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Publication of the mathematical works of René Thom in the collection *Documents mathématiques* of the French Mathematical Society*

Marc Chaperon and François Laudenbach

Volume II has just appeared. We present the whole of the project, initiated by André Haefliger.

René Thom in brief

Thom was one of the first members of the mathematical school created around Henri Cartan at the end of the second world war, and one of the most glorious, but from the start *singular*: provincial, born outside university circles, he had acquired very young an intimate knowledge of the differential calculus as its founders conceived it; unusual for his time, he did not distrust geometry, where he had developed his intuition to the point of "seeing" in four dimensions; finally, having followed Henri Cartan to Strasbourg as a young researcher in the CNRS,[1] he stayed there after the departure of his master, benefiting in particular from the influence of Charles Ehresmann.

Thus blessed with a vision of the world complementary to that of the "Parisians," Thom resolved fundamental questions which others would doubtless not even have thought of. Some landmarks:

– From 1949 to 1956, Thom worked in algebraic topology and elaborated "a completely new way of studying differentiable manifolds" (Milnor), about which he obtained thus definitive results, giving birth to the theory of *cobordism*, for which he was awarded the Fields medal in 1958.

– From 1956 on he concentrated on the singularities of differentiable maps, which appear naturally in his vision of differential topology. Following on from Whitney, he then studied stratifications and introduced the "natural stratification of mapping spaces" made precise later by John Mather.[2]

– From the middle of the 1960's, aware that singularities of maps and transversality aid in understanding all sorts of natural phenomena, he developed a *catastrophe theory*; this met with an extraordinary mediatic period after the appearance of his book *Stabilité structurelle et morphogénèse* in 1972.

– At the end of the 1970's, the sometimes delirious infatuation for catastrophes met with a sudden halt, without doubt hardly more justified than the ups and downs of fashions. Thom moved away then from mathematics in favour of philosophy and a fertile return to Aristotle.

These changes of orientation must not mask a profound unity of thought: already on his entry to the École Normale Supérieure, Thom was tempted by the philosophy of science, from which the director of the École had to dissuade him. His great mathematical results have a strong philosophical connotation, far from the "modern" pretension to separate mathematics from the question of meaning.

*The original French text of this article has been translated by Bernard Teissier and David Trotman.

[1] Centre National de la Recherche Scientifique (National Center of Scientific Research).

[2] This idea, to which Thom was much attached, provided for example the frame for the work of Jean Cerf on pseudo-isotopy and for that of Victor Vassiliev on knot invariants.

Birth of the project

The admiration of André Haefliger for Thom goes back to their common Strasbourg years (1954-1958).[3] Their relations only ceased after the disappearance of Thom in October 2002.

Already possessing documents from a near half-century of exchanges, Haefliger consecrated much time from the end of 2010 to the study of the Thom *archives*, sorted, inventoried and classified from mid-April 2011 in the basements of IHES by its librarian Aurélie Brest, with, at the start, help from Herminia and André Haefliger. He discovered "veritable treasures," for example two quite amazing unpublished mathematical manuscripts, that we will speak of again. He associated Marc Chaperon to this discovery in September 2011, with the idea of putting in place with his help and that of Bernard Teissier a "classical" paper publication, *annotated and commented on*, of the mathematical works of Thom.

Teissier and Étienne Ghys would have liked this edition to be accompanied by the placing online of the complete works (mathematical or not) of Thom, published in the form of a CD-ROM at the beginning of 2003 by IHES.[4] Alas! This quite natural idea was halted by problems of copyright.[5] As to the CD-ROM, perfectly usable despite the obsolescence of its search engine, it is no longer for sale.

A first editorial committee, constituted in October 2011, met at the beginning of the following month; as well as Haefliger, Teissier and Chaperon, it included Alain Chenciner. The project was submitted at the end of November by Teissier to Pierre Colmez and immediately accepted in the collection *Documents mathématiques* newly created by the French Mathematical Society (SMF). François Laudenbach, Jean Petitot, David Trotman and, for volume I, Jean Lannes and Pierre Vogel quickly joined the editorial committee.

Overview

This publication does not pretend to be a substitute for the CD-ROM but to complete it:

– It is concentrated on the mathematical articles (or at least classified as such in *Mathematical Reviews*), which here are provided with mathematical or historical commentaries, justified by later developments and the continued relevance of this often visionary work. Certain commentaries are to be found following the article which they relate to, others serve as introduction to several articles. After each article one finds shorter notes concerning precise details.

– We have chosen to reproduce the originals instead of transcribing into modern mathematical typography, thus avoiding introducing errors and changing the page numbering.[6] An exception comprises unpublished texts (absent from the CD-ROM) whose typed version was not legible enough.

As well as these unpublished texts, we publish a certain number of documents for their historical interest, for example large extracts from the correspondance with Cartan which led to Thom's thesis[7] and fragments of letters written by him to his wife Suzanne during his stay in Princeton in 1951, they contain much information on the genesis of the work which followed.

– The bibliography, like the bibliographical notice, completes, corrects and enriches that of the CD-ROM, using earlier versions of Michèle Porte, Jean Petitot and Aurélie Brest, it covers all Thom's writings, mathematical or not.

[3] Numerous discussions between them about foliated manifolds had then led to the "concrete" part (analytic foliations) of the still famous thesis on which Haefliger worked under the direction of Charles Ehresmann, rarely present in Strasbourg during this period.

[4] Thanks to the efforts of its director, Jean Pierre Bourguignon, and to the unfailing enthusiasm of Michèle Porte, director of this project begun in 1996 and to which Thom had actively collaborated.

[5] Those for the books reproduced were ceded by their publishers only for a limited number of copies.

[6] This choice, even if it has contributed to much delay in the publication of volume I, was imposed all the more since Thom often published in journals or collections of articles which have become almost impossible to find.

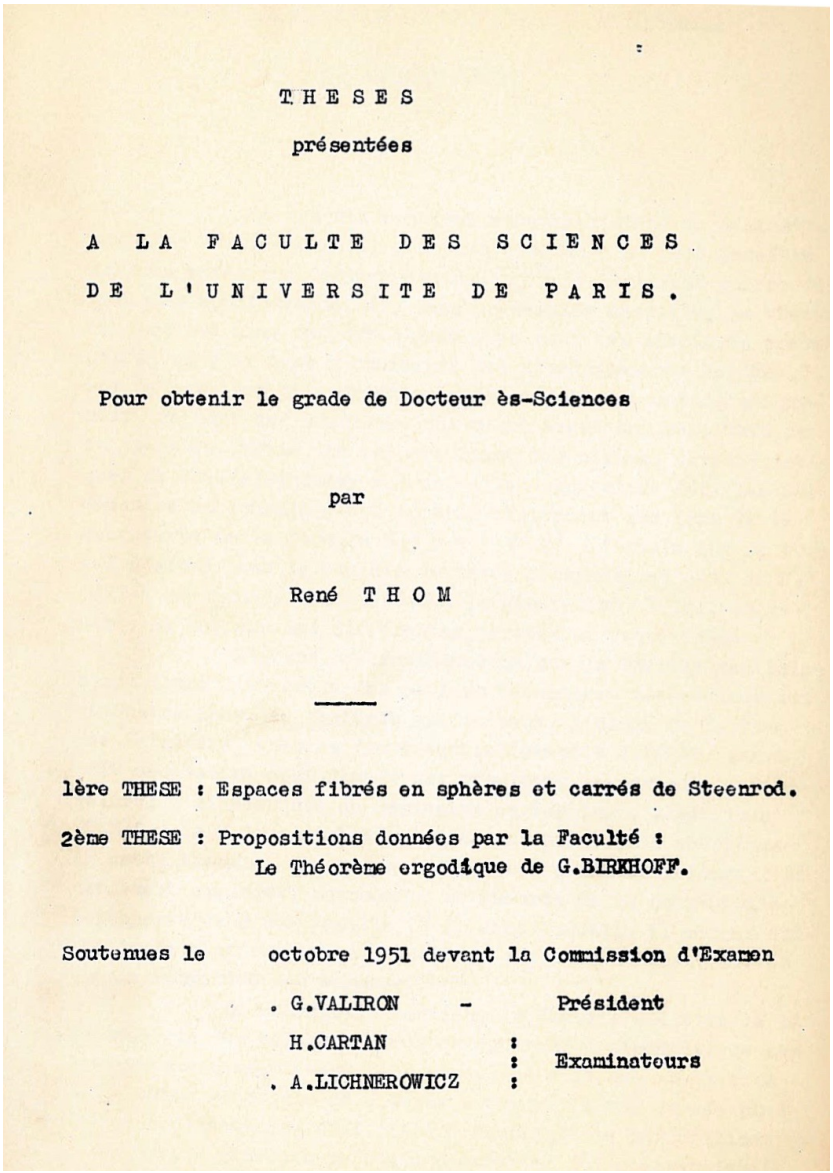
[7] We have unfortunately not been able to take into account in volume I an inestimable document which is part of the archives given since then by the Thom family to the library of the École Normale Supérieure: chapter 6 on cobordism, written by Thom and excluded by Cartan, for whom it seemed to not yet make sense.

Volume I

This volume, of 573 pages, appeared at last in April 2017 as n° 15 of the collection *Documents mathématiques*. It covers the publications of Thom from 1949 to 1959 except one, delayed to the beginning of the following volume. Given the dates, it is not surprising that it contains the major elements of Thom's work. We will evoke them a little more precisely.

The first article of Thom is a note in the Comptes Rendus de l'Académie des Sciences de Paris, 2 pages, entitled *Sur une partition en cellules associée à une fonction sur une variété*. It concerns a Morse function, that is to say a function at least of class C^2 whose critical points have Hessians of maximal rank. This note has had a very rich descendance that we have made explicit by a commentary in volume I. Thom is here curiously mute about what will become, starting in 1954, one of his paradigms: the notion of *genericity*. There lacks indeed the supplementary hypothesis of genericity ensuring that the said partition into cells be what is called a *cellular decomposition*, with its specific properties concerning the attachment of the cells. It is Stephen Smale, ten years later, who will make explicit this hypothesis on the gradient of the function in question, called today the *Morse-Smale condition*.

Next comes Thom's work in algebraic topology that we have preceded by a note of presentation written by J. Lannes and P. Vogel. This covers essentially the work of Thom for his thesis (1951) and his founding article on the *theory of cobordism* (1954). The thesis was published in the Annales de l'École Normale Supérieure (1952), 73 pages.



THESES

présentées

A LA FACULTE DES SCIENCES
DE L'UNIVERSITE DE PARIS.

Pour obtenir le grade de Docteur ès-Sciences

par

René THOM

1ère THESE : Espaces fibrés en sphères et carrés de Steenrod.

2ème THESE : Propositions données par la Faculté :
Le Théorème ergodique de G.BIRKHOFF.

Soutenues le octobre 1951 devant la Commission d'Examen

. G.VALIRON - Président
H.CARTAN :
. A.LICHNEROWICZ : Examinateurs

The article of 1954, 69 pages, has a title turned to the future, *Quelques propriétés globales des variétés différentiables*. It is here that appears for the first time the notion of *transversality*, a property which is generically satisfied, "most of the time," by a sufficiently differentiable map. This article is important in two aspects. On the one hand it opens the way to a new branch of differential topology which will become *homotopic topology*: this starts from the following problem that Thom solves, under what condition on a homology class of a manifold is it realisable by a submanifold. On the other hand, merely by the apparition of transversality—a few pages of this article—it opens the way to the study of singularities. With some exceptions, Thom is going to plunge into this second way. Let us see what happens.

Thom will realise that the way in which he proved his transversality theorem for a submanifold applies at once, without change, to certain situations that one now calls *transversality under constraint*. The first of these situations is the transversality to a submanifold in a *jet space*.

Jet bundles were discovered by Ehresmann. For a function f, let us say real for simplicity, k times differentiable and defined on a manifold M, its jet of order k at a point a of M is, in coordinates, the Taylor development of f at a. Of course, this polynomial depends on the coordinates but that two functions have the same Taylor development to order k does not depend on coordinates. One can then collect all the k-jets of functions at all points of M and we have thus created a new manifold $J^k(M, \mathbf{R})$ which fibres over M by indicating the point a where the k-jet is taken. A *section* of the bundle is rather rarely the collection of Taylor polynomials of a single function at all the points of M: there is said to be an *integrability* condition.

Despite this constraint, the theorem of transversality to a submanifold S of $J^k(M, \mathbf{R})$ holds, not in the space of all sections but among those which are integrable—and of course, one can replace $\mathbf{R}$ by any other manifold. However, the choice of S is decisive for the study of singularities of real functions. This is what Thom does in his article, *Les singularités des applications différentiables*, which appeared in the Annales de l'Institut Fourier (1956). Before this paper, a "cap" written by Haefliger and based on his own archives, presents the work of Thom on singularities in the period 1956-1957. Moreover, with the agreement of the American Mathematical Society we have reproduced the long report of W. S. Massey in Math. Reviews; this provides a useful light on Thom's article.

Because we are in 1957, let us say a word about two unpublished papers. The first is entitled: *Une démonstration d'un théorème de Lefschetz*, and the second *L'homologie des variétés de Stein*. Thom writes at the very beginning of the second: "As a result we have the proof of a conjecture of J.-P. Serre: the homology groups $H_i(V, \mathbf{Z})$ of a Stein manifold of complex dimension n are zero for $i > n$." Thom presented these in a seminar in Chicago in February 1957. Why did they remain unpublished? We will never have the answer. In any case A. Andreotti and T. Frankel published an article, *The Lefschetz theorem on hyperplane sections*, Annals of Math. (1959), 14 pages. In their introduction one may read:

> Recently Thom has given a proof (unpublished) which, as far as we know, is the first to use Morse's theory of critical points. We present in §3, in a slightly more general setting, an alternate proof inspired by Thom's discovery.

Let us finish this look at volume I with a last historical evocation. Thom gave a talk in a CNRS meeting in Lille in 1959. This talk was published in the Bulletin SMF with the title *Remarques sur les problèmes comportant des inéquations différentielles globales*. One cannot avoid relating this title with that of the book by M. Gromov, *Partial Differential Relations*, Springer-Verlag, 1986; moreover, Thom's article is cited there. Thom gives in this article, next to general considerations, a precise statement on the homology of an open set in a jet space. Here we are ten years before the thesis of Gromov (1969) who proves an analogous statement but, with an important difference, Gromov speaks of *homotopy*. As later, the statement of Gromov will be called an h-principle after the initial letter of *homotopy*, one could say that the statement of Thom is an h-principle (only homological) before its time.

The paradox of this affair is that Thom, at the time that S. Smale wrote his (brief) report for Math. Reviews, only believed in his theorem for the jet spaces of order one as shown by Smale's

commentary. However, the commentary of D. Spring in our volume I, p. 562, makes precise that he himself knew how to give a proof using the holonomic approximation theorem of Eliashberg and Mishachev (L'Enseignement Math., 2001). Finally, there is a strong chance that the idea of Thom to introduce small sawtooths in his simplexes can be carried out to the end.

Volume II

As in volume I, the texts of Thom are preceded here by a complete bibliography of his works.[8] It begins with the course on singularities given at Bonn in 1959, as recorded by Harold Levine. Annotated and commented by Haefliger, this lecture course is followed by a translation of the preface and the table of contents of the Russian edition by V. I. Arnold, then by a letter to Haefliger of February 1959; despite a casual error, this shows that Thom had quickly noticed the modules (*moduli*) which complicate the theory.

The volume gathers then the articles of Thom published between 1962 and 1971,[9] in chronological order of their appearance. The editorial committee was solicited more here than in volume I.

– Many of the long articles contained in the latter, written under the gaze of Henri Cartan and his school, required in fact no other commentary than the mention of their whys and wherefores. After his Fields medal, Thom is more alone but does not lose his tranquil audacity, which results in often prophetic (and, very rarely, badly written) mathematical papers requiring completion[10] and, in this volume, commented on in more detail.

– This audacity led him to "leave the frame" with his catastrophe theory, of which the foundational articles figure in this volume II. Containing sometimes very new mathematics, they are part of another story, notably in biology, of which it was important to give an idea: Sara Franceschelli and Jean Petitot took this on with much talent.

Singularities. A large part of this volume is formed of essential papers on singularities of maps—notably on their topological stability and on stratifications—annotated and commented upon by Teissier and Trotman, experts in the field. This includes *La stabilité topologique des applications polynomiales*, which sketches already the celebrated isotopy theorems of Thom-Mather, then *Propriétés différentiables locales des ensembles analytiques (d'après H. Whitney)*, excellent exposition of fundamental results of Whitney on the stratification of analytic sets and a draft of the theory of stratified sets; this is further developed in *Local topological properties of differentiable mappings*, astonishingly clairvoyant, and followed by *On some ideals of differentiable functions*, focusing on the difference between the differentiable and the analytic.

There next comes a "big part," *Ensembles et morphismes stratifiés* which, after inspiring Mather and a pleiad of other experts, continues after fifty years to be the source of active research. *The bifurcation subset of a space of maps* introduces the absolutely fascinating idea of a natural stratification of function spaces, concerning which much remains to be done.

The volume concludes, in a way closer to the two unpublished papers of volume I, by *Un résultat sur la monodromie*, commented on by Norbert A'Campo with Teissier, and by an unpublished manuscript on the monodromy[11] which, despite a *fatal error*, contains important and beautiful ideas. For the anecdote, Thom's article had been accepted in a prestigious journal, which had previously refused at the same time the paper where the young A'Campo gave a counterexample! Happily, Thom withdrew his text and the work of A'Campo, which surprised the most distinguished, established his reputation.

Catastrophes. Preceded by a knowledgeable introduction of Petitot, this part contains five articles. *A dynamic theory for morphogenesis*, "lost" but recovered by Tadashi Tokieda, presents the theory for the first time, insisting on its mathematical aspects—this is in particular the first appearance of

[8] Mildly corrected with respect to volume I, which sometimes changes the numbering.

[9] Except for the Fermi Lectures *Modèles mathématiques de la morphogénèse* given at the Scuola Normale Superiore of Pisa in April 1971, kept for volume III as the lectures in Bonn were kept for volume II.

[10] Mathematicians of the calibre of John Mather and Vladimir Arnold worked on this, but many exciting problems remain open.

[11] In a modern typography of Duco van Straten.

universal unfoldings; the commentary led us to interrogate Mather, whose response, of great interest, is partially reproduced.

Thom presents his ideas in more detail in *Une théorie dynamique de la morphogénèse*, a fundamental article followed by correspondence with the great biologist C. H. Waddington; the whole is commented on by Sara Franceschelli and Petitot, who begin with the relation with *The chemical basis of morphogenesis* of Alan Turing (1952).

Then comes *Topological models in biology*, then *A mathematical approach to morphogenesis: archetypal morphologies* and *Topologie et linguistique*, which illustrates the scale of the project.

Varia. The volume contains other "concrete" applications of the theory of singularities: *Sur la théorie des enveloppes*, written in 1960 to clean up a rather "dirty" domain, is not very readable—we have tried to remedy this in part without departing from Thom's ideas. *Sur les variétés d'ordre fini* sketches notably the proof that, save for rare exceptions, a compact submanifold M of dimension n in $\mathbf{R}^{n+\mu}$ cuts every affine μ-plane P in a finite number of points, bounded as P varies.

More ambitious, the article *Les symétries brisées en physique macroscopique et la mécanique quantique* is confronted by Valentin Poénaru with the later progress in physics. Like the talk *Travaux de Moser sur la stabilité des mouvements périodiques* on "KAM theory," it bares witness to the variety of interests of Thom and to his exceptional insight.

The article *Sur l'homologie des variétés algébriques réelles*, commented on by Ilia Itenberg, is devoted to what is often called since the *Smith-Thom inequality*, very important in real algebraic geometry. This work dedicated to Marston Morse depends on Morse theory, of which a "foliated" version is proposed in *Généralisation de la théorie de Morse aux variétés feuilletées*.

Finally *Jets de Liapunov*, commented on by Krzysztof Kurdyka, examines the implications of the existence of a Liapunov function for a vector field in the neighbourhood of a point.

Volume III

Respecting the name of the collection, this *last* volume[12] essentially contains mathematical writings, a small minority of Thom's publications after 1971, but often remarkable. Let us point out some of them. The first text, the Fermi lectures *Modèles mathématiques de la morphogénèse*[13] is a clear exposition of the stakes of catastrophe theory and a lot of new Mathematics, which have had an important progeny. A critical rereading was therefore in order.

Then comes *Sur le cut-locus d'une variété plongée*, a paper which is very rich mathematically (it takes the functional viewpoint of *The bifurcation subset of a space of maps*) and even beyond Mathematics—one finds there a quite up-to-date model of visual perception.

Phase transitions as catastrophes which is mathematically substantial[14] proposes in particular a proof of the Gibbs phase rule based on the generic structure of *Maxwell sets*.[15] This notion was at the heart of the cut locus paper.

Sur les équations différentielles multiformes et leurs intégrales singulières provides among other things a marvellous introduction to singularities, contact geometry and Pfaffian systems.

Symmetries gained and lost analyses symmetry breakings, in connexion with the work of the physicist Louis Michel and as a continuation of the paper of Volume II.

Introduction à la dynamique qualitative, a mathematico-historico-philosophical text *à la* Thom, evidences his long standing interest for this domain which was for a time not so popular in France, a situation which his seminar around 1970 had remedied, contributing to the orientation of future "leaders" in the subject, such as Michel Herman.

[12] In preparation, we would like to see it published at the end of 2020.

[13] It provides the first three chapters of the book with the same title published in 1974 in the pocket collection 10-18.

[14] and bold, sometimes a little too much so!

[15] The set of values of parameters for which a function depending on those parameters attains its minimum in several points (if one counts them with multiplicities).

Gradients of analytic functions, absent from the CD-ROM and unearthed by our Iranian colleague Massoud Amini, notably states the "gradient conjecture" on the limits at a singular point of the tangents to integral curves of an analytic gradient.[16] Enthusiastic commentary by Kurdyka.

Tectonique des plaques et théorie des catastrophes is the trace of a daring incursion into Claude Allègre's territory.

The Les Houches lectures *Mathematical concepts in the theory of ordered media* is a follow-up to the work of Kléman and Toulouse on dislocations in crystalline media.

The note with Yannick Kergosien *Sur les points paraboliques des surfaces,* completed and corrected with Thomas Banchoff, deals with the apparent contours of a "generic" surface in euclidean space when the direction of projection varies.[17]

The two notes with Peixoto on *Le point de vue énumératif dans les problèmes aux limites pour les équations différentielles ordinaires*, illustrate in particular the very enlightening vision that Thom had of "well posed" problems: they are those in which the d-dimensional manifold of solutions of the differential system is transversal in the function space to the d-codimensional variety defined by the limit conditions of the problem.

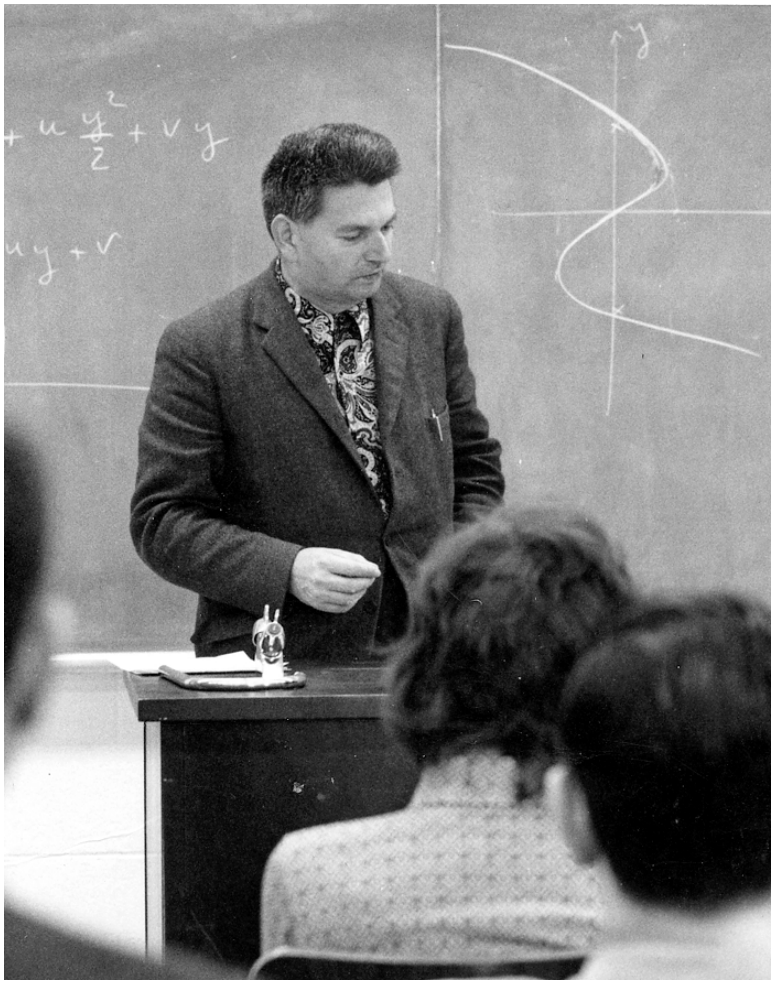
Quid des stratifications canoniques? evidences again their extreme importance in the eyes of Thom.

Finally, the obituaries of Morse and Whitney written for the *Académie des sciences* are obviously of great interest, as well as *La théorie des jets et ses développements ultérieurs* published with Ehresmann's complete works.

We also write a little about "the catastrophe of catastrophe theory", which took place in 1978 and has without any doubt contributed to Thom distancing himself from mathematics, to its detriment.

[16] A version of it has been proved by Kurdyka, Mostowski and Parusiński in a paper published in 2000 in the *Annals of Math.*

[17] The case of a single projection was very well treated in the paper *Sur les équations différentielles multiformes et leurs intégrales singulières.*



Catastrophes, around 1975 ©Archives de l'IHÉS - Droits réservés

Appendix: a modest overview of transversality according to Thom

Transversality plays an essential role in Thom's work, both in mathematics and in catastrophe theory where a key idea is that one can only observe the phenomena that are stable under perturbation. The transversality lemma in jet spaces[18] amazed its first "guinea pig" Whitney who at first could not believe this statement which put under the same umbrella almost all general position arguments. That is why we have thought it would be useful to present an overview of it.[19]

First elementary version.

The following facts seems to correspond to our intuition. If, in $\mathbf{R}^3$, a point p lies on a surface S a jiggling[20] will push p outside of S. If a (compact) curve C is tangent to S at one or several of its points, a jiggling will result in that either C and S are disjoint or, at the possible common points, the tangent to C and the tangent plane to S will intersect transversally. Finally, if S' is a (compact) surface which at some points shares a tangent plane with S, a jiggling of S' will put it in a transversal position with respect to S; at the possible common points the tangent planes will be secant; in fact—and this is less intuitive—if this last condition on the tangent planes is satisfied at each point of the intersection $S \cap S'$, this intersection will be a smooth curve (this last fact is guaranteed by the implicit function theorem).

[18] See Thom's paper *Un lemme sur les applications différentiables* (1956), reproduced and commented in volume I.

[19] Marc Chaperon had treated it very differently in issue 65 of the *Gazette de la SMF*, with many other applications.

[20] A generic small movement; we borrow this term from W. Thurston.

Thom's first statement on transversality is a generalization to all dimensions and all manifolds of those three examples. But one has still to specify what a "jiggling" is. It is a diffeomorphism g of the ambient manifold M, arbitrarily close to the identity (in the C^∞ topology) which one applies to a (compact) submanifold N in order to make it *transversal* to another submanifold N' of M: at any point of $g(N) \cap N'$ the two tangent spaces $T_x g(N)$ and $T_x N'$ generate $T_x M$. In what follows all manifolds will be assumed to be C^∞ (we shall say "smooth").

Elementary transversality theorem (Thom). *The property that $g(N)$ is transversal to N' is* generically *satisfied for g in the group* $\mathrm{Diff}^k(M)$ *of C^k diffeomorphisms of M if $k > \max(0, \dim N - \operatorname{codim} N')$. Here, a property is said to be* generic *if it is satisfied at all points of a countable intersection of dense open subsets of* $\mathrm{Diff}^k(M)$.

Remarks. 1) If N is compact, transversality is an open property in the C^1 topology and therefore in all finer topologies such as the C^∞ on the group $\mathrm{Diff}^\infty(M)$. Moreover, under this assumption, genericity is also true in the subgroup $\mathrm{Diff}^\infty_c(M)$ of diffeomeorphisms with compact support.

2) The same statement is still valid if N is singular (presence of double points, lack of tangent space) or even the image in M of a manifold N by a C^k map.

Proof of Thom's theorem based on theorems of A. Morse and A. Sard.

Theorem (Morse-Sard). *Let X be a manifold. If $f : X \to \mathbf{R}^q$ is C^k, $k > \max(0, \dim X - q)$, then, almost every y in $\mathbf{R}^q$, in the sense of the Lebesgue measure, is a* regular value *of f, which means that for every $x \in f^{-1}(y)$ the differential of f at x is of rank q.*

For instance, if $\dim X < q$, almost no y is in the image of f. By the implicit function theorem, the inverse image of a regular value is a C^k submanifold of codimension q.

The inequality on k in the statement above is necessary. For example, Whitney has built a function $f : \mathbf{R}^2 \to \mathbf{R}$ of class C^1 and an arc A in the plane where f is not constant but along which the differential of f is identically zero (the arc A is continuous but of course not C^1); the interval $f(A)$ contains no regular value of f although it is of positive length.

Here is how Thom uses Sard's theorem to prove his first transversality theorem. With its notations, let us consider the case $k = \infty$ and N compact and concentrate on the density part. Since $\mathrm{Diff}^\infty_c(M)$ is a group one easily shows that it suffices to prove that id_M can be approximated in the C^∞ topology by $g \in \mathrm{Diff}^\infty_c(M)$ such that $g(N)$ is transversal to N'.

Thom's idea is to build, for a certain integer r, a map $G : \mathbf{R}^r \times M \to M$, $(v, x) \mapsto G(v, x)$ of class C^∞, such that:

(1) $G(0, x) = x$ for all $x \in M$;

(2) For all small enough fixed v, $g_v := G(v, -)$ is a diffeomorphism of M with compact support;

(3) $\partial_v G(0, x)$ is of rank $n = \dim M$ for all $x \in N$.

We shall think of G as a family $\{g_v\}_{v \in \mathbf{R}^r}$ depending on r real parameters. Let us assume we have such a family and prove Thom's theorem. Since the rank condition is open, condition (3) implies that $\partial_v G(v, x)$ is of rank n for all (v, x) in a neighborhood U of $\{0\} \times N$. Let us still denote by G its restriction to U. With the rank condition, the implicit function theorem tells us that $W := G^{-1}(N')$ is a submanifold of U of class C^∞.

Let $\Pi : W \to \mathbf{R}^r$ be the projection onto the parameter space. One now applies Sard's theorem to Π. Almost every (r-dimensional) parameter v is a regular value of Π. An easy lemma of linear algebra shows that for such a v, the diffeomorphism g_v pushes N to a position transversal to N'.

There remains only to build the family G. Let us begin with the case $M = \mathbf{R}^n$. We take as parameter space the vector space $\vec{\mathbf{R}}^n$ of translations of $\mathbf{R}^n$. By a well chosen partition of unity, one

extends the translation of the compact N by the vector v into a diffeomorphism g_v with compact support which is id_M if $v = 0$. Conditions (1) - (3) are satisfied.

In the case where M is a manifold, using the compactness of N we cover it by a finite collection $\{B_i\}_{i=1}^s$ of balls, each of which is contained in a coordinate chart $\mathcal{O}_i$ of M.

One chooses the translations v_i in the coordinates of $\mathcal{O}_i$ small enough for $B_i + v_i$ to remain in a fixed compact subset of this chart. One easily extends this translation into a diffeomorphism g_{i,v_i} of M with compact support which is id_M if $v_i = 0$. Let $v := (v_1, ..., v_s)$. Set $g_v = g_{1,v_1} \circ \cdots \circ g_{s,v_s}$. One easily verifies that conditions (1) - (3) are satisfied, which ends the proof.

Transversality and jet spaces.

It is obviously in the jet spaces of order $r \geq 2$ that Thom will give the deepest applications of the transversality theorem to singularities of differentiable mappings. However, one can already realize the innovative aspect of this theorem with jets of order 1 (or 1-jets) of real functions of one real variable.

In order to start with a compact manifold, let us look at the 1-jets of functions on the interval $[0, 1]$, which is not exactly a manifold, but a manifold with boundary. The boundary creates no difficulty for understanding the space of real valued C^∞ functions.

If $f : [0, 1] \to \mathbf{R}$ is C^k, $k \geq 1$, we define its *jet of order one* $j^1 f(a) = (a, f(a), f'(a))$ at any point $a \in [0, 1]$.[21] The map $j^1 f$ is a section (that is, a right inverse) of the projection $(x, y, z) \mapsto x$ of $J^1([0, 1])$ to $[0, 1]$; thus, it identifies with its image which is a C^{k-1} submanifold of $J^1([0, 1])$.

Given $a \in [0, 1]$, any $P \in \mathbf{R}[T]$ of degree one is equal to $j^1_a f$, where $f(x) = P(x - a)$. However a global section $s(x) = (x, y(x), z(x))$ is of the form $j^1 f(x)$ if and only if

$$z(x) = \frac{dy}{dx}(x) \quad \text{(integrability condition)}.$$

This condition is quite rarely satisfied. Thus, if Σ is a submanifold of $J^1([0, 1])$, the proof given for the transversality theorem (elementary version) will produce from $j^1 f$ at best a section transversal to Σ, with practically no chance to produce an integrable section.

We remember that everything lies in the choice of the family G of maximal rank with respect to the parameters. It is Thom's clairvoyance which made him see that the same proof worked in the subspace of integrable sections if we take as parameter space the (finite dimensional) space of degree one real polynomials, here in one variable. At the same time we shall replace the diffeomorphism group of $J^1([0, 1])$ by its subgroup, which one might call the *gauge group* consisting of those diffeomorphisms which preserve not only each fiber $J^1_x([0, 1])$ but also the distribution of *contact planes*, i.e., the kernels of the differential form $dy - zdx$.[22]

If P is a polynomial function of degree one, the translation (in the fibers)

$$(x, y, z) \mapsto (x, y + P(x), z + P'(x)) \quad \text{in other words} \quad j^1 f(x) \mapsto j^1(f + P)(x)$$

is a gauge transformation, sending each integrable section $j^1 f$ to the integrable section $j^1(f + P)$. This family of translations verifies conditions (1) - (3), once truncated to obtain (2).

The same approach works in any dimension of base manifold and for any jet order.[23] Thom thus obtains the following theorem:

Transversality theorem in a jet space (Thom).

Let N be an n-dimensional manifold, let $J^r(N)$ be its space of r-jets of real functions and let Σ be a submanifold of $J^r(N)$ of codimension q and of class C^ℓ, $\ell > \max\{0, n - q\}$. For any $f \in C^k(N, \mathbf{R})$, $k - r > \max\{0, n - q\}$, generically for g in the C^{k-r} gauge group, the submanifold $g(j^r f(N))$ is transversal to Σ.

[21] In other words $j^1 f(a) = (a, j^1_a f)$, where $j^1_a f \in \mathbf{R}[T]$ is the Taylor polynomial $f(a) + f'(a)T$.

[22] A "contactologist" tradition exchanges y and z, giving $dz - ydx$.

[23] Even replacing the target space $\mathbf{R}$ by another manifold.

Here is a first application to Morse functions. At the beginning of the 1930's, Marston Morse showed the importance of the functions which now bear his name for the understanding of manifolds. Recall that a Morse function $f : N \to \mathbf{R}$ is a function, of class at least C^2, whose critical points (points where $df = 0$) are all non degenerate (in coordinates, the matrix of second partial derivatives is of maximum rank).

Corollary. *Morse functions of class C^k on N are dense in $C^k(N, \mathbf{R})$ for $k \geq 2$.*

Indeed, a simple computation shows that f is a Morse function if and only if $j^1 f$ is transversal to the zero section $\Sigma := \{j^1\varphi(x) : d_x\varphi = 0\}$ of the projection $j^1\varphi(x) \mapsto j^0\varphi(x) := (x, \varphi(x))$. According to the theorem (here with $q = n$ and $r = 1$), generically $j^1 f$ is transversal to Σ, hence the density.