in \mathbb{N}^*$,

$$\begin{aligned} &\mathbb{E} \left[\left(\Psi_{F^{\text{Vol}(E)}, f^E}(M_{t+s}) - \Psi_{F^{\text{Vol}(E)}, f^E}(M_t) - \int_t^{t+s} \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M_u) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}) \right) \right] \\ &= \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(\Psi_{F^{\text{Vol}(E)}, f^E}(M_{t+s}) - \Psi_{F_n^{\text{Vol}(E)}, f^E}(M_{t+s}) \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}) \right) \right] \\ &\quad + \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(\Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_{t+s}) - \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_{t+s}) \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}) \right) \right] \end{aligned}$$

$$\begin{aligned}
& - \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(\Psi_{F^{\text{Vol}(E)}, f^E}(M_t) - \Psi_{F_n^{\text{Vol}(E)}, f^E}(M_t) \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}) \right) \right] \\
& - \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(\Psi_{F_n^{\text{Vol}(E)}, f^E}(M_t) - \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_t) \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}) \right) \right] \\
& - \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(\int_t^{t+s} \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E)}, f^E}(M_u) - \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M_u) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}) \right) \right] \\
& - \lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(\int_t^{t+s} \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f^E}(M_u) - \mathcal{L}_\mu^\infty \Psi_{F_n^{\text{Vol}(E)}, f_n^E}(M_u) du \right) \times \left(\prod_{i=1}^l h_i(M_{t_i}) \right) \right]
\end{aligned}$$

under the condition that all these limits exist.

By Eq. (12), Lemma 34 and Lemma 35, all the terms inside the expectations converge to 0 when $n \rightarrow +\infty$. Using the bounds given by Eq. (8), (9), (10) and Lemma 33, we can apply the dominated convergence theorem and obtain the desired result. \square

4.2 Extended martingale problem for the ∞ -parent ancestral process

In this section, we prove the following result.

Lemma 36. *Let μ be a σ -finite measure on $(0, +\infty)$ satisfying Condition (5). Let $\Xi^0 \in \mathcal{M}^{cf}$, and let $(\Xi_t^\infty)_{t \geq 0} = (m(E_t))_{t \geq 0}$ be the ∞ -parent ancestral process associated to μ with initial condition Ξ^0 .*

Then, for all measurable function $f : \mathbb{R}^d \rightarrow \{0, 1\}$,

$$\left(\Phi_{\delta_0, f}(\Xi_t^\infty) - \Phi_{\delta_0, f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_s^\infty) ds \right)_{t \geq 0}$$

is a martingale, where $\Phi_{\delta_0, f} : M \in \mathcal{M}^{cf} \rightarrow \Phi_{\delta_0, f}(M)$ is the function defined by

$$\forall m(E) \in \mathcal{M}^{cf}, \Phi_{\delta_0, f}(m(E)) := \delta_0 \left(\int_E f(x) dx \right).$$

Proof. Let $(\mathcal{F}_t)_{t \geq 0}$ be the filtration generated by $(\Xi_t^\infty)_{t \geq 0}$. Let $(F_n)_{n \in \mathbb{N}^*} \in C_b^1(\mathbb{R})$ be a sequence of functions converging pointwise to δ_0 such that

- (A) $\forall n \in \mathbb{N}^*$, F_n is increasing on \mathbb{R}_- and decreasing on \mathbb{R}_+ ,
- (B) $\forall n \in \mathbb{N}^*$, $F_n(0) = 1$ and $\forall x \in \mathbb{R}$, $0 \leq F_n(x) \leq 1$,
- (C) $\forall n \in \mathbb{N}^*$, $\text{Supp}(F_n) \subseteq [-n^{-3}, n^{-3}]$.

The interest of this sequence lies in the fact that for all $n \in \mathbb{N}^*$ and for all measurable function

$f : \mathbb{R}^d \rightarrow \{0, 1\}$,

$$\left(\Phi_{F_n, f}(\Xi_t^\infty) - \Phi_{F_n, f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_s^\infty) ds \right)_{t \geq 0}$$

is a martingale.

Let $f : \mathbb{R}^d \rightarrow \{0, 1\}$ be a measurable function, and let $0 \leq s \leq t$. $\Phi_{\delta_0, f}$ is bounded by 1, and by Hypothesis (B), the functions $(\Phi_{F_n, f})_{n \in \mathbb{N}^*}$ are bounded by 1 as well. Moreover, since $u \rightarrow \text{Vol}(\Xi_u^\infty)$ is increasing, and as there exists $C_t > 0$ such that for all $\mathcal{R} > 0$,

$$\text{Vol}(S^{\mathcal{R}}(\Xi_t^\infty)) \leq C_t \times (\mathcal{R}^d \vee 1),$$

we can deduce that for all $u \in [0, t]$ and for all $n \in \mathbb{N}^*$, by Hypothesis (B),

$$\begin{aligned} \left| \mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_s^\infty) \right| &\leq \int_0^\infty 2 \times \text{Vol}(S^{\mathcal{R}}(\Xi_s^\infty)) \mu(d\mathcal{R}) \\ &\leq \int_0^\infty 2 \times \text{Vol}(S^{\mathcal{R}}(\Xi_t)) \mu(d\mathcal{R}) \\ &\leq 2C_t \times \int_0^\infty (\mathcal{R}^d \vee 1) \mu(d\mathcal{R}). \end{aligned}$$

Similarly, we obtain that

$$\left| \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_s^\infty) \right| \leq 2C_t \times \int_0^\infty (\mathcal{R}^d \vee 1) \mu(d\mathcal{R}).$$

Since μ satisfies Condition (4), both quantities are finite. Therefore, by Fubini's theorem, for all $n \in \mathbb{N}^*$,

$$\begin{aligned} &\mathbf{E} \left[\Phi_{\delta_0, f}(\Xi_t^\infty) - \Phi_{\delta_0, f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right] \\ &= \mathbf{E} \left[\Phi_{\delta_0, f}(\Xi_t^\infty) - \Phi_{F_n, f}(\Xi_t^\infty) \middle| \mathcal{F}_s \right] + \mathbf{E} \left[\Phi_{F_n, f}(\Xi_0^\infty) - \Phi_{\delta_0, f}(\Xi_0^\infty) \middle| \mathcal{F}_s \right] \\ &\quad + \int_0^t \mathbf{E} \left[\mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_u^\infty) - \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_u^\infty) \middle| \mathcal{F}_s \right] du \\ &\quad + \mathbf{E} \left[\Phi_{F_n, f}(\Xi_t^\infty) - \Phi_{F_n, f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right]. \end{aligned}$$

Using Proposition 30, we obtain that

$$\begin{aligned} &\mathbf{E} \left[\Phi_{\delta_0, f}(\Xi_t^\infty) - \Phi_{\delta_0, f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right] \\ &= \Phi_{F_n, f}(\Xi_s^\infty) - \Phi_{F_n, f}(\Xi_0^\infty) - \int_0^s \mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_u^\infty) du \\ &\quad + \mathbf{E} \left[\Phi_{\delta_0, f}(\Xi_t^\infty) - \Phi_{F_n, f}(\Xi_t^\infty) \middle| \mathcal{F}_s \right] + \mathbf{E} \left[\Phi_{F_n, f}(\Xi_0^\infty) - \Phi_{\delta_0, f}(\Xi_0^\infty) \middle| \mathcal{F}_s \right] \end{aligned}$$

$$+ \int_0^t \mathbf{E} \left[\mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_u^\infty) - \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_u^\infty) \middle| \mathcal{F}_s \right] du.$$

Since this is true for all $n \in \mathbb{N}^*$,

$$\begin{aligned} & \mathbf{E} \left[\Phi_{\delta_0, f}(\Xi_t^\infty) - \Phi_{\delta_0, f}(\Xi_0^\infty) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right] \\ &= \lim_{n \rightarrow +\infty} \Phi_{F_n, f}(\Xi_s^\infty) - \lim_{n \rightarrow +\infty} \Phi_{F_n, f}(\Xi_0^\infty) - \lim_{n \rightarrow +\infty} \int_0^s \mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_u^\infty) du \\ & \quad + \mathbf{E} [\Phi_{\delta_0, f}(\Xi_t^\infty) - \Phi_{F_n, f}(\Xi_t^\infty) | \mathcal{F}_s] + \mathbf{E} [\Phi_{F_n, f}(\Xi_0^\infty) - \Phi_{\delta_0, f}(\Xi_0^\infty) | \mathcal{F}_s] \\ & \quad + \int_0^t \mathbf{E} \left[\mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_u^\infty) - \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_u^\infty) \middle| \mathcal{F}_s \right] du, \end{aligned}$$

under the condition that all these limits exist.

First, since $\Phi_{F_n, f}$ converges pointwise to $\Phi_{\delta_0, f}$,

$$\lim_{n \rightarrow +\infty} \Phi_{F_n, f}(\Xi_s^\infty) - \Phi_{F_n, f}(\Xi_0^\infty) = \Phi_{\delta_0, f}(\Xi_s^\infty) - \Phi_{\delta_0, f}(\Xi_0^\infty),$$

and by the dominated convergence theorem,

$$\lim_{n \rightarrow +\infty} \mathbf{E} [\Phi_{\delta_0, f}(\Xi_t^\infty) - \Phi_{F_n, f}(\Xi_t^\infty) | \mathcal{F}_s] = \lim_{n \rightarrow +\infty} \mathbf{E} [\Phi_{\delta_0, f}(\Xi_0^\infty) - \Phi_{F_n, f}(\Xi_0^\infty) | \mathcal{F}_s] = 0.$$

Moreover, since for all $n \in \mathbb{N}^*$,

$$\int_0^s |\mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_u^\infty)| du \leq 2s \times C_t \times \int_0^\infty (\mathcal{R}^d \vee 1) \mu(d\mathcal{R}),$$

again by the dominated convergence theorem, we obtain

$$\lim_{n \rightarrow +\infty} \int_0^s \mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_u^\infty) du = \int_0^s \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_u^\infty) du.$$

Then, let $n \in \mathbb{N}^*$. Recalling that Ξ_u^∞ is also denoted $m(E_u)$,

$$\begin{aligned} & \int_0^t \mathbf{E} \left[\mathcal{G}_\mu^\infty(\Xi_u^\infty) - \mathcal{G}_\mu^\infty(\Xi_u^\infty) \middle| \mathcal{F}_s \right] du \\ &= \int_0^t \mathbf{E} \left[\int_0^\infty \int_{S^{\mathcal{R}(E)}} (F_n(\langle m(E_u \cup B(x, \mathcal{R}), f \rangle) \rangle) - \delta_0(\langle m(E_u \cup B(x, \mathcal{R}), f \rangle) \rangle) dx \mu(d\mathcal{R}) \middle| \mathcal{F}_s \right] du \\ & \quad + \int_0^t \mathbf{E} \left[\int_0^\infty \int_{S^{\mathcal{R}(E)}} (\delta_0(\langle m(E_u), f \rangle) \rangle) - F_n(\langle m(E_u), f \rangle) \rangle) dx \mu(d\mathcal{R}) \middle| \mathcal{F}_s \right] du. \end{aligned}$$

Since for all $x \in \mathbb{R}^d$, $u \in [0, t]$ and $\mathcal{R} > 0$,

$$\lim_{n \rightarrow +\infty} F_n (\langle m(E_u \cup B(x, \mathcal{R})), f \rangle) = \delta_0 (\langle m(E_u \cup B(x, \mathcal{R})), f \rangle)$$

and

$$\lim_{n \rightarrow +\infty} F_n (\langle m(E_u), f \rangle) = \delta_0 (\langle m(E_u), f \rangle),$$

using the dominated convergence theorem, we obtain that

$$\lim_{n \rightarrow +\infty} \int_0^t \mathbf{E} \left[\mathcal{G}_\mu^\infty \Phi_{F_n, f}(\Xi_u^\infty) - \Phi_{\delta_0, f}(\Xi_u^\infty) \middle| \mathcal{F}_s \right] = 0,$$

and we can conclude that

$$\begin{aligned} & \mathbf{E} \left[\Phi_{\delta_0, f}(\Xi_t^\infty) - \Phi_{\delta_0, f}(\Xi_0^\infty(\Xi_0^\infty)) - \int_0^t \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_u^\infty) du \middle| \mathcal{F}_s \right] \\ &= \Phi_{\delta_0, f}(\Xi_s^\infty) - \Phi_{\delta_0, f}(\Xi_0^\infty(\Xi_0^\infty)) - \int_0^s \mathcal{G}_\mu^\infty \Phi_{\delta_0, f}(\Xi_u^\infty) du. \end{aligned}$$

□

4.3 Uniqueness of the solution to the martingale problem characterizing the ∞ -parent SLFV

We now use the extended martingale problem in order to prove Proposition 13, i.e, that the ∞ -parent ancestral process is the dual of the ∞ -parent SLFV.

Proof. (Proposition 13) For all $t \geq 0$, let ω_t be a density of M_t^∞ . Let $(E_t)_{t \geq 0}$ such that $(\Xi_t^\infty)_{t \geq 0} = (m(E_t))_{t \geq 0}$.

For all $s, t \geq 0$, we set :

$$F(s, t) = \mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} [D(M_s^\infty, \Xi_t^\infty)] \right].$$

Then,

$$F(s, t) = \mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} [\Phi_{\delta_0, 1-\omega_s}(\Xi_t^\infty)] \right]$$

and by Lemma 36,

$$F(s, t) = \mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} [\Phi_{\delta_0, 1-\omega_s}(\Xi_0)] \right] + \mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} \left[\int_0^t \mathcal{G}_\mu^\infty \Phi_{\delta_0, 1-\omega_s}(\Xi_u^\infty) du \right] \right].$$

By Fubini's theorem, we obtain

$$F(s, t) = F(s, 0) + \int_0^t \mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} \left[\mathcal{G}_\mu^\infty \Phi_{\delta_0, 1-\omega_s}(\Xi_u^\infty) \right] \right] du.$$

Then,

$$\begin{aligned} F(s, t) &= \mathbf{E}_{m(E^0)} \left[\mathbb{E}_{M^0} \left[\tilde{D}(M_s^\infty, \Xi_t^\infty) \right] \right] \\ &= \mathbf{E}_{m(E^0)} \left[\mathbb{E}_{M^0} \left[\Psi_{F^{\text{Vol}(E_t)}, fE_t}(M_s^\infty) \right] \right], \end{aligned}$$

and by Lemma 31,

$$\begin{aligned} F(s, t) &= \mathbf{E}_{m(E^0)} \left[\mathbb{E}_{M^0} \left[\Psi_{F^{\text{Vol}(E_t)}, fE_t}(M_0^\infty) \right] \right] + \mathbf{E}_{m(E^0)} \left[\mathbb{E}_{M^0} \left[\int_0^t \Psi_{\delta_{F^{\text{Vol}(E_t)}, fE_t}}(M_u^\infty) du \right] \right] \\ &= F(0, t) + \mathbf{E}_{m(E^0)} \left[\mathbb{E}_{M^0} \left[\int_0^t \Psi_{F^{\text{Vol}(E_t)}, fE_t}(M_u^\infty) du \right] \right]. \end{aligned}$$

Again by Fubini's theorem, we obtain

$$F(s, t) = F(0, t) + \int_0^t \mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} \left[\mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E_t)}, fE_t}(M_u^\infty) \right] \right] du.$$

Combining both expressions for $F(s, t)$, by Lemma 4.4.10 in [20], we obtain :

$$\begin{aligned} &F(t, 0) - F(0, t) \\ &= \int_0^t \left(\mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} \left[\mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E_{t-u}), fE_{t-u}}}(M_u^\infty) \right] \right] - \mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} \left[\mathcal{G}_\mu^\infty \Phi_{\delta_0, 1-\omega_u}(\Xi_{t-u}^\infty) \right] \right] \right) du. \end{aligned}$$

Let $u \in [0, t]$. We have

$$\begin{aligned} &\mathcal{G}_\mu^\infty \Phi_{\delta_0, 1-\omega_u}(\Xi_{t-u}^\infty) \\ &= \int_0^\infty \int_{S^{\mathcal{R}}(E_{t-u})} \left(\delta_0 \left(\int_{E_{t-u} \cup B(x, \mathcal{R})} (1 - \omega_u(z)) dz \right) - \delta_0 \left(\int_{E_{t-u}} (1 - \omega_u(z)) dz \right) \right) dx \mu(d\mathcal{R}) \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E_{t-u}), fE_{t-u}}}(M_u^\infty) &= \int_0^\infty \int_{S^{\mathcal{R}}(E_{t-u})} \left(1 - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_u(z)) dz \right) \right) \\ &\quad \times \left[\delta_0 \left(\text{Vol}(E_{t-u}) - \langle \Theta_{x, \mathcal{R}}^-(\omega_u), \mathbf{1}_{E_{t-u}} \rangle \right) - \delta_0 \left(\text{Vol}(E_{t-u}) - \langle \omega_u, \mathbf{1}_{E_{t-u}} \rangle \right) \right] dx \\ &= \int_0^\infty \int_{S^{\mathcal{R}}(E_{t-u})} \left(1 - \delta_0 \left(\int_{B(x, \mathcal{R})} (1 - \omega_u(z)) dz \right) \right) \end{aligned}$$

$$\times \left[\delta_0 \left(\text{Vol}(E_{t-u}) - \langle \Theta_{x,\mathcal{R}}^-(\omega_u), \mathbf{1}_{E_{t-u}} \rangle \right) - \delta_0 \left(\int_{E_{t-u}} (1 - \omega_u(z)) dz \right) \right] dx.$$

For all $\mathcal{R} > 0$ and $x \in S^{\mathcal{R}}(E_{t-u})$,

$$\begin{aligned} \delta_0 \left(\text{Vol}(E_{t-u}) - \langle \Theta_{x,\mathcal{R}}^-(\omega_u), \mathbf{1}_{E_{t-u}} \rangle \right) &= \delta_0 \left(\text{Vol}(E_{t-u}) - \int_{E_{t-u} \setminus B(x,\mathcal{R})} \omega_u(z) dz \right) \\ &= \delta_0 \left(\text{Vol}(E_{t-u} \cap B(x,\mathcal{R})) + \int_{E_{t-u} \setminus B(x,\mathcal{R})} (1 - \omega_u(z)) dz \right). \end{aligned}$$

Since $x \in S^{\mathcal{R}}(E_{t-u})$, $\text{Vol}(E_{t-u} \cap B(x,\mathcal{R})) \neq 0$, and hence

$$\delta_0 \left(\text{Vol}(E_{t-u}) - \langle \Theta_{x,\mathcal{R}}^-(\omega_u), \mathbf{1}_{E_{t-u}} \rangle \right) = 0.$$

Moreover, notice that

$$\delta_0 \left(\int_{E_{t-u} \cup B(x,\mathcal{R})} (1 - \omega_u(z)) dz \right) = \delta_0 \left(\int_{E_{t-u}} (1 - \omega_u(z)) dz \right) \times \delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega_u(z)) dz \right).$$

Therefore,

$$\begin{aligned} &\mathcal{L}_\mu^\infty \Psi_{F^{\text{Vol}(E_{t-u}), f^{E_{t-u}}}}(M_u^\infty) \\ &= \int_0^\infty \int_{S^{\mathcal{R}}(E_{t-u})} \delta_0 \left(\int_{B(x,\mathcal{R})} (1 - \omega_u(z)) dz \right) \times \delta_0 \left(\int_{E_{t-u}} (1 - \omega_u(z)) dz \right) dx \mu(d\mathcal{R}) \\ &\quad - \int_0^\infty \int_{S^{\mathcal{R}}(E_{t-u})} \delta_0 \left(\int_{E_{t-u}} (1 - \omega_u(z)) dz \right) dx \mu(d\mathcal{R}) \\ &= \int_0^\infty \int_{\mathbb{R}^d} \mathbf{1}_{x \in S^{\mathcal{R}}(E_{t-u})} \times \left[\delta_0 \left(\int_{E_{t-u} \cup B(x,\mathcal{R})} (1 - \omega_u(z)) dz \right) - \delta_0 \left(\int_{E_{t-u}} (1 - \omega_u(z)) dz \right) \right] dx \mu(d\mathcal{R}), \end{aligned}$$

which is equal to $\mathcal{G}_\mu^\infty \Phi_{\delta_0, 1-\omega_u}(\Xi_{t-u}^\infty)$. Thus

$$F(t, 0) = F(0, t)$$

$$\begin{aligned} \text{i.e.} \quad &\mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} \left[\tilde{D}(M_t^\infty, \Xi_0^\infty) \right] \right] = \mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} \left[\tilde{D}(M_0^\infty, \Xi_t^\infty) \right] \right] \\ \iff &\mathbb{E}_{M^0} \left[\mathbf{E}_{m(E^0)} \left[\delta_0 \left(\int_{E_0} (1 - \omega_t(x)) dx \right) \right] \right] = \mathbb{E}_{M^0} \left[\mathbf{E}_{m(E_t)} \left[\delta_0 \left(\int_{E_0} (1 - \omega_0(x)) dx \right) \right] \right]. \end{aligned}$$

Therefore

$$\mathbb{E}_{M^0} \left[\delta_0 \left(\int_{E_0} (1 - \omega_t(x)) dx \right) \right] = \mathbf{E}_{m(E^0)} \left[\delta_0 \left(\int_{E_t} (1 - \omega_0(x)) dx \right) \right]$$

and we can conclude. \square

Finally, we can prove the second part of Theorem 10, i.e, the uniqueness of the solution to the martingale problem satisfied by the ∞ -SLFV when μ satisfies Condition (5). The first part of this theorem was proved in Section 3 (Proposition 22).

Proof. (Theorem 10)

Let $(M_t^1)_{t \geq 0}$ and $(M_t^2)_{t \geq 0}$ two solutions to the martingale problem $(\mathcal{L}_\mu^\infty, \delta_{M^0})$. Then, due to the form of the operator \mathcal{L}_μ^∞ , there exists densities $(\omega_t^1)_{t \geq 0}$ and $(\omega_t^2)_{t \geq 0}$ of $(M_t^1)_{t \geq 0}$ and $(M_t^2)_{t \geq 0}$ such that

$$\forall t \geq 0, \forall x \in \mathbb{R}^d, \omega_t^1(x) \in \{0, 1\} \text{ et } \omega_t^2(x) \in \{0, 1\}.$$

Then, let $t \geq 0$, let $E \in \mathcal{E}^{cf}$ and let $(\Xi_t^\infty)_{t \geq 0}$ be the ∞ -parent ancestral process associated to $m(E)$.

We have

$$\begin{aligned} \mathbb{P}_{M^0} \left(\delta_0 \left(\int_E (1 - \omega_t^1(x)) dx \right) = 1 \right) &= \mathbb{E}_{M^0} \left[\delta_0 \left(\int_E (1 - \omega_t^1(x)) dx \right) \right] \\ &= \mathbb{E}_{M^0} \left[D(M_\infty^1, \Xi_0^\infty) \right] \\ &= \mathbf{E}_{m(E)} \left[D(M^0, \Xi_t^\infty) \right] \text{ by Proposition 13} \\ &= \mathbb{E}_{M^0} \left[D(M_\infty^2, \Xi_0^\infty) \right] \text{ by the same proposition} \\ &= \mathbb{E}_{M^0} \left[\delta_0 \left(\int_E (1 - \omega_t^2(x)) dx \right) \right] \\ &= \mathbb{P}_{M^0} \left(\delta_0 \left(\int_E (1 - \omega_t^2(x)) dx \right) = 1 \right), \end{aligned}$$

using Proposition 13 to pass from line 2 to line 3, and from line 3 to line 4. We can conclude $(M_t^1)_{t \geq 0} = (M_t^2)_{t \geq 0}$. \square

5 Technical lemmas

5.1 Properties of the operators \mathcal{L}_μ^k and \mathcal{L}_μ^∞

The goal of this section is to show that the operators \mathcal{L}_μ^k and \mathcal{L}_μ^∞ introduced in Section 1 are well-defined, as well as to prove some properties they satisfy.

In all that follows, let $F \in C^1(\mathbb{R})$, $f \in C_c(\mathbb{R}^d)$, and $M \in \mathcal{M}_\lambda$. Let $\omega : \mathbb{R}^d \rightarrow \{0, 1\}$ be a measurable function, let μ be a σ -finite measure on \mathbb{R}_+^* satisfying Condition (4), and let $k \geq 2$. Since f is of compact support, there exist constants $C_1, C_2 > 0$ such that for all $\mathcal{R} > 0$,

$$\text{Vol}(\text{Supp}^{\mathcal{R}}(f)) \leq C_2 \times (\mathcal{R}^d \vee 1), \quad (13)$$

and for all $\tilde{\omega} : \mathbb{R}^d \rightarrow \{0, 1\}$ measurable,

$$\left| \langle \mathbb{1}_{B(x, \mathcal{R})} \times \tilde{\omega}, f \rangle \right| \leq C_1 \times \|f\|_\infty \times (\mathcal{R}^d \wedge 1). \quad (14)$$

Lemma 37. For all $x \in \mathbb{R}^d$ and for all $\mathcal{R} > 0$,

$$\begin{aligned} \left| \langle \Theta_{x, \mathcal{R}}^+(\omega), f \rangle - \langle \omega, f \rangle \right| &\leq \|f\|_\infty \times \text{Vol}(\text{Supp}(f)) \\ \text{and} \quad \left| \langle \Theta_{x, \mathcal{R}}^-(\omega), f \rangle - \langle \omega, f \rangle \right| &\leq \|f\|_\infty \times \text{Vol}(\text{Supp}(f)). \end{aligned}$$

Proof. Let $x \in \mathbb{R}^d$ and $\mathcal{R} > 0$.

$$\begin{aligned} \left| \langle \Theta_{x, \mathcal{R}}^+(\omega), f \rangle - \langle \omega, f \rangle \right| &\leq \left| \langle \mathbb{1}_{B(x, \mathcal{R})^c} \times \omega, f \rangle + \langle \mathbb{1}_{B(x, \mathcal{R})}, f \rangle - \langle \mathbb{1}_{B(x, \mathcal{R})^c} \times \omega, f \rangle - \langle \mathbb{1}_{B(x, \mathcal{R})} \times \omega, f \rangle \right| \\ &\leq \left| \langle \mathbb{1}_{B(x, \mathcal{R})} \times (1 - \omega), f \rangle \right| \\ &\leq \left| \int_{B(x, \mathcal{R})} (1 - \omega(y)) \times f(y) dy \right| \\ &\leq \int_{B(x, \mathcal{R})} |f(y)| dy \\ &\leq \|f\|_\infty \times \text{Vol}(\text{Supp}(f)). \end{aligned}$$

We can similarly show the corresponding result for $\left| \langle \Theta_{x, \mathcal{R}}^-(\omega), f \rangle - \langle \omega, f \rangle \right|$. □

Lemma 38. For all $\mathcal{R} > 0$, for all $x \in \mathbb{R}^d \setminus \text{Supp}^{\mathcal{R}}(f)$,

$$\langle \Theta_{x, \mathcal{R}}^+(\omega), f \rangle - \langle \omega, f \rangle = \langle \Theta_{x, \mathcal{R}}^-(\omega), f \rangle - \langle \omega, f \rangle = 0$$

Proof. Let $\mathcal{R} > 0$, and let $x \in \mathbb{R}^d \setminus \text{Supp}^{\mathcal{R}}(f)$,

$$\begin{aligned} \left| \langle \Theta_{x, \mathcal{R}}^+(\omega), f \rangle - \langle \omega, f \rangle \right| &= \left| \langle \mathbb{1}_{B(x, \mathcal{R})} \times (1 - \omega), f \rangle \right| \\ &\leq \int_{B(x, \mathcal{R})} |f(y)| dy \\ &= 0 \end{aligned}$$

since $x \in \mathbb{R}^d \setminus \text{Supp}^{\mathcal{R}}(f)$. Similarly,

$$\begin{aligned} \left| \langle \Theta_{x, \mathcal{R}}^-(\omega), f \rangle - \langle \omega, f \rangle \right| &= \left| \langle \mathbb{1}_{B(x, \mathcal{R})} \times \omega, f \rangle \right| \\ &\leq \int_{B(x, \mathcal{R})} |f(y)| dy \\ &= 0 \end{aligned}$$

for the same reason, and we can conclude. \square

Lemma 39. For all $x \in \mathbb{R}^d$ and for all $\mathcal{R} > 0$,

$$\begin{aligned} & \left| F \left(\langle \Theta_{x,\mathcal{R}}^+(\omega), f \rangle \right) - F \left(\langle \omega, f \rangle \right) \right| \leq C_1 \times \|f\|_\infty \times \left(\mathcal{R}^d \wedge 1 \right) \times C(F, f) \\ \text{and} \quad & \left| F \left(\langle \Theta_{x,\mathcal{R}}^-(\omega), f \rangle \right) - F \left(\langle \omega, f \rangle \right) \right| \leq C_1 \times \|f\|_\infty \times \left(\mathcal{R}^d \wedge 1 \right) \times C(F, f) \end{aligned}$$

where

$$C(F, f) = \sup_{z \in [-\|f\|_\infty \text{Vol}(\text{Supp}(f)), \|f\|_\infty \text{Vol}(\text{Supp}(f))]} |F'(z)|.$$

Proof. Let $x \in \mathbb{R}^d$ and $\mathcal{R} > 0$. First, we notice that as in the proof of Lemma 37, we only need to show the result for $\Theta_{x,\mathcal{R}}^+(\omega)$.

By Taylor-Lagrange inequality and by Lemma 37,

$$\begin{aligned} \left| F \left(\langle \Theta_{x,\mathcal{R}}^+(\omega), f \rangle \right) - F \left(\langle \omega, f \rangle \right) \right| & \leq \left| \langle \Theta_{x,\mathcal{R}}^+(\omega), f \rangle - \langle \omega, f \rangle \right| \times C(F, f) \\ & \leq \left| \langle \mathbb{1}_{B(x,\mathcal{R})} \times (1 - \omega), f \rangle \right| \times C(F, f) \\ & \leq C_1 \times \|f\|_\infty \times \left(\mathcal{R}^d \wedge 1 \right) \times C(F, f) \end{aligned}$$

by Eq. (14). \square

We can now show that the operator \mathcal{L}_μ^k is well-defined.

Lemma 40. The operator \mathcal{L}_μ^k is well-defined. Moreover, the function $\mathcal{L}_\mu^k \Psi_{F,f} : \mathcal{M}_\lambda \rightarrow \mathbb{R}$ is bounded.

Proof. Let $M \in \mathcal{M}_\lambda$. Then

$$\begin{aligned} & \left| \mathcal{L}_\mu^k \Psi_{F,f}(M) \right| \\ & \leq \left| \int_{\mathbb{R}^d} \int_0^\infty \int_{B(x,\mathcal{R})^k} \frac{1}{V_{\mathcal{R}}^k} \times \left(\prod_{j=1}^k \omega_M(y_j) \right) \times \left(F \left(\langle \Theta_{x,\mathcal{R}}^+(\omega_M), f \rangle \right) - F \left(\langle \omega_M, f \rangle \right) \right) dy_1 \dots dy_k \mu(d\mathcal{R}) dx \right| \\ & \quad + \left| \int_{\mathbb{R}^d} \int_0^\infty \int_{B(x,\mathcal{R})^k} \frac{1}{V_{\mathcal{R}}^k} \times \left(\prod_{j=1}^k (1 - \omega_M(y_j)) \right) \times \left(F \left(\langle \Theta_{x,\mathcal{R}}^-(\omega_M), f \rangle \right) - F \left(\langle \omega_M, f \rangle \right) \right) dy_1 \dots dy_k \mu(d\mathcal{R}) dx \right| \\ & \leq \left| \int_{\mathbb{R}^d} \int_0^\infty \int_{B(x,\mathcal{R})^k} \frac{1}{V_{\mathcal{R}}^k} \times \left(F \left(\langle \Theta_{x,\mathcal{R}}^+(\omega_M), f \rangle \right) - F \left(\langle \omega_M, f \rangle \right) \right) dy_1 \dots dy_k \mu(d\mathcal{R}) dx \right| \\ & \quad + \left| \int_{\mathbb{R}^d} \int_0^\infty \int_{B(x,\mathcal{R})^k} \frac{1}{V_{\mathcal{R}}^k} \times \left(F \left(\langle \Theta_{x,\mathcal{R}}^-(\omega_M), f \rangle \right) - F \left(\langle \omega_M, f \rangle \right) \right) dy_1 \dots dy_k \mu(d\mathcal{R}) dx \right| \\ & \leq \left| \int_0^\infty \int_{\text{Supp}^{\mathcal{R}}(f)} \int_{B(x,\mathcal{R})^k} \frac{1}{V_{\mathcal{R}}^k} \times \left(F \left(\langle \Theta_{x,\mathcal{R}}^+(\omega_M), f \rangle \right) - F \left(\langle \omega_M, f \rangle \right) \right) dy_1 \dots dy_k dx \mu(d\mathcal{R}) \right| \\ & \quad + \left| \int_0^\infty \int_{\text{Supp}^{\mathcal{R}}(f)} \int_{B(x,\mathcal{R})^k} \frac{1}{V_{\mathcal{R}}^k} \times \left(F \left(\langle \Theta_{x,\mathcal{R}}^-(\omega_M), f \rangle \right) - F \left(\langle \omega_M, f \rangle \right) \right) dy_1 \dots dy_k dx \mu(d\mathcal{R}) \right|. \end{aligned}$$

Using Lemma 39,

$$\begin{aligned}
& \left| \mathcal{L}_\mu^k \Psi_{F,f}(M) \right| \\
& \leq \int_0^\infty \int_{\text{Supp}^{\mathcal{R}}(f)} \int_{B(x,\mathcal{R})^k} \frac{2}{V_{\mathcal{R}}^k} \times C_1 \times \|f\|_\infty \times (\mathcal{R}^d \wedge 1) \times C(F, f) dy_1 \dots dy_k dx \mu(d\mathcal{R}). \\
& \leq \int_0^\infty 2\text{Vol}(\text{Supp}^{\mathcal{R}}(f)) \times C_1 \times \|f\|_\infty \times (\mathcal{R}^d \wedge 1) \times C(F, f) \mu(d\mathcal{R}),
\end{aligned}$$

and by Eq. (13),

$$\begin{aligned}
& \left| \mathcal{L}_\mu^k \Psi_{F,f}(M) \right| \\
& \leq \int_0^\infty 2C_1 C_2 \times \|f\|_\infty \times C(F, f) \times (\mathcal{R}^d \wedge 1) \times (\mathcal{R}^d \vee 1) \mu(d\mathcal{R}) \\
& \leq 2C_1 C_2 \|f\|_\infty \times C(F, f) \times \int_0^\infty \mathcal{R}^d \mu(d\mathcal{R}) \\
& < +\infty
\end{aligned}$$

since μ satisfies Condition (4).

The second part of the lemma is a direct consequence of the fact that

$$2C_1 C_2 \|f\|_\infty \times C(F, f) \times \int_0^\infty \mathcal{R}^d \mu(d\mathcal{R})$$

does not depend on the choice of M . □

A consequence of this lemma and of Lemma 38 is that for all $M \in \mathcal{M}_\lambda$, $\mathcal{L}_\mu^k \Psi_{F,f}(M)$ can be rewritten as :

$$\begin{aligned}
\mathcal{L}_\mu^k \Psi_{F,f}(M) = \int_0^\infty \int_{\text{Supp}^{\mathcal{R}}(f)} \int_{B(x,\mathcal{R})^k} \frac{1}{V_{\mathcal{R}}^k} \times & \left[\prod_{j=1}^k \omega_M(y_j) \times F(\langle \Theta_{x,\mathcal{R}}^+(\omega_M), f \rangle) \right. \\
& + (1 - \prod_{j=1}^k \omega_M(y_j)) \times F(\langle \Theta_{x,\mathcal{R}}^-(\omega_M), f \rangle) \\
& \left. - F(\langle \omega_M, f \rangle) \right] dy_1 \dots dy_k dx \mu(d\mathcal{R}).
\end{aligned}$$

We now prove that the operator \mathcal{L}_μ^∞ is well-defined.

Lemma 41. *The operator \mathcal{L}_μ^∞ is well-defined. Moreover, the function $\mathcal{L}_\mu^\infty \Psi_{F,f} : \mathcal{M}_\lambda \rightarrow \mathbb{R}$ is bounded.*

Proof. Let $M \in \mathcal{M}_\lambda$. Then,

$$\left| \mathcal{L}_\mu^\infty \Psi_{F,f}(M) \right|$$

$$\begin{aligned}
&\leq \int_0^\infty \int_{\text{Supp}^{\mathcal{R}}(f)} \left| \left(1 - \delta_0 \left(\int_{B(x, \mathcal{R})} 1 - \omega_M(z) dz \right) \right) \times \left[F \left(\langle \Theta_{x, \mathcal{R}}^-(\omega_M), f \rangle \right) - F \left(\langle \omega_M, f \rangle \right) \right] \right| dx \mu(d\mathcal{R}) \\
&\leq \int_0^\infty \int_{\text{Supp}^{\mathcal{R}}(f)} C_1 \times \|f\|_\infty \times (\mathcal{R}^d \wedge 1) \times C(F, f) dx \mu(d\mathcal{R}) \\
&\leq \int_0^\infty \text{Vol}(\text{Supp}^{\mathcal{R}}(f)) C_1 \times \|f\|_\infty \times (\mathcal{R}^d \wedge 1) \times C(F, f) dx \mu(d\mathcal{R}) \\
&\leq C_1 C_2 C(F, f) \times \|f\|_\infty \times \int_0^\infty (\mathcal{R}^d \wedge 1) \times (\mathcal{R}^d \vee 1) \mu(d\mathcal{R}) \\
&\leq C_1 C_2 C(F, f) \times \|f\|_\infty \times \int_0^\infty \mathcal{R}^d \mu(d\mathcal{R}) \\
&< +\infty
\end{aligned}$$

since μ satisfies Condition (4). Here we used Lemma 39 to pass from the second to the third line, and Lemma 37 to pass from the fourth to the fifth line.

As before, the second part of the lemma is the consequence of the fact that

$$C_1 C_2 C(F, f) \times \|f\|_\infty \times \int_0^\infty \mathcal{R}^d \mu(d\mathcal{R})$$

does not depend on the choice of M . □

5.2 Properties of the operator \mathcal{G}_μ^∞

In all the following, let μ be a σ -finite measure on \mathbb{R}_+^* satisfying Condition (5), let $F \in C_b^1(\mathbb{R})$ and let $f \in \mathcal{B}(\mathbb{R}^d)$.

Lemma 42. *The operator \mathcal{G}_μ^∞ is well-defined, and the function $\mathcal{G}_\mu^\infty \Phi_{F, f}$ is bounded.*

Proof. Let $m(E) \in \mathcal{M}^{cf}$. Then,

$$\begin{aligned}
\left| \mathcal{G}_\mu^\infty \Phi_{F, f}(m(E)) \right| &\leq \int_0^\infty \int_{S^{\mathcal{R}}(E) \cup \text{Supp}^{\mathcal{R}}(f)} |F(\langle m(E \cup B(x, \mathcal{R})), f \rangle) - F(\langle m(E), f \rangle)| dx \mu(d\mathcal{R}) \\
&\leq \int_0^\infty \int_{S^{\mathcal{R}}(E) \cup \text{Supp}^{\mathcal{R}}(f)} 2\|F\|_\infty dx \mu(d\mathcal{R}) \\
&\leq 2\|F\|_\infty \int_0^\infty \text{Vol}(S^{\mathcal{R}}(E) \cup \text{Supp}^{\mathcal{R}}(f)) \mu(d\mathcal{R}) \\
&\leq 2\|F\|_\infty \int_0^\infty \text{Vol}(\text{Supp}^{\mathcal{R}}(f)) \mu(d\mathcal{R}) \\
&\leq 2\|F\|_\infty \int_0^\infty C_2 \times (\mathcal{R}^d \vee 1) \mu(d\mathcal{R}) \\
&< +\infty,
\end{aligned}$$

since μ satisfies Condition (4). □

Lemma 43. Let $\Xi \in \mathcal{M}^{cf}$, and let $(\Xi_t)_{t \geq 0}$ be the ∞ -parent ancestral process associated to μ with initial condition Ξ . Then, for all $t \geq 0$,

$$\mathbf{E} \left[\int_0^t \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_s) ds \right] = \int_0^t \mathbf{E} \left[\mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_s) \right] ds.$$

Proof. Let $t \geq 0$, and let $\tilde{\mathcal{R}} > 0$ such that μ satisfies Condition (5). Then, since $u \rightarrow \text{Vol}(\Xi_u)$ is increasing,

$$\begin{aligned} \mathbf{E} \left[\int_0^t \left| \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_s) \right| ds \right] &\leq 2\|F\|_\infty \times \mathbf{E} \left[\int_0^t \int_0^\infty \text{Vol}(S^{\mathcal{R}}(\Xi_s)) \mu(d\mathcal{R}) ds \right] \\ &\leq 2\|F\|_\infty \times \mathbf{E} \left[\int_0^t \int_0^\infty \text{Vol}(S^{\mathcal{R}}(\Xi_t)) \mu(d\mathcal{R}) ds \right] \\ &\leq 2\|F\|_\infty \times t \times \mathbf{E} \left[\int_0^\infty C_t \times \text{Vol}(\mathcal{B}(0, \mathcal{R})) \mu(d\mathcal{R}) \right], \end{aligned}$$

with $(C_t)_{t \geq 0}$ being the $\tilde{\mathcal{R}}$ -covering process associated to $(\Xi_t)_{t \geq 0}$. Therefore,

$$\begin{aligned} \mathbf{E} \left[\int_0^t \left| \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_s) \right| ds \right] &\leq 2\|F\|_\infty \times t \times \int_0^\infty V_1 \times (\tilde{\mathcal{R}} + \mathcal{R})^d \mu(d\mathcal{R}) \times \mathbf{E}[C_t] \\ &\leq 2\|F\|_\infty \times t \times \int_0^\infty V_1 \times (\tilde{\mathcal{R}} + \mathcal{R})^d \mu(d\mathcal{R}) \times \mathbf{E}[Y_t], \end{aligned}$$

where Y_t is the branching process associated to $(C_t)_{t \geq 0}$ introduced in Section 3. Hence

$$\mathbf{E} \left[\int_0^t \left| \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_s) \right| ds \right] < +\infty.$$

We conclude by applying Fubini's theorem. □

Lemma 44. Let $\Xi \in \mathcal{M}^{cf}$, and let $(\Xi_t)_{t \geq 0}$ be the ∞ -parent ancestral process associated to μ with initial condition Ξ . Then, for all $t \geq 0$,

$$\mathbf{E} \left[\mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_t) \right] = \frac{d}{du} \left[\Phi_{F,f}(\Xi_u) \right] \Big|_{u=t}.$$

Proof. Let $t \geq 0$. We have

$$\frac{d}{du} \mathbf{E} \left[\Phi_{F,f}(\Xi_t) \right] \Big|_{t=0} = \mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi),$$

so for all $s \in [0, t]$,

$$\mathbf{E} \left[\mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_s) \right] = \mathbf{E} \left[\frac{d}{du} \mathbf{E} \left[\Phi_{F,f}(\Xi_t) \mid \Xi_s \right] \Big|_{t=s} \right].$$

Since F' is bounded, by the dominated convergence theorem,

$$\begin{aligned}\mathbf{E} \left[\mathcal{G}_\mu^\infty \Phi_{F,f}(\Xi_s) \right] &= \frac{d}{du} \mathbf{E} \left[\mathbf{E} [\Phi_{F,f}(\Xi_t) | \Xi_s] \right] \Big|_{t=s} \\ &= \frac{d}{du} \mathbf{E} [\Phi_{F,f}(\Xi_t)] \Big|_{t=s}\end{aligned}$$

and we can conclude. \square

5.3 Properties of the densities of coupled k -parent SLFVs

The goal of this section is to prove technical lemmas about the density of coupled k -parent SLFVs, which will be used in Section 3 in order to construct the ∞ -parent SLFV.

In all that follows, let μ be a σ -finite measure on $(0, +\infty)$ satisfying Condition (4), and let Π^c be a Poisson point process on $\mathbb{R} \times \mathbb{R}^d \times (0, +\infty) \times U$ with intensity

$$dt \otimes dx \otimes \mu(d\mathcal{R}) \otimes \tilde{u}(d(p_n)_{n \geq 1}).$$

Lemma 45. *For all $k \geq 2$, for all $0 \leq s \leq t$ and for all $x \in \mathbb{R}^d$,*

$$A \left(\Xi_{k,t}^{\Pi^c, t, \delta_x} \right) = \bigcup_{x' \in A \left(\Xi_{k,t-s}^{\Pi^c, t, \delta_x} \right)} A \left(\Xi_{k,s}^{\Pi^c, s, \delta_{x'}} \right).$$

Proof. Let $k \geq 2$, let $0 \leq s \leq t$ and let $x \in \mathbb{R}^d$. Let $y \in A \left(\Xi_{k,t}^{\Pi^c, t, \delta_x} \right)$. Then, we can construct a chain of reproduction events linking the point x at time t to the point y at time 0. We can split it into two chains :

- one linking the point x at time t to a point $y' \in \mathbb{R}^d$ at time s ,
- one linking the point y' at time s to the point y at time 0.

Therefore, $y' \in A \left(\Xi_{k,t-s}^{\Pi^c, t, \delta_x} \right)$ and $y \in A \left(\Xi_{k,s}^{\Pi^c, s, \delta_{y'}} \right)$, which means that

$$y \in \bigcup_{x' \in A \left(\Xi_{k,t-s}^{\Pi^c, t, \delta_x} \right)} A \left(\Xi_{k,s}^{\Pi^c, s, \delta_{x'}} \right).$$

Conversely, let y belonging to this set. It means that there exists $x' \in A \left(\Xi_{k,t-s}^{\Pi^c, t, \delta_x} \right)$ such that $y \in A \left(\Xi_{k,s}^{\Pi^c, s, \delta_{x'}} \right)$. Therefore, we can construct two chains of reproduction events, linking the point x at time t to the point x' at time s , and the point x' at time s to the point y at time 0. Hence $y \in A \left(\Xi_{k,t}^{\Pi^c, t, \delta_x} \right)$, and we can conclude. \square

Lemma 46. For all $k \geq 2$, for all $0 \leq s \leq t$ and for all $x \in rd$,

$$\omega_{k,t}^{\Pi,\omega}(x) = \prod_{x' \in A(\Xi_{k,t-s}^{\Pi^c,t,\delta x})} \omega_{k,s}^{\Pi^c,\omega}(x').$$

Proof. Let $k \geq 2$, let $0 \leq s \leq t$ and let $x \in rd$. By definition,

$$\omega_{k,t}^{\Pi^c,\omega}(x) = \prod_{y \in A(\Xi_{k,t}^{\Pi^c,t,\delta x})} \omega(y) \quad (15)$$

and

$$\prod_{x' \in A(\Xi_{k,t-s}^{\Pi^c,t,\delta x})} \omega_{k,s}^{\Pi^c,\omega}(x') = \prod_{x' \in A(\Xi_{k,t-s}^{\Pi^c,t,\delta x})} \prod_{y \in A(\Xi_{k,s}^{\Pi^c,t,\delta x'})} \omega(y). \quad (16)$$

Since by Lemma 45

$$A(\Xi_{k,t}^{\Pi^c,t,\delta x}) = \bigcup_{x' \in A(\Xi_{k,t-s}^{\Pi^c,t,\delta x})} A(\Xi_{k,s}^{\Pi^c,t,\delta x'}),$$

the same terms appear in both products. However, some terms may appear more than once in Eq. (16), while they can appear only once in Eq. (15). But ω is $\{0, 1\}$ -valued, so for all $y \in \mathbb{R}^d$ and $j \in \mathbb{N}^*$, $\omega^j(y) = \omega(y)$, and we can conclude. \square

Lemma 47. For all $\tilde{k} \geq 2$, for all $0 \leq s \leq t$ and for all $x \in \mathbb{R}^d$,

$$\lim_{k \rightarrow +\infty} \prod_{x' \in A(\Xi_{k,t-s}^{\Pi^c,t,\delta x})} \omega_{k,s}^{\Pi^c,\omega}(x') \leq \prod_{x' \in A(\Xi_{\tilde{k},t-s}^{\Pi^c,t,\delta x})} \lim_{k \rightarrow +\infty} \omega_{k,s}^{\Pi^c,\omega}(x').$$

Proof. Let $\tilde{k} \geq 2$, let $0 \leq s \leq t$ and let $x \in \mathbb{R}^d$. Since both quantities are $\{0, 1\}$ -valued, we only need to show that if

$$\prod_{x' \in A(\Xi_{k,t-s}^{\Pi^c,t,\delta x})} \lim_{k \rightarrow +\infty} \omega_{k,s}^{\Pi^c,\omega}(x') = 0$$

then

$$\lim_{k \rightarrow +\infty} \prod_{x' \in A(\Xi_{\tilde{k},t-s}^{\Pi^c,t,\delta x})} \omega_{k,s}^{\Pi^c,\omega}(x') = 0.$$

Assume that the first equality is true. Then, there exists $x' \in A(\Xi_{\tilde{k},t-s}^{\Pi^c,t,\delta x})$ such that $\lim_{k \rightarrow +\infty} \omega_{k,s}^{\Pi^c,\omega}(x') = 0$. But since $(\omega_{k,s}^{\Pi^c,\omega}(x'))_{k \geq 2}$ is decreasing and $\{0, 1\}$ -valued, there exists $k' \geq 2$ such that for all $k \geq k'$,

$\omega_{k,s}^{\Pi^c,\omega}(x') = 0$. Therefore, for all $k \geq k'$,

$$\prod_{x' \in A\left(\Xi_{k,t-s}^{\Pi^c,t,\delta_x}\right)} \omega_{k,s}^{\Pi^c,\omega}(x') = 0,$$

which means that

$$\lim_{k \rightarrow +\infty} \prod_{x' \in A\left(\Xi_{k,t-s}^{\Pi^c,t,\delta_x}\right)} \omega_{k,s}^{\Pi^c,\omega}(x') = 0.$$

□

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