# Basic aspects of differential geometry

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This is a very partial description of differential geometry as elaborated by Élie Cartan and expressed in a suitable language by Charles Ehresmann. I am entirely responsable for the selection of materials and for the mistakes, if any.

The framework is that of smooth[1] (finite dimensional) manifolds and maps, whose definition is taken for granted—most of the notions we consider "pass" without any problem to the real analytic and (replacing $\mathbb{R}$ by $\mathbb{C}$) complex and/or Banach categories. The $k^{\text{th}}$ derivative of a map $f$ is denoted by $D^k f$ as in [10]. Paths are defined on intervals.

## Jets

Introduced by Ehresmann [14], curiously almost absent from [11, 12], they are at the very beginning of modern differential geometry, as they generalize Taylor expansions to maps between manifolds. Recall the *Faà di Bruno formula* giving the $k^{th}$ derivative of the composed map of two $C^k$ maps between open subsets of Banach spaces:

$$\frac{1}{k!}D^k(g\circ f)(x)v^k = \sum D^{|p|}g(f(x))\left(\frac{1}{p_1!}\Big(\frac{1}{1!}D^1f(x)v^1\Big)^{p_1},\dots,\frac{1}{p_k!}\Big(\frac{1}{k!}D^kf(x)v^k\Big)^{p_k}\right),$$

where $x$ lies in the definition domain of $g\circ f$, the vector $v$ in the ambient Banach space, $v^k := (\overbrace{v,\dots,v}^{k\text{ times}})$ and the sum is on all $p=(p_1,\dots,p_k)\in\mathbb{N}^k$ with $\sum j\,p_j = k$, setting $|p| = \sum p_j$.

This formula is obtained by "composition of $k^{th}$ order Taylor expansions" [8]. Its author, born in Alessandria in 1825, was an officer in the Italian Royal Army before studying mathematics in Paris under the supervision of Cauchy and Le Verrier and taking up the position of Professor of Mathematics at the university of Turin. He was beatified in 1988, one century after his death, for his work as a social reformer, most notably the foundation of the Minim Sisters of St. Zita. Also a musician, he had been ordained in 1876.

For each integer $k$, two $C^k$ maps $f$ and $g$, defined in the neighbourhood of a point $a$ in a manifold $M$, taking their values in a manifold $N$, have the same $k^{\text{th}}$ *order jet* at $a$, denoted $j^k_a f = j^k_a g$, when they take the same value $b$ at $a$ and there exist local charts $\varphi : (M,a)\to\mathbb{R}^n$ and $\psi : (N,b)\to\mathbb{R}^p$ such that $\psi\circ f\circ\varphi^{-1}$ and $\psi\circ g\circ\varphi^{-1}$ have the same $k^{\text{th}}$ order Taylor expansion at $\varphi(a)$; fortunately for this definition, the Faà di Bruno formula implies that such is then the case for *all* local charts $\varphi$ and $\psi$ at $a$ and $b$ respectively.

Let $J^k(M,N)$ be the set of $k^{\text{th}}$ order jets $j^k_a f$ of maps of $M$ into $N$. If $M,N$ are open subsets $U,V$ of $\mathbb{R}^n,\mathbb{R}^p$ respectively, $J^k(U,V)$ identifies to the open subset $U\times V\times J^k(n,p)$ of the finite dimensional vector space

$$J^k(\mathbb{R}^n,\mathbb{R}^p) = \mathbb{R}^n\times\mathbb{R}^p\times J^k(n,p) := \mathbb{R}^n\times\prod_{j=0}^{k} L^j_s(\mathbb{R}^n,\mathbb{R}^p),$$

where $L^j_s(\mathbb{R}^n,\mathbb{R}^p)$ is the space of symmetric $j$-linear maps of $(\mathbb{R}^n)^j$ into $\mathbb{R}^p$ and $L^0_s(\mathbb{R}^n,\mathbb{R}^p) := \mathbb{R}^p$; indeed, $j^k_a f$ is then naturally identified to $\Big(a,(D^jf(a))_{0\le j\le k}\Big)$, and this identification is bijective since every $(a,b_0,\dots,b_k)\in U\times V\times J^k(n,p)$ is of the form $j^k_a f$ for $f(x)=\sum_0^k \frac{1}{j!}b_j(x-a)^j$.

In the general case, it follows from the Faà di Bruno formula that $J^k(M,N)$ is endowed with a smooth manifold structure by the *natural charts* $\Phi^k_{\varphi,\psi}$ associated to pairs of local charts $\varphi$ of $M$ and $\psi$ of $N$ as follows:

[1] That is $C^\infty$ or "smooth enough", the word being implicit when nothing is specified

- the definition domain $\mathbf{dom}\,\Phi^k_{\varphi,\psi}$ of $\Phi^k_{\varphi,\psi}$ is the set of $j^k_a f$ with $a \in \mathbf{dom}\,\varphi$ and $f(a) \in \mathbf{dom}\,\psi$
- the chart $\Phi^k_{\varphi,\psi}$ is given by the formula

$$\Phi^k_{\varphi,\psi}(j^k_a f) := j^k_{\varphi(a)}(\psi \circ f \circ \varphi^{-1})$$

  implying that the transition maps are $\Phi^k_{\varphi_1,\psi_1} \circ (\Phi^k_{\varphi,\psi})^{-1} = \Phi_{\varphi_1\circ\varphi^{-1},\psi_1\circ\psi^{-1}}$
- its range $\mathbf{im}\,\Phi^k_{\varphi,\psi}$ therefore is $J^k(\mathbf{im}\,\varphi, \mathbf{im}\,\psi)$.

**Examples and "derived products"** The manifold $J^0(M,N)$ is of course identified to $M \times N$ by the diffeomorphism $j^0_a f \mapsto (a, f(a))$.

The set of all $j^1_0 f \in J^1(\mathbb{R}, N)$ is a submanifold, the *tangent bundle* $TN$ of $N$: each natural chart $\Phi^1_{\mathrm{id}_\mathbb{R},\psi}$ is an adapted chart for $TN$ and restricts to the chart $T\psi : j^1_0\gamma \mapsto (\psi\circ\gamma(0), (\psi\circ\gamma)'(0))$; moreover, $J^1(\mathbb{R}, N)$ is identified to $\mathbb{R} \times TN$ by the map $j^1_t\gamma \mapsto (t, j^1_0(\gamma \circ \tau_{-t}))$, where $\tau_{-t}(x) = x + t$. One calls $j^1_0(\gamma \circ \tau_{-t})$ the *velocity* $\dot\gamma(t)$ of the path $\gamma$ at time $t$ (the knowledge of this velocity includes that of the position $\gamma(t)$, but not that of the time $t$).

Symmetrically, the set of all $j^1_a f \in J^1(M, \mathbb{R})$ with $f(a) = 0$ is a submanifold, the *cotangent bundle* $T^*M$ of $M$: each natural chart $\Phi^1_{\varphi,\mathrm{id}_\mathbb{R}}$ is an adapted chart for $T^*M$ and restricts to the chart $T^*\varphi : j^1_a f \mapsto \big(\varphi(a), D(f \circ \varphi^{-1})(\varphi(a))\big)$; moreover, $J^1(M,\mathbb{R})$ is identified to $T^*M \times \mathbb{R}$ by the map $j^1_a f \mapsto (j^1_a(\tau_{f(a)} \circ f), f(a))$. One calls $j^1_a(\tau_{f(a)} \circ f)$ the *differential* $d_a f$ of $f$ at $a$ (its knowledge includes that of $a$, but not of $f(a)$).

The natural charts endow $J^k(M,N)$ with much more than just a manifold structure, since the projections $j^k_a f \mapsto a$ ("source projection"), $j^k_a f \mapsto f(a)$ ("target projection") and $j^k_a f \mapsto j^\ell_a f$, $0 \le \ell < k$, are fibrations, as we shall now see.

## Submersions and fibrations

A map $\begin{array}{c} E \\ \downarrow \pi \\ B \end{array}$ between manifolds is a *submersion* when "it is locally *in* $E$ the projection onto the first factor of a product": for every $a \in E$, there exist an open subset $U$ of $\mathbb{R}^n$, an open subset $V$ of $\mathbb{R}^r$, a local chart $\tilde\varphi$ of $E$ at $a$ and a local chart $\varphi$ of $B$ at $\pi(a)$ such that $\mathbf{im}\,\tilde\varphi = U \times V$, $\mathbf{im}\,\varphi = U$ and $\varphi \circ \pi = \mathrm{pr}_1 \circ \tilde\varphi$, where $\mathrm{pr}_1 : U \times V \to U$ denotes the projection onto the first factor. One then calls $\tilde\varphi$ a *fibred chart* of the submersion *over* $\varphi$.

Similarly, $\pi$ is a *locally trivial fibration* when "it is locally *in* $B$ the projection onto the first factor of a product": for every $b \in B$, there exist a local chart $\varphi$ of $B$ at $b$, a manifold $F$ and a diffeomorphism $\tilde\varphi$ of $\pi^{-1}(\mathbf{dom}\,\varphi)$ onto $\mathbf{im}\,\varphi \times F$ such that $\varphi \circ \pi = \mathrm{pr}_1 \circ \tilde\varphi$, where $\mathrm{pr}_1 : \mathbf{im}\,\varphi \times F \to \mathbf{im}\,\varphi$ is the projection onto the first factor.

One can avoid the use of $\varphi$ via an equivalent definition: for every $b \in B$, there exist an open subset $\Omega \ni b$ of $B$ and a diffeomorphism $h$ of $\pi^{-1}(\Omega)$ onto $\Omega \times F$ such that $\pi|_{\pi^{-1}(\Omega)}$ is the first component of the *local trivialisation* $h$ of $\pi$.

Clearly (taking local charts of $F$) a fibration is a submersion and (by the very definition of a submanifold) the *fibres* $\pi^{-1}(b)$ of a submersion are submanifolds. When $\pi$ is a fibration, one calls $E$ (the *total space* of) a *fibre bundle over* $B$ (called its *base space*) with *projection* $\pi$.

When $F$ is an open subset of $\mathbb{R}^r$, the diffeomorphism $\tilde\varphi$ in the definition of a fibre bundle (which determines $\varphi$) is a chart of $E$. A *vector bundle* is defined by an atlas of such charts $\tilde\varphi$ with $F = \mathbb{R}^r$ (or a vector space), such that the transition maps $\tilde\varphi_1 \circ \tilde\varphi^{-1}$ are linear with respect to $F$ ("atlas of vector bundle"). Il follows that the fibres $E_b = \pi^{-1}(b)$ are endowed with a structure of vector space isomorphic to $F$. Replacing "linear" and "vector" by "affine", on gets the notion of an *affine bundle*, whose fibres are affine spaces.

**Sections** With the previous notation, a *smooth section of the submersion $\pi$ over the open subset $U$ of $B$* is a smooth map $\sigma$ of $U$ into $\pi^{-1}(U)$ such that $\pi \circ \sigma = \mathrm{id}_U$; if $U = B$, it is called a *section* of $\pi$. In the same way as a map is determined by its graph, a section is determined by its *image* $\sigma(U)$, that is a submanifold (it appears as a graph in the fibred charts $\tilde{\varphi}$). It is therefore natural—hence the terminology—to consider that a smooth section of $\pi$ over $U$ is a submanifold meeting each fibre of $\pi|_{\pi^{-1}(U)}$ at a unique point and *transversally* (see the sequel).

**The case of jets** Il is immediate that the projections $\pi_k^\ell : J^k(M,N) \to J^\ell(M,N)$ defined for $\ell \leq k$ by $\pi_k^\ell(j_a^k f) = j_a^\ell f$ are fibrations, whose typical fibre $F$ is the vector space $\prod_{\ell<j\leq k} L_s^j(\mathbb{R}^n, \mathbb{R}^p)$: just take $\tilde{\varphi} = \Phi_{\varphi,\psi}^k$ and $\varphi := \Phi_{\varphi,\psi}^\ell$ in the definition. Similarly, taking $\tilde{\varphi} = \Phi_{\varphi,\psi}^k$ and $\varphi = \varphi$ (resp. $\varphi := \psi$) in the definition of a submersion, one sees that the source projection $s_k : j_a^k f \to a$ and the target projection $b_k : j_a^k f \to f(a)$ are submersions. By the Faà di Bruno formula,

- this defines on $J^1(M,N)$ a vector bundle structure with base space $J^0(M,N) = M \times N$, projection $\pi_1^0$ and typical fibre $L(\mathbb{R}^n, \mathbb{R}^p)$
- thus the tangent bundle $TN$ is a vector bundle over $N$ with typical fibre $\mathbb{R}^p = L(\mathbb{R}, \mathbb{R}^p)$, and the cotangent bundle $T^*M$ a vector bundle over $M$ with typical fibre $\mathbb{R}^{n*} = L(\mathbb{R}^n, \mathbb{R})$
- for $k > 1$, the fibre bundle $J^k(M,N)$ is an *affine* bundle with typical fibre $L_s^k(\mathbb{R}^n, \mathbb{R}^p)$ over $J^{k-1}(M,N)$
- for $\ell < k \leq 2\ell + 1$, the space $J^k(M,N)$ is endowed by the charts $\Phi_{\varphi,\psi}^k$ with an *affine* bundle structure over $J^\ell(M,N)$
- such is not the case for $k > 2\ell+1$, the transition maps between natural charts being polynomial of degree at least 2 with respect to the typical fibre, *but*
- if $N$ is a vector space, $J^k(M,N)$ is endowed for $0 \leq \ell < k$ with a structure of affine bundle over $J^\ell(M,N)$ (*vector* bundle if $\ell = 0$) by the charts $\Phi_{\varphi,\mathrm{id}_N}^k$.

The fibre $T_aM$ of $TM$ over $a \in M$ is the *tangent space of $M$ at $a$.*

Though it is a *vector* space, it should be pictured genuinely tangent to $M$ at $a$ when $M$ is a submanifold of $\mathbb{R}^d$: indeed, $T_aM$ is obtained by looking at $M$ through a microscope centred at $a$, taken as the origin of the *affine* space $\mathbb{R}^d$.

The fibre $T_a^*M$ of $T^*M$ identifies naturally to the dual space $(T_aM)^*$, the duality form being $(\dot{\gamma}(a), d_a f) \mapsto (f \circ \gamma)'(a)$.

The source projection $s_k : j_a^k f \to a$ and the target projection $b_k : j_a^k f \to f(a)$ are in fact *fibrations*, whose typical fibres are respectively the set $J_0^k(\mathbb{R}^n, N)$ of all $j_0^k f \in J^k(\mathbb{R}^n, N)$ and the set $J^k(M, \mathbb{R}^p)_0$ of all $j_a^k f \in J^k(M, \mathbb{R}^p)$ with $f(a) = 0$.

The proof is the same as for the tangent and cotangent bundles: to each chart $\varphi$ of $M$ one can associate the diffeomorphism $\tilde{\varphi}$ of $s_k^{-1}(\mathbf{dom}\,\varphi)$ onto $\mathbf{im}\,\varphi \times J_0^k(\mathbb{R}^n, N)$ mapping $j_a^k f$ to $\big(\varphi(a), j_0^k(f \circ \varphi^{-1} \circ \tau_{-\varphi(a)})\big)$; similarly, to each chart $\psi$ of $N$ is associated the diffeomorphism $\tilde{\psi}$ of $b_k^{-1}(\mathbf{dom}\,\psi)$ onto $\mathbf{im}\,\psi \times J^k(M, \mathbb{R}^p)_0$ mapping $j_a^k f$ to $\big(\psi \circ f(a), j_a^k(\tau_{\psi \circ f(a)} \circ \psi \circ f)\big)$.

**Examples of sections** For every smooth map $f$ of an open subset $U$ of a manifold $M$ into a manifold $N$, the map $a \mapsto j_a^k f$ is a section $j^k f$ of the source projection $J^k(M,N) \to M$ over $U$, the $k^{th}$ *order jet of $f$*, clearly a section of the source projection $J^k(U,N) \to U$; such sections are called *holonomic*.

A section of the tangent bundle $TM \to M$ over $U$ is called a *vector field* on $U$ (at every point $a$ of $U$ one grows a vector $X_a \in T_aU = T_aM$).

For every smooth real function $f$ on an open subset $U$ of $M$, the map $df : a \mapsto d_a f$ is a section of the cotangent bundle $T^*M \to M$ over $U$ or, equivalently, a section of the cotangent bundle $T^*U \subset T^*M$; a section of the cotangent bundle $T^*U \to U$ is called a "field of covectors" or *Pfaffian form* (or *differential form of degree* 1, or *differential* 1*-form*, or 1*–form*) on $U$.

More generally to each smooth map $f : M \to N$ is associated the map $Tf$ of $TM$ in $TN$ defined by $Tf(\dot\gamma(a)) = \dot{\overline{f \circ \gamma}}(a)$; its restriction $T_a f$ to each fibre $T_a M$ is a *linear* map into $T_{f(a)}M$ ("linear map tangent to $f$ at $a$"): this is expressed by calling Tf a *homomorphism of vector bundles*.

Of course, $T_a f$ is identified to $j^1_a f$. In the seventies, some authors [11, 12] would replace for example $j^2 f$ by $T(Tf)$, but the ensuing inflation of dimensions and redondance are unreasonable.

**Infinitesimal characterisation of submersions, vertical and horizontal spaces and sections**

It follows easily from the inverse mapping theorem that a smooth map $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$ between manifolds is a submersion in the neighbourhood of $a \in E$ if and only if the tangent linear map $T_a\pi$ is onto; therefore, $\pi$ is a submersion if and only if $T_a\pi$ is onto for every $a \in E$.

For each $a \in E$, setting $b = \pi(a)$, the tangent space at $a$ to the fibre $\pi^{-1}(b)$ of the submersion $\pi$ is the kernel $\mathbf{ker}\, T_a\pi$; it is called the *vertical space* $\mathcal{V}_a$ of $\pi$ at $a$; in the case of a vector bundle, it therefore identifies to the vector space $E_b$; for an affine bundle, it is identified to the underlying vector space $\vec{E}_b$ of the fibre.

We can now characterise the smooth sections $\sigma$ of the submersion $\pi$ over an open subset $U$ of $B$ as submanifolds: they are the submanifolds $W$ of $\pi^{-1}(U)$ that meet each fibre $\pi^{-1}(b)$ with $b \in U$ at a unique point $a$, such that the tangent space $T_aW$ is *horizontal*, i. e., a complement in $T_aE$ of the vertical space $\mathcal{V}_a$; in other words, $\pi|_W$ is a diffeomorphism of $W$ onto $U$ and the corresponding section $\sigma$ is the composed map of $(\pi|_W)^{-1}$ and the inclusion $W \hookrightarrow \pi^{-1}(U)$.

**Remarks** In the case of the tangent bundle, one should therefore imagine the fibres $T_aM$ as vertical, transversal to $M$ (identified to the zero section). This somewhat contradicts the geometric intuition of submanifolds in $\mathbb{R}^d$, for which $T_aM$ lies along $M$, but one must understand that by identifying each $T_aM$ to the affine subspace so obtained, one gets a *very bad* representation of $TM$: in the case where $M$ is a curve in $\mathbb{R}^3$, for example, the surface of $\mathbb{R}^3$ so obtained admits $M$ as a cuspidal line at points where the curve is "truly spatial", i. e., with nonegative curvature and torsion, even though these are the least singular points of the surface lying in $M$.

Similarly, the geodesics of a surface $S$ in Euclidean space $\mathbb{R}^3$ are the parametrised curves $\gamma$ with values in $S$ whose acceleration $\gamma''(t)$ is *normal* to the surface for every $t$, whereas the second derivative $\ddot\gamma(t)$ is *horizontal* for the Levi-Civita connection (see the sequel). One has to get used to it...

Worse: the *rank* of a fibre bundle is the dimension of its fibre, i. e., the *corank* of its projection.

**More fibre bundles** The datum of a basis ("reference frame") $(e_1, \dots, e_n)$ of a real vector space $E$ is equivalent to that of the isomorphism $(x^1, \dots, x^n) \mapsto x^1e_1 + \cdots + x^ne_n$ of $\mathbb{R}^n$ onto $E$. An essential object, introduced (in a different language) by Élie Cartan, is the *frame bundle* of a manifold $M$ of dimension $n$, whose fibre over $a \in M$ is the set of (linear) *isomorphisms* $A_a$ of $\mathbb{R}^n$ onto $T_aM$; therefore, it is a dense open subset of the vector bundle over $M$ (generalising $TM$) consisting of all $j^1_0 f \in J^1(\mathbb{R}^n, M)$, and obviously a fibre bundle whose typical fibre is the linear group $GL_n(\mathbb{R})$ ($L_n$ in Ehresmann's notation): this can be seen by restricting the natural charts $\Phi_{\mathrm{id}_{\mathbb{R}^n},\varphi}$ of $J^1(\mathbb{R}^n, M)$.

This frame bundle, denoted by $\mathrm{Isom}(M \times \mathbb{R}^n, TM)$ in [12] (this is a little misleading, as it might make one believe that the sphere of dimension 2 is *parallelisable* in the sense given hereafter), is naturally endowed with the action $(B, A_a) \mapsto A_a \circ B^{-1}$ of $GL_n(\mathbb{R})$, which is free and transitive in each fibre: this is expressed by calling it a *principal bundle* with *structural group* $GL_n(\mathbb{R})$.

Ehresmann's "regular infinitesimal structures" are "principal subbundles of the frame bundle".

For example, the datum of a Riemannian metric on $M$ (i. e., a scalar product in each tangent space $T_aM$, depending smoothly on $a$ in the sense that the real function which to $v \in TM$ associates its scalar square is smooth) is equivalent to the datum of the subbundle of the frame bundle consisting of those $A_a$ which map the canonical basis of $\mathbb{R}^n$ to an orthonormal basis for the scalar product in $T_aM$. This is a principal bundle whose structural group is the orthogonal group $O_n$, the *orthonormal frame bundle* of the Riemannian manifold. The scalar product on $T_aM$ is the image of the standard Euclidean scalar product on $\mathbb{R}^n$ by any of of those "orthonormal frames" $A_a$.

Similarly, given a closed subgroup $H$ of $GL_n(\mathbb{R})$, the datum of a principal subbundle of the frame bundle, with structural group $H$, is equivalent to the datum, for each $a \in M$, of *one* of the frames $A_a$, the others being determined by the action of $H$. The "structure" preserved (or defined) by $H$ is then tranferred to $T_aM$ by any of the $A_a$'s.

If one wishes frames $A_a$ to depend smoothly on $a$, one must stay at the local level: otherwise, one would get an isomorphism of the trivial vector bundle $M \times \mathbb{R}^n$ onto $TM$, an isomorphism that does *not* exist [24] in the case of manifolds as respectable as the sphere of dimension 2: they are not *parallelisable*.

For each $A_a$, the $n$ components of $A_a^{-1}$ (coordinate functions in the frame $A_a$) are linear forms on $T_aM$; they constitute the "coframe" mentioned by Élie Cartan and Ehresmann; given a section of the frame bundle under study over the open $U$ of $M$, i. e., for each $a \in U$, the choice of *one* frame $A_a$ in the fibre, the components of $a \mapsto A_a^{-1}$ are therefore Pfaffian forms on $U$.

## Pfaffian systems and systems of partial differential (in)equations

The space $J^k(M,N)$ is not only a fibre bundle in many ways: for $k > 0$, it is also endowed with a canonical Pfaffian system, easy to understand when $M = \mathbb{R}^n$ and $N = \mathbb{R}^p$.

A section $\sigma$ of the source projection of $J^k(\mathbb{R}^n, \mathbb{R}^p) = \mathbb{R}^n \times \prod_0^k L_s^j(\mathbb{R}^n, \mathbb{R}^p)$ over an open subset $U$ of $\mathbb{R}^n$ is a map of $U$ into $J^k(\mathbb{R}^n, \mathbb{R}^p)$ that writes $\sigma(x) = (x, y_0(x), \ldots, y_k(x))$; clearly, it is holonomic (i. e. of the form $j^k f$) if and only if, modulo the canonical identification of $L\left(\mathbb{R}^n, L^j(\mathbb{R}^n, \mathbb{R}^p)\right)$ to $L^{j+1}(\mathbb{R}^n, \mathbb{R}^p)$ familiar in differential calculus, $Dy_j(x) = y_{j+1}(x)$ for $0 \le j < k$ for all $x \in U$.

Let us express this viewing $\sigma$ as the submanifold $W = \sigma(U)$: if one writes $z = (x, y_0, \ldots, y_k)$ the points of $J^k := J^k(\mathbb{R}^n, \mathbb{R}^p)$, the section is holonomic if and only if, at every point $z$ of $W$, the tangent space $T_zW$ (in other words, the image of $D\sigma(x)$) is contained in the subspace $\mathcal{K}_z^k = \mathcal{K}_z^k(\mathbb{R}^n, \mathbb{R}^p)$ of $T_zJ^k \simeq J^k$ defined by the equations

$$dy_j = y_{j+1}\, dx \quad \text{pour} \quad 0 \le j < k, \tag{1}$$

i. e. consisting of those vectors $\delta z = (\delta x, \delta y_0, \ldots, \delta y_k)$ such that, modulo the canonical identification just mentioned, $\delta y_j = y_{j+1}\, \delta x$ for $0 \le j < k$; here, $y_{j+1}\, \delta x$ is the interior product ("contraction") of $y_{j+1}$ by $\delta x$, i. e., the symmetric $j$–linear map $(\delta x_1, \ldots, \delta x_j) \mapsto y_{j+1}(\delta x, \delta x_1, \ldots, \delta x_j)$.

One calls (1) the *canonical Pfaffian system* or *Cartan system* (or *canonical contact structure*) of $J^k(\mathbb{R}^n, \mathbb{R}^p)$; equivalently, one can give the same name to the field of vector subspaces ("plane field") $z \mapsto \mathcal{K}_z^k$, that can be seen geometrically as the sub-vector bundle $\mathcal{K}^k = \mathcal{K}^k(\mathbb{R}^n, \mathbb{R}^p)$ of $TJ^k \simeq J^k \times J^k$ union of the subsets $\{z\} \times \mathcal{K}_z^k$.

One can see that, for each $z \in J^k$, the "plane" $\mathcal{K}_z^k$ is the closure[2] of the union of all $T_zW$ when $W$ varies among the holonomic sections through $z$; using the natural charts, this yields the following fact: given now two manifolds $M$ and $N$, one defines a *Pfaffian system* $\mathcal{K}^k(M,N)$ on $J^k(M,N)$, i. e. a sub-vector bundle of the tangent bundle $TJ^k(M,N)$, by the fact that its fibre over $z \in J^k(M,N)$ is the closure in $T_zJ^k(M,N)$ of the union of the tangent spaces at $z$ to holonomic sections through $z$. Naturally,

- it is called the *canonical Pfaffian system* or *Cartan system* (or *canonical contact structure*) of $J^k(M,N)$
- one has $T_z\Phi(\mathcal{K}_z(M,N)) = \mathcal{K}_{\Phi(z)}(\mathbb{R}^n, \mathbb{R}^p)$ for every natural chart $\Phi$ of $J^k(M,N)$ and every jet $z \in \mathbf{dom}\,\Phi$, implying that $\mathcal{K}^k(M,N)$ is indeed a sub-vector bundle of $TJ^k(M,N)$.

The reader has understood that a *Pfaffian system* on a manifold $V$ can be defined as a sub-vector bundle $\mathcal{P}$ of the tangent bundle $TV$.

In "real life", we are going to see that the notion can be more complicated: the manifold $V$ may have singular points, the dimension of the fibre $\mathcal{P}_z$ may vary at some points $z \in V$, etc.

An *integral manifold* of $\mathcal{P}$ is a submanifold $W$ of $V$ verifying $T_zW \subset \mathcal{P}_z$ for every $z \in W$; in this langage, a section of the source projection of $J^k(M,N)$ is holonomic if and only if, seen as a submanifold, it is an integral manifold of the Cartan system—which admits other integral manifolds, for example the fibres of the projection onto $J^{k-1}(M,N)$.

[2] One has to "catch" also the vertical vectors for the projection onto $J^{k-1}$.

**Example** If $\dim N = 1$, the Cartan system $\mathcal{K}^1(M,N)$ is a field of hyperplanes, authentic contact structure in today's restrictive sense, and its integral manifolds of dimension $n$ are called *Legendre submanifolds*, a terminology due to V.I. Arnold. In particular, (1) consists of one equation, and the Pfaffian form $\alpha = dy_0 - y_1\, dx$ on $J^1(\mathbb{R}^n, \mathbb{R})$ is a *contact form*, meaning that $d\alpha_z$ induces a nondegenerate bilinear form on $\mathcal{K}^1_z = \mathbf{ker}\, \alpha_z$; according to a theorem of Darboux [8], up to diffeomorphism, all contact forms in dimension $2n+1$ are *locally* equal to $\alpha$.

**Systems of partial differential equations** A system of $q$ partial differential equations of degree $k$ in $p$ unknown functions of $n$ variables writes in a condensed way $F(j^k_x y) = 0$, where $F$ is a map of an open subset of $J^k(\mathbb{R}^n, \mathbb{R}^p)$ into $\mathbb{R}^q$, the variable is $x \in \mathbb{R}^n$ and the unknown function $y$ (with values in $\mathbb{R}^p$). A solution $f$ of the system defined in an open subset of $\mathbb{R}^n$ is identified to $j^k f$, i. e. to a holonomic section of the source projection $J^k(\mathbb{R}^n, \mathbb{R}^p) \to \mathbb{R}^n$ over $U$ that takes its values in $E = F^{-1}(0)$ or, in other words, to an integral manifold of the canonical contact structure contained in $E$ and projecting diffeomorphically onto $U$.

A system of partial differential equations therefore identifies to a Pfaffian system, provided the name is given to the pair consisting of (1) and of the equation $F(z) = 0$. To use our first definition, one should take as a manifold $V$ the smooth part of $E$ (when $F$ is analytic, this makes sense) and as a Pfaffian system $\mathcal{P}_z := \mathcal{K}_z \cap T_z V$, a "fibre bundle" whose rank may have an unfortunate propension to jump (for example, if $k = n = p = q = 1$, it may well happen that $\mathcal{K}_z = T_z V$ at some points, which should be excluded from $V$ if one is looking for a genuine sub-vector bundle).

Of course, all this extends to the case where $E$ is a submanifold of codimension $q$ of $J^k(M,N)$, not necessarily defined globally by $q$ real equations.

For $k = p = q = 1$, it is fruitful to first forget the projection $J^1 \to J^0$ and consider the "geometric solutions" of the equation, i. e. the Legendre submanifolds contained in $E$, whether they are or not sections of the source projection. They sometimes have a physical meaning: for example, caustics are the projections into $J^0$ of such geometric solutions. This case, whose local theory goes back to the nineteenth century, still gives rise to new global developments.

**Systems of partial differential inequations** The spaces of jets also serve as the framework of the homotopy principle or $h$–principle [18], introduced by Gromov (following Thom [25]) in his thesis as an astounding abstraction of Smale's classification of immersions. The idea is dual to what has just been done: in the case of immersions of a manifold $M$ into a manifold $N$, one considers in $J^1(M,N)$ the open subset $\Omega$ consisting of jets of immersions, i. e. $j^1_a f$ such that $T_a f$ is injective. Given two immersions $f_0, f_1$ of $M$ into $N$, the question is whether they are *regularly homotopic*, i. e. whether there exists a smooth path $[0,1] \ni t \mapsto f_t$ joining them *in the space of immersions*; in other words, one wonders whether there exists a path of holonomic sections $j^1 f_t$ of $J^1(M,N) \to M$ joining $j^1 f_0$ to $j^1 f_1$ and such that all these sections take their values in $\Omega$. Naturally, the same problem can be posed for various subsets $\Omega$ of various $J^k(M,N)$'s; the *homotopy principle* (when it is true) states that the question admits a positive answer if and only if this is the case *forgetting the contact structure but not the source projection*, meaning that one can join the two holonomic sections by a path in the set of not necessarily holonomic sections with values in $\Omega$. With time, this has become astonishingly simple [15], back to Thom in fact (see Laudenbach's comment of [25] in [26]).

## Connections

Here again, Ehresmann did a good job. The problem is that a submersion $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$ does not allow even locally the unique lifting of paths, except when it is a local diffeomorphism at every point (in which case, if it is a fibration, one calls it a *covering*): if $\tilde\varphi$ is a fibred chart of $\pi$, with image $U \times V$, over a chart $\varphi$ of $B$, then, for every path $\gamma$ with values in $\mathbf{dom}\, \varphi$, any path $\tilde\gamma$ with values in $\mathbf{dom}\, \tilde\varphi$ of the form $\tilde\gamma(t) = \tilde\varphi^{-1}(\varphi \circ \gamma(t), f(t))$ with $\mathbf{dom}\, \tilde\gamma = \mathbf{dom}\, \gamma$ is a *lift(ing)* of $\gamma$, meaning that $\pi \circ \tilde\gamma = \gamma$;

therefore, even if one imposes to $\tilde{\gamma}$ a given value $a \in \pi^{-1}(\gamma(t_0))$ for $t = t_0$, there are many possible choices $f$, none of which is *a priori* better than the others. The datum of a connection suppresses this indeterminacy and provides (at least locally) a unique lifting $\tilde{\gamma}$ of $\gamma$ such that $\tilde{\gamma}(t_0) = a$.

For example, if $E$ is the frame bundle of $B$ (or a principal subbundle), a connection allows one to obtain along $\gamma$ a *moving frame* $\tilde{\gamma}(t)$, well determined by its value at $t_0$. If the connection is better than the others, so will be this moving frame.

**Definition** A *connection* on the submersion $\pi$ is a *field of horizontal spaces*, i. e., a Pfaffian system $\mathcal{H}$ on $E$ such that $\mathcal{H}_a$ is, for every $a \in E$, a complementary subspace in $T_aE$ of the vertical space $\mathcal{V}_a = \mathbf{ker}\, T_a\pi = T_a\left(\pi^{-1}(a)\right)$; in other words, $T_a\pi|_{\mathcal{H}_a}$ is an isomorphism onto $T_{\pi(a)}B$.

The datum of $\mathcal{H}_a$ is equivalent to that of the projection of $T_aE$ onto $\mathcal{V}_a$ parallel to $\mathcal{H}_a$, chosen by Dieudonné [12] to define a connection; it can be denoted $\mathbf{v} \mapsto \mathbf{v}_V$ (*vertical component* of the tangent vector $\mathbf{v}$). The unique lifting ("horizontal lifting") $\tilde{\gamma}$ announced will be defined by the initial condition and by the fact that the derivative $\dot{\tilde{\gamma}}(t)$ is *horizontal* for every $t$, which writes (notation of [23])

$$\frac{D\tilde{\gamma}}{dt} := \dot{\tilde{\gamma}}(t)_V = 0. \tag{2}$$

Indeed, the connection $\mathcal{H}$ "reads" as follows in a fibred chart $\tilde{\varphi}$ of $E$ over $\varphi$, with image the open $U \times V$ of $\mathbb{R}^n \times \mathbb{R}^r$: for every $a \in \mathbf{dom}\,\tilde{\varphi}$, if $\tilde{\varphi}(a) = (x, y)$, the image of $\mathcal{H}_a$ by $T_a\tilde{\varphi}$ is the graph of a linear map $-\Gamma(x, y)$ of $\mathbb{R}^n$ into $\mathbb{R}^r$: one defines in that way the *Christoffel map* $\Gamma : U \times V \to L(\mathbb{R}^n, \mathbb{R}^r)$ of the connection $\mathcal{H}$ in the fibred chart $\tilde{\varphi}$, and it is smooth because $\mathcal{H}$ is; the equation $\mathbf{v}_V = 0$ expressing that $\mathbf{v} \in TE$ is horizontal therefore writes $\delta y + \Gamma(x, y)\delta x = 0$, where $((x, y), (\delta x, \delta y)) = T\tilde{\varphi}(\mathbf{v})$. Hence, if $\gamma$ is a path in $\mathbf{dom}\,\varphi$ and $x(t) := \varphi \circ \gamma(t)$, a lifting $\tilde{\gamma}(t) = \tilde{\varphi}^{-1}(x(t), y(t))$ of $\gamma$ with values in $\mathbf{dom}\,\tilde{\varphi}$ is horizontal if and only if the path $t \mapsto y(t)$ verifies the differential equation

$$y'(t) + \Gamma(x(t), y(t))x'(t) = 0$$

expressing (2); this enables one to use Cauchy's theorem on differential equations to obtain the local existence and uniqueness of the lifting $\tilde{\gamma}$ taking a given value at time $t_0$.

Its *global* existence is ensured for example when $\pi$ is *proper*, i. e., when $\pi^{-1}(K)$ is compact for every compact $K$ of $B$: indeed, in that case, the solution $\tilde{\gamma}$ of (2) can not "go to infinity" at time $t \in \mathbf{dom}\,\gamma$. Let us deduce from this a fundamental result in differential topology:

**Theorem (Ehresmann)** *If the submersion $\pi$ is proper, then it is a fibration.*

*Proof* For every $b \in B$, there exist an open subset $\Omega \ni b$ of $B$ and a connection $\mathcal{H}$ on $\pi|_{\pi^{-1}(\Omega)}$: to see it, cover the compact manifold $\pi^{-1}(b)$ by the domains of finitely many fibred charts $\tilde{\varphi}_j$ and take $\Omega = \bigcap \mathbf{dom}\,\varphi_j$, where the $\varphi_j$'s are the charts of $B$ defined by the $\tilde{\varphi}_j$'s; restricting the $\tilde{\varphi}_j$'s, we may assume $\mathbf{dom}\,\varphi_j = \Omega$ for every $j$, so that the $\mathbf{dom}\,\tilde{\varphi}_j$'s form a finite cover of $\pi^{-1}(\Omega)$ and that there exists [11] a smooth partition of unity $\theta_j$ subordinate to this cover; for each $j$, there is a connection $\mathcal{H}_j$ on $\pi|_{\mathbf{dom}\,\tilde{\varphi}_j}$, for example that whose Christoffel map in the fibred chart $\tilde{\varphi}_j$ is identically zero; denoting by $\mathbf{v} \mapsto \mathbf{v}_{j,V}$ the corresponding projection, one can then take the connection $\mathcal{H}$ whose projection $T_aE \to \mathcal{V}_a$ is defined by $\mathbf{v}_V := \sum_j \theta_j(a)\mathbf{v}_{j,V}$ for each $a \in \pi^{-1}(\Omega)$ (as usual, the sum is on those $j$'s such that $a \in \mathbf{dom}\,\tilde{\varphi}_j$).

Restricting $\Omega$, one may assume that there exists a chart $\varphi$ of $B$ with $\mathbf{dom}\,\varphi = \Omega$ such that $\varphi(\Omega)$ is an open ball of centre $0 = \varphi(b)$ in $\mathbb{R}^n$. For each $y \in \Omega$, one therefore defines a path $\gamma_y : [0,1] \to \Omega$ joining $y$ to $b$ by $\gamma_y(t) := \varphi^{-1}\big((1-t)\varphi(y)\big)$; for all $x \in \pi^{-1}(y)$, the path $\gamma_y$ admits a unique horizontal lift $\tilde{\gamma}_x : [0,1] \to E$ such that $\tilde{\gamma}_x(0) = x$, and the map $x \mapsto \tilde{\gamma}_x(1)$ of $\pi^{-1}(y)$ in $\pi^{-1}(b)$, called *parallel transport from time* $0$ *to time* $1$ *along the path* $\gamma_y$ *for the connection* $\mathcal{H}$, is obviously bijective (its inverse is obtained by lifting $t \mapsto \gamma_y(1-t)$); as solutions of differential equations depend smoothy on initial conditions and parameters, the map $x \mapsto \tilde{\gamma}_x(1)$ is a diffeomorphism, and so is the map $h$ of $\pi^{-1}(\Omega)$ onto $\Omega \times \pi^{-1}(b)$ given by $h(x) := \big(\pi(x), \tilde{\gamma}_x(1)\big)$, that is the required local trivialisation.

**Remarks** Conversely, a fibration with compact fibres is obviously proper. As in the definition of a fibration, if one wants the typical fibre to be unique up to diffeomorphism, $B$ must be assumed connected.

This very robust theorem holds, with the same proof, in the Banach framework. Proceeding as in the first part of the proof, one can see that a submersion defined on a paracompact manifold (as in real life) admits a connection, which can be used in the second part of the proof, $\Omega$ being the domain of any chart $\varphi$ vanishing at $b$ whose image is a ball.

**An example** The contact structure $\mathcal{K}^1(M,\mathbb{R})$ is a connection for the fibration $\pi : j^1_a f \mapsto d_a f$ of $J^1(M,\mathbb{R})$ onto $T^*M$. We shall go back to it in the section on curvature.

## Integral of differential forms, pullbacks, exterior derivative

**Direct images of paths, curvilinear integral, pullback of functions and 1-forms** Let $g$ be a smooth map of a manifold $M$ into a manifold $N$.

The (direct) *image* under $g$ of a path $\gamma$ in $M$ is the path $g_*\gamma := g \circ \gamma$ in $N$; similarly, the *inverse image* (or *pullback*) by $g$ of a real function $f$ on $N$ is the real function $g^*f := f \circ g$ on $M$.

When $\gamma$ is defined on a segment $[t_0, t_1]$ ($\gamma$ is then called an *arc*), the (curvilinear) *integral along* $\gamma$ *of a Pfaffian form* $\alpha$ on $M$ is by definition

$$\int_\gamma \alpha := \int_{t_0}^{t_1} \alpha_{\gamma(t)}(\dot\gamma(t))\, dt,$$

where $\alpha_{\gamma(t)} \in T^*_{\gamma(t)}M = (T_{\gamma(t)}M)^*$ denotes the value of $\alpha$ at $\gamma(t)$. This integral is invariant under parameter changes : if $\varphi : [s_0, s_1] \to [t_0, t_1]$ verifies $\varphi(s_j) = t_j$, then $\int_{\gamma\circ\varphi} \alpha = \int_\gamma \alpha$; when $\alpha$ is the differential $df$ of a real function $f$ on $M$, since $df_{\gamma(t)}(\dot\gamma(t)) = (f \circ \gamma)'(t)$,

$$\int_\gamma df = f(\gamma(t_1)) - f(\gamma(t_0)) \quad \text{(mean value formula).} \tag{3}$$

A Pfaffian form $\alpha$ is determined by the integrals $\int_\gamma \alpha$.

Indeed, for every $x \in M$ and every $\mathbf{v} \in T_xM$, there exists an arc $\gamma : [0,1] \to M$ such that $\dot\gamma(0) = \mathbf{v}$ (take a chart $\varphi$ of $M$ such that $\varphi(a) = 0$ and a path of the form $\varphi \circ \gamma(t) = \theta(t\,\varphi_*\mathbf{v})\, t\, \varphi_*\mathbf{v}$, where $\varphi_*\mathbf{v} = T_x\varphi(\mathbf{v})$ and $\theta : \mathbf{im}\,\varphi \to [0,1]$ is $C^\infty$ with compact support, equal to 1 near 0). If $\gamma_\varepsilon : [0,1] \to M$ is given by $\gamma_\varepsilon(t) := \gamma(\varepsilon t)$, then $\lim_{\varepsilon\to 0} \varepsilon^{-1} \int_{\gamma_\varepsilon} \alpha = \lim_{\varepsilon\to 0} \int_0^1 \alpha_{\gamma(\varepsilon t)}\big(\dot\gamma(\varepsilon t)\big)\, dt = \alpha_{\gamma(0)}\big(\dot\gamma(0)\big) = \alpha_x\mathbf{v}$.

The *pullback by* $g$ *of a Pfaffian form* $\beta$ on $N$ is the Pfaffian form $g^*\beta$ on $M$ such that $\int_\gamma g^*\beta = \int_{g_*\gamma} \beta$ for every arc $\gamma$ in $M$; it is given by the formula

$$(g^*\beta)_x = \beta_{g(x)} \circ T_x g.$$

For $f : M \to \mathbb{R}$, the chain rule, in intrinsic terms

$$T(f \circ g) = (Tf) \circ Tg,$$

therefore writes $g^*df = d(g^*f)$.

**Differential forms, their integral on parametrised rectangles and their pullbacks** A *differential form of degree* $k$ or *differential* $k$*-form*, or $k$*-form* $\alpha$ on a manifold $M$ is a field of *alternate* $k$-linear forms $\alpha_x : (T_xM)^k \to \mathbb{R}$, i. e., a smooth section of the vector bundle $\bigwedge^k T^*M$ over $M$ whose fibre over $x \in M$ is the space $L^k_{\text{alt}}(T_xM,\mathbb{R})$ of alternate $k$-linear forms on $T_xM$; an atlas of this vector bundle consists (naturally) of the *natural charts* $\bigwedge^k T^*\varphi : \alpha_x \mapsto (\varphi(x), (T_x\varphi)_*\,\alpha_x) \in \mathbf{im}\,\varphi \times L^k_{\text{alt}}(\mathbb{R}^n,\mathbb{R})$, where $\varphi$ is a chart of $M$ with values in $\mathbb{R}^n$ (the linear tangent map $T_x\varphi$ therefore maps $T_xM$ onto $T_{\varphi(x)}\mathbb{R}^n = \mathbb{R}^n$), $\alpha_x \in L^k_{\text{alt}}(T_xM,\mathbb{R})$ and $(T_x\varphi)_*\,\alpha_x(\mathbf{v}_1,\dots,\mathbf{v}_k) := \alpha_x\left((T_x\varphi)^{-1}\mathbf{v}_1,\dots,(T_x\varphi)^{-1}\mathbf{v}_k\right)$ for $\mathbf{v}_1,\dots,\mathbf{v}_k \in \mathbb{R}^n$.

For every smooth map $\rho : [0,1]^k \to M$, the *integral* of $\alpha$ along the *parametrised rectangle* $\rho$ of dimension $k$ is by definition

$$\int_\rho \alpha := \int_{[0,1]^k} \alpha_{\rho(t)}(\partial_1\rho(t),\dots,\partial_k\rho(t))\, dt$$

(integral with respect to Lebesgue measure), where $\partial_j\rho(t) \in T_{\rho(t)}M$ is the partial derivative of $\rho$ with respect to the $j^{\text{th}}$ factor and $\alpha_{\rho(t)} \in L^k_{\text{alt}}(T_{\rho(t)}M,\mathbb{R})$ denotes the value of $\alpha$ at $\rho(t)$.

A $k$-form $\alpha$ is determined by the integrals $\int_\rho \alpha$.

Indeed, for $x \in M$ and $\mathbf{v}_1,\dots,\mathbf{v}_k \in T_xM$, there exists (same proof as for $k=1$, replacing $t\varphi_*\mathbf{v}$ by $\sum t_j\varphi_*\mathbf{v}_j$) a parametrised rectangle $\rho : [0,1]^k \to M$ such that $\partial_j\rho(0) = \mathbf{v}_j$ for every $j$; if $\rho_\varepsilon : [0,1]^k \to M$ is given for $0 < \varepsilon \le 1$ by $\rho_\varepsilon(t) := \rho(\varepsilon t)$, then $\lim_{\varepsilon\to 0} \varepsilon^{-k} \int_{\rho_\varepsilon} \alpha = \alpha_x(\mathbf{v}_1,\dots,\mathbf{v}_k)$ as for $k=1$.

Given a smooth map $g : M \to N$ between manifolds, the *pullback by $g$ of a $k$-form $\beta$* on $N$ is the $k$-form $g^*\beta$ on $M$ such that $\int_\rho g^*\beta = \int_{g_*\rho} \beta$ for every parametrised rectangle $\rho$ of dimension $k$ in $M$, using the notation $g_*\rho := g \circ \rho$; it is given by the formula

$$(g^*\beta)_x = \beta_{g(x)} \circ (T_x g)^k,$$

where $(T_x g)^k(\mathbf{v}_1, \dots, \mathbf{v}_k) := (T_x g(\mathbf{v}_1), \dots, T_x g(\mathbf{v}_k))$ for $\mathbf{v}_1, \dots, \mathbf{v}_k \in T_x M$.

**The exterior derivative** That of a Pfaffian form $\alpha$ on $M$ is the 2-form $d\alpha$ on $M$ such that

$$\int_\rho d\alpha = \int_{\partial\rho} \alpha \tag{4}$$

for every $C^2$ parametrised rectangle $\rho : [0,1]^2 \to M$, where $\partial\rho$ denotes the *oriented boundary* of $\rho$, obtained by concatenation of the paths $[0,1] \ni s \mapsto \rho(s,0)$, $[0,1] \ni s \mapsto \rho(1,s)$, $[0,1] \ni s \mapsto \rho(1-s,1)$ and $[0,1] \ni s \mapsto \rho(0,1-s)$; it is given par

$$d\alpha_{\rho(t)}(\partial_1\rho(t), \partial_2\rho(t)) = \partial_1(\alpha_{\rho(t)}\partial_2\rho(t)) - \partial_2(\alpha_{\rho(t)}\partial_1\rho(t)). \tag{5}$$

More generally, for each $k \geq 1$, the *exterior derivative of a $k$-form* $\alpha$ on $M$ is the $(k+1)$-form $d\alpha$ on $M$ verifying (4) for every parametrised rectangle $\rho$ of dimension $k+1$, setting

$$\int_{\partial\rho} \alpha := \sum_{i=1}^{k+1} (-1)^{i+1} \Big( \int_{\partial\rho_i^1} \alpha - \int_{\partial\rho_i^0} \alpha \Big),$$

where the "faces" $\partial\rho_i^j$ of $\rho$ are the parametrised rectangles of dimension $k$ defined by

$$\partial\rho_i^j(s) := \rho((s_\ell)_{\ell<i}, j, (s_\ell)_{\ell\geq i}),\ s = (s_1, \dots, s_k) \in [0,1]^k,\ j = 0,1;$$

the identity (5) is the particular case $k = 1$ of the formula

$$d\alpha_{\rho(t)}(\partial_1\rho(t), \dots, \partial_{k+1}\rho(t)) = \sum_{i=1}^{k+1} (-1)^{i+1} \partial_i \left( \alpha_{\rho(t)}\Big( (\partial_\ell\rho(t))_{\ell<i}, (\partial_\ell\rho(t))_{\ell>i} \Big) \right), \tag{6}$$

valid when $\rho$ is a $C^2$ map with values in $M$ defined on an open subset or an "open subset with corners" of $\mathbb{R}^k$, for example $[0,1]^k$.

This formula follows from (4), the mean value formula and the Fubini theorem. indeed, if one alleviates notation by setting for example $\alpha\big(\partial_1\rho(t), \dots, \partial_k\rho(t)\big) := \alpha_{\rho(t)}\big(\partial_1\rho(t), \dots, \partial_k\rho(t)\big)$, then

$$\begin{aligned} \int_{\partial\rho_i^1} \alpha - \int_{\partial\rho_i^0} \alpha &= \int_{[0,1]^k} \left( \alpha\Big( \partial_{s_j}\rho\big((s_\ell)_{\ell<i}, 1, (s_\ell)_{\ell\geq i}\big) \Big)_{1\leq j\leq k} - \alpha\Big( \partial_{s_j}\rho\big((s_\ell)_{\ell<i}, 0, (s_\ell)_{\ell\geq i}\big) \Big)_{1\leq j\leq k} \right) ds \\ &= \int_{[0,1]^k} \int_0^1 \partial_\tau \alpha\Big( \partial_{s_j}\rho\big((s_\ell)_{\ell<i}, \tau, (s_\ell)_{\ell\geq i}\big) \Big)_{1\leq j\leq k} d\tau\, ds \\ &= \int_{[0,1]^{k+1}} \partial_i \alpha\Big( \big(\partial_\ell\rho(t)\big)_{\ell<i}, \big(\partial_\ell\rho(t)\big)_{\ell>i} \Big)\, dt \end{aligned}$$

where $t := \big((s_\ell)_{\ell<i}, \tau, (s_\ell)_{\ell\geq i}\big)$. Naturally, the "miracle" is that the right-hand side of (6) depends only on the $\partial_j\rho(t)$'s: this can be checked in a chart, which reduces the problem to the case where $M$ is an open subset $U$ of $\mathbb{R}^n$, and using the fact that, then, $\partial_i\partial_\ell\rho = \partial_\ell\partial_i\rho$. Indeed, in that case, $\alpha$ is identified to a map of $U$ into $L^k_{\text{alt}}(\mathbb{R}^n, \mathbb{R})$ (its second component) and $d\alpha : U \to L^{k+1}_{\text{alt}}(\mathbb{R}^n, \mathbb{R})$ is given by $d\alpha(x)(\mathbf{v}_1, \dots, \mathbf{v}_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} D\alpha(x)(\mathbf{v}_i)\big((\mathbf{v}_\ell)_{\ell<i}, (\mathbf{v}_\ell)_{\ell>i}\big)$, $x \in U$, $\mathbf{v}_1, \dots, \mathbf{v}_{k+1} \in \mathbb{R}^n$.

This definition of the exterior derivative is not too intrinsic, but it shows that a $k$-form is meant to be integrated on objects of dimension $k$, exterior derivation appearing as the dual ("coboundary") of the "oriented boundary" $\partial$ via the Stokes formula (4)—which generalises (3) and yields easily the other "Stokes formulae". Whitney even *constructed* the theory of differential forms out of it [29].

It follows at once from the definitions of the pullback and the exterior derivative that

$$d(g^*\beta) = g^* d\beta \tag{7}$$

for every smooth map $g : M \to N$ between manifolds and every differential form $\beta$ on $N$.

Moreover, for every differential $k$-form $\alpha$ on $M$,

$$dd\alpha = 0. \tag{8}$$

Indeed, the integral of $dd\alpha$ on every parametrised rectangle $\rho : [0,1]^{k+2} \to M$ is zero since by definition $\int_\rho dd\alpha = \sum_{i=1}^{k+2}(-1)^{i+1}\Big(\int_{\partial\rho_i^1} d\alpha - \int_{\partial\rho_i^0} d\alpha\Big) = \sum_{i=1}^{k+2}(-1)^{i+1}\Big(\int_{\partial\partial\rho_i^1} \alpha - \int_{\partial\partial\rho_i^0} \alpha\Big)$, in other words

$$\int_\rho dd\alpha = \sum_{i=1}^{k+2}(-1)^{i+1}\sum_{j=1}^{k+1}(-1)^{j+1}\Big(\int_{\partial(\partial\rho_i^1)_j^1}\alpha - \int_{\partial(\partial\rho_i^1)_j^0}\alpha - \int_{\partial(\partial\rho_i^0)_j^1}\alpha + \int_{\partial(\partial\rho_i^0)_j^0}\alpha\Big),$$

a sum where "each face of dimension $k$ of $\rho$ appears twice and with opposite signs " as

$$\partial(\partial\rho_i^\ell)_j^m = \partial(\partial\rho_j^m)_{i-1}^\ell, \quad 1 \le j < i \le k+2, \quad \ell, m \in \{0,1\}.$$

If $k = 1$, these faces correspond to the edges of the cube $[0,1]^3$.

A differential form $\beta$ is *closed* when $d\beta = 0$; it is *exact* when it is the exterior derivative $\beta = d\alpha$ of a differential form, called a *primitive* of $\beta$ and obviously unique up to the addition of a closed form (when one adds two sections $\alpha$ and $\beta$ of a vector bundle $E$, it is of course fibrewise addition, i. e., $(\alpha+\beta)(x) = \alpha(x) + \beta(x)$ in $E_x$); the formula (8) therefore means that *every exact form is closed.*

## Flows, Lie derivative and Lie bracket

**Flows and Lie derivative** To every smooth vector field $X$ on the manifold $M$ is associated its *flow* or *one-parameter* (pseudo)*group* $g_X^t$, defined as follows: for every $a \in M$, the map $t \mapsto g_X^t(a)$ is the path in $M$ that is the *maximal* solution of the differential equation $\dot{x} = X(x)$ ("integral curve of $X$ ") passing through $a$ at time $t = 0$. Here, *maximal* means "defined on an *interval* as large as possible."

**Note** As the integral curves are *parametrised,* they are not merely one-dimensional integral manifolds.

By the theory of differential equations, the definition domain of $g_X : (t,a) \mapsto g_X^t(a)$ is an open subset of $\mathbb{R} \times M$ and $g_X$ is as smooth as $X$; clearly, $g_X^s(g_X^t(a)) = g_X^{s+t}(a)$ when the left-hand side makes sense or, equivalently, for $a \in \mathbf{dom}(g_X^t) \cap \mathbf{dom}(g_X^{s+t})$; in particular, since $g_X^0 = \mathrm{id}_M$, each $g_X^t$ is a diffeomorphism of the open subset $\mathbf{dom}\, g_X^t \subset M$ onto the open subset $\mathbf{dom}\, g_X^{-t}$, and $(g_X^t)^{-1} = g_X^{-t}$.

If $X$ has compact support, the solutions of $\dot{x} = X(x)$ cannot "go to infinity in finite time"; therefore, $\mathbf{dom}\, g_X = \mathbb{R} \times M$ and $g_X$ is a smooth action of the additive group $\mathbb{R}$ on $M$, meaning that $t \mapsto g_X^t$ is a homomorphism of $\mathbb{R}$ into the group of diffeomorphisms of $M$ onto itself; in that case, $X$ (or its flow) is said to be *complete.*

The *Lie derivative* of a tensor field $\tau$ on $M$ (here, a differential form of degree $k$ or, as a little further, a vector field) with respect to $X$ is by definition

$$\mathcal{L}_X\tau := \frac{d}{dt} g_X^{t\,*}\tau\Big|_{t=0}, \tag{9}$$

that is a tensor field of the same nature as $\tau$; for example, the Lie derivative of a real function $f$ on $M$ is the real function on $M$ which is the (interior) *product* or *contraction* $df(X)$ of $df$ by $X$:

$$\mathcal{L}_X f = df(X) : x \mapsto d_x f(X_x).$$

For $k > 0$, the Lie derivative of a differential $k$-form $\alpha$ on $M$ verifies the *Cartan formula*

$$\mathcal{L}_X\alpha = d(\alpha X) + (d\alpha)X, \tag{10}$$

where $\alpha X$ and $(d\alpha)X$ denote the *interior products* (or *contractions*) $x \mapsto \alpha_x X_x$ and $x \mapsto (d\alpha_x)X_x$ of $\alpha$ and $d\alpha$ by $X$, a notation introduced when we wrote the Cartan system of $J^k(\mathbb{R}^n, \mathbb{R}^p)$.

Though Élie undoubtfully knew and used the Cartan formula [4], it took some time for the Lie derivative—as for many primitive notions—to be recognised as such and it is Henri who wrote (10) under this form. One can, if one really wants to, take it as an intrinsic but incomprehensible definition of the exterior derivative.

Its proof is very easy: for all $x \in M$ and $(\mathbf{v}_1, \ldots, \mathbf{v}_k) \in T_xM$, there exists $\rho_1 : (\mathbb{R}^k, 0) \to (M, x)$ such that $\mathbf{v}_j = \partial_j \rho_1(0)$, and one can take $\rho(t) := g_X^{t_1} \circ \rho_1(t_2, \ldots, t_{k+1})$ and $t = 0$ in (6). Here is an important application:

**Poincaré lemma** *Every closed differential form $\alpha$ of degree $k \geq 1$ on $M$ is* locally *exact: each $a \in M$ has an open neighbourhood $\Omega$ such that $\alpha|_\Omega$ is exact.*

Indeed, if $\Omega$ is the domain of a chart $\varphi$ vanishing at $a$ whose image is a ball $B$ of $\mathbb{R}^n$, let $X$ be the vector field on $\Omega$ that is the pullback by $\varphi$ of the radial field $Y_y := y$ on $B$; for every $x \in \Omega$, the points $g_X^t(x) = \varphi^{-1}\big(e^t \varphi(x)\big)$ with $t \leq 0$ are well defined and, by (10), since $d\alpha = 0$,

$$\begin{aligned} \alpha_x &= (g_X^{0*}\alpha)_x = (g_X^{0*}\alpha)_x - \lim_{t \to -\infty} (g_X^{t*}\alpha)_x = \int_{-\infty}^0 \frac{d}{dt}(g_X^{t*}\alpha)_x \, dt = \int_{-\infty}^0 (g_X^{t*}\mathcal{L}_X\alpha)_x \, dt \\ &= \int_{-\infty}^0 \big(g_X^{t*} d(\alpha X)\big)_x \, dt = \int_{-\infty}^0 d\left(g_X^{t*}(\alpha X)\right)_x dt = \left(d\int_{-\infty}^0 g_X^{t*}(\alpha X)\, dt\right)_x , \end{aligned}$$

where the last integral is in each fibre (one can find it more secure to work in the chart $\varphi$ and take as variable $s = e^t$).

**The de Rham cohomology** For $k > 0$, the quotient of the vector space of closed forms of degree $k$ on $M$ by the vector space of exact forms of degree $k$ is the $k^{\text{th}}$ de Rham cohomology space $H^k(M, \mathbb{R})$; as every alternate $k$-linear form on a space of dimension $< k$ is zero, $H^k(M, \mathbb{R}) = \{0\}$ for $k > \dim M$; one denotes by $H^0(M, \mathbb{R})$ the space of locally constant functions on $M$ and $H^\bullet(M, \mathbb{R}) := \bigoplus_{k \geq 0} H^k(M, \mathbb{R})$.

**Pullback of vector fields, Lie brackets** Given a smooth map $h : M \to N$ between manifolds, a *pullback* of a vector field $Y$ on $N$ by $h$, if it exists, is a vector field $X$ on $M$ such that $h$ "maps the integral curves of $X$ onto those of $Y$ ", meaning that $h \circ g_X^t = g_Y^t \circ h$; as this relation holds for $t = 0$, it is equivalent to the one obtained by derivating it with respect to time, which writes $T_xh(X_x) = Y_{h(x)}$ for every $x \in X$; one therefore sees that if $h$ is *etale*, i. e., if all the $T_xh$'s are isomorphisms, then $Y$ has a unique pullback by $h$, denoted by $h^*Y$ and given by the formula

$$(h^*Y)_x = (T_xh)^{-1} Y_{h(x)}.$$

The formula (9) therefore has a meaning when $\tau$ is a vector field $Y$ on $M$, and

$$\mathcal{L}_X Y = [X, Y]$$

is the *Lie bracket of the vector fields $X$ and $Y$*, such that

$$\mathcal{L}_{[X,Y]} f = \mathcal{L}_X \mathcal{L}_Y f - \mathcal{L}_Y \mathcal{L}_X f$$

for every real function $f$ on $M$ ("derivation of a product": $\mathcal{L}_X \mathcal{L}_Y f = \mathcal{L}_{\mathcal{L}_X Y} f + \mathcal{L}_Y \mathcal{L}_X f$).

The *Jacobi identity* $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ follows, making the $C^\infty$ vector fields on $M$ an archetypical Lie algebra.

By the formula for the derivation of a product and (9), for every choice of the function $f$, the tensor field $\tau$, the vector fields $X, Y$ and the differential form $\alpha$ of degree $k > 0$ on $M$, one has

$$\begin{aligned} \mathcal{L}_X(f\tau) &= (\mathcal{L}_X f)\tau + f\mathcal{L}_X \tau \\ \mathcal{L}_X(\alpha Y) &= (\mathcal{L}_X \alpha)Y + \alpha \mathcal{L}_X Y \\ &= d(\alpha X)Y + (d\alpha)XY + \alpha[X, Y]. \end{aligned} \tag{11}$$

If $\varphi$ is a chart of $M$ with values in $\mathbb{R}^n$, setting $X_\varphi(x) := T_x\varphi(X_{\varphi^{-1}(x)}) \in T_x\mathbb{R}^n = \mathbb{R}^n$ for every vector field $X$ on $M$ and every $x \in \mathbf{im}\,\varphi$, one has

$$[X, Y]_\varphi(x) = DY_\varphi(x)X_\varphi(x) - DX_\varphi(x)Y_\varphi(x). \tag{12}$$

## Some applications of the Cartan formula

**Infinitesimal contact transformations** Let $\alpha$ be a *contact form* on a manifold $V$—recall that this means that $T_xM = \mathbf{ker}\,\alpha_x \oplus \mathbf{ker}\,d\alpha_x$ for every $x \in V$; let $\mathcal{K}$ be the associated *contact structure* $\mathcal{K}_x := \mathbf{ker}\,\alpha_x$. An *infinitesimal contact transformation* or *Lie field* for $\mathcal{K}$ is a vector field $X$ on $V$ whose flow $g^t := g^t_X$ preserves $\mathcal{K}$, meaning that $T_x g^t(\mathcal{K}_x) = \mathcal{K}_{g^t(x)}$ for every $(t,x) \in \mathbf{dom}\,g_X$: this is expressed by calling the maps $g^t$ *contact transformations* or (local) *automorphisms* of $\mathcal{K}$.

**Theorem (Libermann)** *Under these hypotheses, a Lie field $X$ is determined by its* Hamiltonian *$-\alpha X$ with respect to $\alpha$, and every $C^2$ real function $F$ on $V$ is the Hamiltonian of a $C^1$ Lie field $X_F$. In particular, if $\alpha$ is $C^\infty$, the map $F \mapsto X_F$ is an isomorphism of $C^\infty(V,\mathbb{R})$ onto the space of $C^\infty$ Lie fields for $\mathcal{K}$, an isomorphism whose inverse is $X \mapsto -\alpha X$.*

Indeed, $X$ is a Lie field if and only if its flow $g^t$ verifies $(g^{t*}\alpha)_x = \mu_t(x)\alpha_x$ for every $x \in \mathbf{dom}\,g^t$, which (after derivation with respect to $t$) writes $\mathcal{L}_X\alpha = \lambda\alpha$, where $\lambda$ is a real function on $V$; by (10), the relations between $X$ and $F := -\alpha X$ are therefore expressed for each $x \in V$ by the two equations

$$-\alpha_x X_x = F(x) \qquad (13)$$

$$-d_xF + d\alpha_x\, X_x = \lambda(x)\alpha_x; \qquad (14)$$

if $X_x = Y_x + Z_x$ in the decomposition $T_xV = \mathcal{K}_x \oplus \mathbf{ker}\,d\alpha_x$, (13) determines $Z_x$ knowing $F(x)$ and vice versa since $\alpha_x|_{\mathbf{ker}\,d\alpha_x}$ is an isomorphism; as for (14), it writes

$$\begin{aligned} -d_xF|_{\mathbf{ker}\,d\alpha_x} &= \lambda(x)\alpha_x|_{\mathbf{ker}\,d\alpha_x} \\ d_xF|_{\mathcal{K}_x} &= (d\alpha_x\, Y_x)|_{\mathcal{K}_x}; \end{aligned}$$

the first equation determines $\lambda(x)$ knowing $d_xF|_{\mathbf{ker}\,d\alpha_x}$ and vice versa, and the second yields $Y_x$ knowing $d_xF|_{\mathcal{K}_x}$ and vice versa, as the nondegenerate bilinear form $d\alpha_x|_{(\mathcal{K}_x)^2}$ induces the isomorphism $\mathbf{v} \mapsto (d\alpha_x\mathbf{v})|_{\mathcal{K}_x}$ of $\mathcal{K}_x$ onto its dual.

Having always [9] attributed this result to Sophus Lie, I nearly asked who was that Bermann the first time it was rightly [20] credited to Paulette Libermann in my presence. It implies that the group of automorphisms of $\mathcal{K}$ is huge, the vector fields $X_F$ with $F$ compactly supported being complete.

**Application: local theory of first order partial differential equations** Under these hypotheses, given $F : V \to \mathbb{R}$, let $E := F^{-1}(0)$. Two preliminary observations:

i) as there is no nondegenerate alternate bilinear form on a space of odd dimension, $V$ is of odd dimension $2n+1$

ii) an integral manifold $W$ of $\mathcal{K}$ is of dimension at most $n$; indeed, if $\iota : W \hookrightarrow V$ is the inclusion, the relation $\iota^*\alpha = 0$ expressing that $W$ is integral implies that $\iota^*d\alpha = d(\iota^*\alpha) = 0$, i. e., that each tangent space $T_xW$ is included in its orthogonal for the nondegenerate bilinear form $d\alpha_x|_{(\mathcal{K}_x)^2}$, hence $\dim T_xW \le 2n - \dim T_xW$; the integral manifolds of dimension $n$ are the *Legendre manifolds* of $\mathcal{K}$.

For every $x \in E$,

iii) the previous proof shows that $X = X_F$ vanishes at $x$ if $d_xF = 0$, since then $Y_x = Z_x = 0$

iv) it follows from (13)–(14) and from the antisymmetry of $d\alpha_x$ that $d_xF(X_x) = 0$; hence, $X_F$ is tangent at $x$ to $E$ for $d_xF \neq 0$ ($F$ is a submersion in an open neighbourhood $U$ of $x$, therefore $U \cap E$ is a submanifold of codimension 1 with tangent space $\mathbf{ker}\,d_xF$ at $x$)

v) it follows from (13) that $X_x$ belongs to $\mathcal{K}_x$.

Assertions (iii)–(iv) imply that one has $g^t_X(E \cap \mathbf{dom}\,g^t_X) \subset E$ for every $t$; assertion (v), together with the fact that the maps $g^t_X$ preserve $\mathcal{K}$, therefore yields the following facts:

vi) for every integral manifold $W_0 \subset E$ of $\mathcal{K}$ and every $a \in W_0$ with $X_a \notin T_aW_0$, there exists an open subset $\Omega \ni (0,a)$ of $\mathbb{R} \times W_0$ such that the map $j : \Omega \to E$ defined by $j(t,x) := g^t_X(x)$ is a diffeomorphism onto an integral manifold $W$ of $\mathcal{K}$, which therefore verifies $\dim W = \dim W_0 + 1$

vii) this imposes $\dim W_0 < n$ by (ii); hence, a *geometric solution* of the *generalised partial differential equation* $E$, i. e., a Legendre manifold $L$ contained in $E$, verifies $X_x \in T_xL$ for every $x \in L$

viii) if $\dim W_0 = n-1$ (one then calls $(E, W_0)$ a *generalised Cauchy problem, well-posed at $a$*), then $W$ is a geometric solution of $E$

ix) conversely, by (vii), every geometric solution $W$ of $E$ is obtained in this fashion in the neighbourhood of each $a \in W$ where $X_a$ is nonzero (just take for $W_0$ a hypersurface of $W$ passing through $a$ with $X_a \notin T_a W_0$); this proves the local existence and uniqueness of the solution of a generalised Cauchy problem.

If $V = J^1(\mathbb{R}^n, \mathbb{R})$, $\mathcal{K} = \mathcal{K}^1(\mathbb{R}^n, \mathbb{R})$ and, denoting by $(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1}$ the points of $\mathbb{R}^n$, the equation $E$ is of the form $\partial_t y = g(t, x, y, \partial_x y)$, a well-posed classical Cauchy problem is the datum of the value $y_0(x)$ of the unknown function for $t = 0$; this does determine the generalised Cauchy datum $W_0 = \big\{(0, x, y_0(x), g(0, j^1_x y_0), Dy_0(x))\big\} \subset E$, which defines at each of its points a well-posed problem whose local generalised solutions $W$ are holonomic sections of the source projection $J^1(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n$ contained in $E$, jets order 1 of the local solutions of the Cauchy problem.

**Hamiltonian vector fields on a symplectic manifold** Paulette Libermann is not foreign [20] to their intrinsic definition. A symplectic manifold is the pair consisting of a manifold $V$ and a *symplectic form* on $V$, i. e., a *closed* 2-form $\omega$ such that every $\omega_x \in L^2_{\mathrm{alt}}(T_x V, \mathbb{R})$ is *nondegenerate* (the dimension of $V$ must therefore be even). A vector field $X$ on $V$ is *symplectic* when its flow $g^t = g^t_X$ preserves $\omega$, meaning that $g^{t\,*}\omega = \omega$ in $\mathbf{dom}\, g^t$ for every $t$ (the maps $g^t$ are therefore *symplectic tranformations* of $\omega$). As this relation is verified if $t = 0$, this amounts to saying that $0 = \frac{d}{dt} g^{t\,*}\omega = g^{t*}\mathcal{L}_X \omega$ for every $t$, i. e., that $\mathcal{L}_X \omega = 0$; since $\omega$ is closed, it follows from (10) that this is the case if and only if the Pfaffian form $\omega X$ is *closed.*

When it is exact, $\omega X = dH$, one says that $X$ is *Hamiltonian* and that the function $H$ is a *Hamiltonian* of $X$; il determines $X$, and each real function $H$ on $V$ is the Hamiltonian of a unique Hamiltonian vector field $X_H$: indeed, for each $x \in V$, the equation $\omega_x \mathbf{v} = d_x H$ has a unique solution $\mathbf{v} \in T_x V$ since $\omega_x$ is nondegenerate. The group of (global) symplectic transformations of $\omega$ therefore is huge too, since it contains the maps $g^t_{X_H}$ with $H$ compactly supported.

As $\mathcal{L}_{X_H} H = dH(X_H) = \omega(X_H, X_H) = 0$, the flow of $X = X_H$ preserves $H$, meaning that $H(g^t_X(x)) = H(x)$ for every $(t, x) \in \mathbf{dom}\, g_X$ ("conservation of energy"); one also calls $H$ a *first integral* of $X_H$. Since $\mathcal{L}_{X_H} K = dK(X_H) = \omega(X_K, X_H) = -\mathcal{L}_{X_K} H$ for all real functions $H$ and $K$ on $V$, the *Poisson bracket* $\{H, K\} := \mathcal{L}_{X_H} K$ ("Poisson parentheses") is antisymmetric; this yields the (trivial but quite useful) Hamiltonian version of a theorem by Emmy Noether: if "$X_K$ is an infinitesimal symmetry of $H$ ", meaning that $H$ is a first integral of $X_K$, then $K$ is a first integral of $X_H$. The Poisson bracket lifts to functions the Lie bracket of vector fields in the sense that $X_{\{H,K\}} = [X_H, X_K]$; it (therefore) satisfies the Jacobi identity, endowing $C^\infty(V, \mathbb{R})$ with a Lie algebra structure if $\omega$ is $C^\infty$.

Similarly, if $\alpha$ is a contact form, the Lie bracket of Lie fields can be lifted to real functions by (the inverse of) the isomorphism $X \mapsto \alpha X$, the bracket so obtained being called the *Lagrange bracket*, it seems.

In the "concrete" case studied since Lagrange at least [21], $V$ is the cotangent bundle ("phase space") $T^*M$ of a manifold ("configuration space") $M$, endowed with its *canonical symplectic structure* $\omega_M$, unique 2-form on $T^*M$ whose pullback by the projection $J^1(M, \mathbb{R}) \to T^*M$ is the exterior derivative of the canonical contact form $dy_0 - y_1\, dx$ defining $\mathcal{K}^1(M, \mathbb{R})$.

## Curvature

**Curvature of a connection** If $\mathcal{H}$ is a connection on a submersion $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$, every vector field $X$ on an open subset $U \subset B$ lifts to a unique *horizontal* vector field $\tilde{X}$ on $\pi^{-1}(U)$, given by $\tilde{X}_a = (T_a\pi|_{\mathcal{H}_a})^{-1} X_{\pi(a)}$. A remarkable fact of Nature is that, if $Y$ is another vector field on $U$, the vertical component $([\tilde{X}, \tilde{Y}]_a)_V$ of the Lie bracket $[\tilde{X}, \tilde{Y}]$, at each point $a \in \pi(U)$, depends only on $\tilde{X}_a, \tilde{Y}_a \in \mathcal{H}_a$, i. e., on $X_{\pi(a)}, Y_{\pi(a)} \in T_{\pi(a)} B$; hence one defines an alternate bilinear map $R_a : T_{\pi(a)} B \times T_{\pi(a)} B \to \mathcal{V}_a$, the *curvature tensor of $\mathcal{H}$ at $a$*, by the formula

$$R_a(X_{\pi(a)}, Y_{\pi(a)}) := ([\tilde{X}, \tilde{Y}]_a)_V. \tag{15}$$

If $\Gamma$ is the Christoffel map of $\mathcal{H}$ in a fibred chart $\tilde{\varphi}$ of $\pi$ over the chart $\varphi$ of $B$, it follows from (12) that

$$R_a(\mathbf{v}_1, \mathbf{v}_2) = D\Gamma(z)(\mathbf{x}_2, -\Gamma(z)\mathbf{x}_2)\mathbf{x}_1 - D\Gamma(z)(\mathbf{x}_1, -\Gamma(z)\mathbf{x}_1)\mathbf{x}_2\,, \quad \text{where} \quad \begin{cases} z := \tilde{\varphi}(a) \\ \mathbf{x}_j := T_{\pi(a)}\varphi(\mathbf{v}_j), \end{cases} \tag{16}$$

which proves our "fact of Nature" (see the next paragraph for a nicer argument).

When $E$ is a vector bundle over $B$, the identification of $\mathcal{V}_a$ to the fibre $E_{\pi(a)}$ makes $R_a$ into an element of $L^2_{\text{alt}}(T_{\pi(a)}B, E_{\pi(a)})$; in particular, if $E = TB$, one is in the perhaps more familiar situation where $R_a$ takes its values in $T_{\pi(a)}B$.

For a general vector bundle, when $\mathcal{H}$ is *linear*, i. e., when the parallel transport from one time to another along any path is (which amounts to saying that the Christoffel maps $\Gamma(x, y)$ in the charts of the vector bundle are linear in $y$), it follows from (16) that the curvature $R_a$ depends linearly of $a$ viewed as an element of $E_{\pi(a)}$; setting $b = \pi(a)$, $R_a(\mathbf{v}, \mathbf{w})$ therefore is the value at $(a, \mathbf{v}, \mathbf{w}) \in E_b \times T_bB \times T_bB$ of a *trilinear* map $R_b$ with values in $E_b$; if $E = TB$, the familiar monster of Riemannian geometry [23] is the *quadrilinear* form $(T_bB)^4 \ni (a, \mathbf{v}, \mathbf{w}, \mathbf{h}) \mapsto R_b(a, \mathbf{v}, \mathbf{w}) \cdot \mathbf{h}$ (scalar product).

If $E$ is an affine bundle, $R_a$ takes its values in the vector space $\vec{E}_{\pi(a)}$ underlying the fibre. More generally, when $E$ is a principal bundle with structural group $G$, the datum of $a$ enables one to identify $E_{\pi(a)}$ to $G$ by the inverse of the bijection $G \ni g \mapsto ga$, and therefore identify $\mathcal{V}_a$ to the *Lie algebra* of $G$ (the tangent space $\mathfrak{g} := T_1G$ of $G$ at 1) by the inverse of the differential at 1 of the previous bijection; in this identification, one therefore has $R_a \in L^2_{\text{alt}}(T_{\pi(a)}B, \mathfrak{g})$.

**"Curvature" of a Pfaffian system** If $\mathcal{P}$ is a Pfaffian system on a manifold $V$ (that is, a sub-vector bundle of the tangent bundle $TV$, the stupid cases of $TV$ and its zero section being excluded), one can replace in the previous construction the "concrete" vertical space $\mathcal{V}_a$ by its "abstract" version

$$\nu\mathcal{P}_a := T_aV/\mathcal{P}_a$$

(which defines a vector bundle $\nu\mathcal{P}$ over $V$, the *normal bundle* of $\mathcal{P}$) and denote by $\mathbf{v} \mapsto \mathbf{v}_\nu$ the canonical projection $T_aV \to \nu\mathcal{P}_a$. The previous fact of Nature generalises: one defines the *"curvature tensor"* $R_a \in L^2_{\text{alt}}(\mathcal{P}_a, \nu\mathcal{P}_a)$ *of the Pfaffian system* $\mathcal{P}$ *at* $a \in V$ by the formula

$$R_a(X_a, Y_a) := ([X, Y]_a)_\nu, \tag{17}$$

where $X, Y$ vary among the *sections of the vector bundle* $\mathcal{P}$ over open subsets $U \ni a$ of $V$ (vector fields on $U$ verifying $X_x, Y_x \in \mathcal{P}_x$ or, equivalently, $(X_x)_\nu = (Y_x)_\nu = 0$ for every $x$).

To prove our "fact of Nature", one can consider locally $\mathcal{P}$ as a connection (see the proof of the Frobenius theorem hereafter) and use (16) or, in a more elegant way, remark that if one multiplies for example $Y$ by a real function $f$ defined near $a$, (11) yields $[X, fY]_a = f(a)[X, Y]_a + \mathcal{L}_X f(a)Y_a$ and therefore $([X, fY]_a)_\nu = f(a)([X, Y]_a)_\nu$ since $(Y_a)_\nu = 0$, hence $([X, fY]_a)_\nu = ([X, Y]_a)_\nu$ if $f(a) = 1$.

**Proposition** *For every integral manifold* $W$ *of* $\mathcal{P}$*, the curvature tensor* $R_a$ *is identically zero on* $T_aW \times T_aW$ *for all* $a \in W$*.*

Indeed, if $X, Y$ are vector fields on a neighbourhood of $a$ in $W$, it is easy to extend them locally to sections $\bar{X}, \bar{Y}$ of $\mathcal{P}$ defined in the neighbourhood of $a$ in $V$; by definition, $\bar{X}_a = X_a$, $\bar{Y}_a = Y_a$ and, moreover, $[\bar{X}, \bar{Y}]_a = [X, Y]_a \in T_aW \subset \mathcal{P}_a$ since, near $a$, the flow of $\bar{X}$ coincides on $W$ with that of $X$. It follows that $R_a(X_a, Y_a) = R_a(\bar{X}_a, \bar{Y}_a) = ([\bar{X}, \bar{Y}]_a)_\nu = ([X, Y]_a)_\nu = 0$, hence the proposition since $(X_a, Y_a)$ can be any pair of vectors tangent to $W$ at $a$.

**Definition** An *integral element* of $\mathcal{P}$ at $a \in V$ is a plausible candidate to be the tangent space at $a$ of an integral manifold of $\mathcal{P}$, i. e., a vector subspace $I_a$ of $\mathcal{P}_a$ such that $R_a|_{I_a \times I_a} = 0$.

The *Cartan-Kähler theorem* for Pfaffian systems [7, 12, 1, 22] asserts that, in the analytic case, every "generic" integral element $I_a$ of $\mathcal{P}$ is indeed of the form $I_a = T_aW$ for at least one (analytic) integral manifold $W$ of $\mathcal{P}$. This statement is more Cartan than Kähler [2]; it is astounding that Élie Cartan, from three examples, could have the idea of so general a result and *see* how to "corner" the required integral manifold. Here are two extreme examples where this general result is not needed.

**Example 1. Completely integrable Pfaffian systems** They are those Pfaffian system $\mathcal{P}$ such that $R_a = 0$ for every $a \in V$ (in other words, $\mathcal{P}_a$ is an integral element). For example, the Pfaffian system $\mathcal{V}$ defined by the vertical spaces of a submersion is completely integrable (and completely integrated, the fibres being integral manifolds). A completely integrable connection is sometimes said to be *flat* since its curvature is everywhere zero.

**Frobenius theorem** *If a Pfaffian system $\mathcal{P}$ on $V$ is completely integrable, there does exist, for every $a \in V$, an integral manifold $W$ of $\mathcal{P}$ such that $T_aW = \mathcal{P}_a$* (hence, for dimensional reasons, $T_xW = \mathcal{P}_x$ for every $x \in W$ if $W$ is connected)*; moreover, this integral manifold is* locally unique*: if $W'$ is another one, there exists an open neighbourhood $U$ of $a$ in $V$ such that $W \cap U = W' \cap U$* (in words, $W$ and $W'$ have the same *germ*[3] at $a$)*.*

Hence, the relation "there exists a connected integral manifold of $\mathcal{P}$ containing $a$ and $a'$ " between points $a, a'$ of $V$ is an equivalence relation, whose equivalence classes are called the *leaves* of the *foliation* of $V$ defined by $\mathcal{P}$; they inherit from their definition a structure of connected manifold (injectively immersed) of the same dimension as the $\mathcal{P}_a$'s, but they are not (embedded) submanifolds in general. Even for $\dim \mathcal{P}_a = 1$ ("line field", always completely integrable since the $R_a$'s are alternate), the global study of foliations is a very difficult subject to which, after Ehresmann and Reeb, contributed Haefliger, Bott, Novikov, Thurston among others and, in the case of line fields, all the great names of dynamical systems since Poincaré. Indeed, the theory includes the study of the *orbits* of a vector field $X$ on $V$ (considering the line field $x \mapsto \mathbb{R}X_x$ on the open subset of $V$ where $X$ does not vanish), which are the images of its integral curves.

*Local structure of the foliation defined by a completely integrable Pfaffian system* For every $a \in V$, there exist open subsets $U \subset \mathbb{R}^n$, $U' \subset \mathbb{R}^p$ and a chart ("plaque family") $\psi$ of $V$ with $a \in \mathbf{dom}\,\psi$ and $\mathbf{im}\,\psi = U \times U'$ such that the leaves of the foliation of $\mathbf{dom}\,\psi$ defined by $\mathcal{P}$ are the subsets $\psi^{-1}(U \times \{y_0\})$ with $y_0 \in U'$; each of these local leaves ("plaques") is obviously contained in one of the leaves of the global foliation, but this global leaf can come back and cut $\mathbf{dom}\,\psi$ following other plaques, whose union can even be dense in $\mathbf{dom}\,\psi$: for example, if $\alpha$ is an irrationnal number, all the orbits of the constant vector field $X_x := (1, \alpha) \in \mathbb{R}^2 = T_x\mathbb{T}^2$ on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ are dense.

*Proof à la Dieudonné [10] of the Frobenius theorem and of the existence of plaque families* Let $\varphi$ be an arbitrary chart of $V$ at $a$; composing it with a translation and a permutation of coordinates, one can assume that it takes its values in $\mathbb{R}^n \times \mathbb{R}^p$, that $\varphi(a) = 0$ and that $T_a\varphi(\mathcal{P}_a)$ is horizontal, i. e., complementary of the vertical space $\{0\} \times \mathbb{R}^p$ of the projection $\pi : (x, y) \mapsto x$. Restricting $\mathbf{dom}\,\varphi$, it follows that *all* the spaces $\mathcal{H}_{\varphi(z)} := T_z\varphi(\mathcal{P}_z)$ are horizontal; therefore, there exists a Christoffel map $\Gamma : \mathbf{im}\,\varphi \to L(\mathbb{R}^n, \mathbb{R}^p)$, such that $\mathcal{H}_{(x,y)}$ is the graph of $-\Gamma(x, y)$ for every $(x, y) \in \mathbf{im}\,\varphi$. The integral manifolds of maximal dimension of $\mathcal{P}$ in $\mathbf{dom}\,\varphi$ are the images by $\varphi^{-1}$ of those of the connection $\mathcal{H}$ so defined, which integral manifolds are *locally* the graphs of solutions $y = f(x)$ of the "total differential equation"

$$\frac{dy}{dx} + \Gamma(x, y) = 0; \tag{18}$$

if such a solution $f$ takes the value $y_0$ at 0, then, for every $x \in \mathbb{R}^n$ such that the segment $[0, x]$ is contained in $\mathbf{dom}\,f$, it follows that $f(tx)$ is for $0 \le t \le 1$ the value $R^t(x, y_0)$ at time $t$ of the solution of the differential equation $\dfrac{dy}{dt} + \Gamma(tx, y)x = 0$ equal to $y_0$ at $t = 0$. As $R^t(x, y_0)$ exists for every $t$ if $x = 0$, the theory of differential equations [8] tells us that there are open balls $U \subset \mathbb{R}^n$ and $U' \subset \mathbb{R}^p$ centred at 0 such that, for $x \in U$, the map $y_0 \mapsto R^1(x, y_0)$ is a diffeomorphism of $U'$ onto an open subset of $\mathbb{R}^p$; in other words, $h : (x, y_0) \mapsto \big(x, R^1(x, y_0)\big)$ is a diffeomorphism of $U \times U'$ onto an open subset of $U \times \mathbb{R}^p$.

We now just have to check that, for every $y_0 \in U'$, the unique candidate $f : x \mapsto R^1(x, y_0)$ to be in $U$ the solution of (18) equal to $y_0$ for $x = 0$ is indeed a solution of (18): one will get the plaque family $\psi := h^{-1} \circ \varphi$ and, for $y_0 = 0$, the Frobenius theorem. Now, derivating with respect to $x$ the identity $\frac{\partial}{\partial t} f(tx) + \Gamma\big(tx, f(tx)\big)x = 0$ and using (16), one can see that $t \mapsto tDf(tx)$ and $t \mapsto -t\Gamma\big(tx, f(tx)\big)$ verify the same differential equation on $[0, 1]$ and take the same value $0 \in L(\mathbb{R}^n, \mathbb{R}^p)$ at $t = 0$; therefore, they are equal, hence the required result for $t = 1$.

**Remarks** For line fields, this is just the theory of "time dependent" differential equations. The construction performed in general (before the final verification, which uses curvature) is a local version of the proof of Ehresmann's theorem. The vanishing of curvature is imposed by the symmetry of the second derivative of solutions of (18). Dieudonné's proof works in infinite dimensions.

[3] In the beginning, Ehresmann used the word *jet*, little recommendable in this case except in the analytic framework.

**Example 2. Fields of hyperplanes and contact structures** *If $\alpha$ is a nowhere vanishing Pfaffian form on $V$ and $\mathcal{K}_z := \mathbf{ker}\,\alpha_z$, the curvature at $z$ of the Pfaffian system $\mathcal{K}$ identifies to $-d\alpha_z|_{\mathcal{K}_z}$ by the isomorphism of $\nu\mathcal{K}_z = T_zV/\mathcal{K}_z$ onto $\mathbb{R}$ induced by $\alpha_z$.*

Indeed, for all local sections $X, Y$ of the vector bundle $\mathcal{K}$ in the neighbourhood of $z$, one has $\alpha X = \alpha Y = 0$, and therefore $\alpha_z[X_z, Y_z] = -d\alpha_z(X_z, Y_z)$ by (11).

A contact structure therefore is "completely non integrable", its curvature being at every point a nondegenerate bilinear form.

The canonical contact structure $\mathcal{K} = \mathcal{K}^1(M,\mathbb{R})$ of $J^1(M,\mathbb{R})$ is a connection on the trivial fibre bundle $J^1(M,\mathbb{R}) = T^*M \times \mathbb{R}$ over $T^*M$; therefore, it has an intrinsic "Christoffel map": denoting the points of $T^*M$ by $x = (q,p)$ ($p \in T^*_qM$), as in mechanics, and by $z = (q,p,y)$ those of $J^1(M,\mathbb{R})$, each $\mathcal{K}_z$ is defined by the equation $dy = p\,dq$; hence, it is the graph of the linear form $p\,dq$ on $T_x(T^*M)$; the Pfaffian form $\lambda = \lambda_M$ on $T^*M$ given by $\lambda_x = p\,dq$ is called the *Liouville form* of $T^*M$.

*The curvature of the connection $\mathcal{K}^1(M,\mathbb{R})$ on the trivial fibre bundle $J^1(M,\mathbb{R}) = T^*M \times \mathbb{R}$ over $T^*M$ identifies therefore to the 2-form $d\lambda_M$ on $T^*M$: one obtains again the canonical symplectic form $\omega_M = -d\lambda_M$ of $T^*M$.*

**Remarks** To obtain Hamilton's equations under their historical form, one has the choice between our sign conventions and those of [21], according to which $\omega_M = d\lambda_M$ and $\omega_M X_H = -dH$.

Every etale map $g$ between open subsets of $M$ lifts to the map $T^*g$ of $T^*\,\mathbf{dom}\,g$ onto $T^*\,\mathbf{im}\,g$ given by $T^*g(q,p) := \big(g(q), p\circ(T_qg)^{-1}\big)$, which is obviously symplectic (it preserves the Liouville form); if $X$ is a vector field on $M$, each $T^*g^t_X$ is the time $t$ of the flow of the *Hamiltonian* vector field with Hamiltonian $K(q,p) = pX_q$; the first integrals of classical mechanics obtained by applying the "Hamiltonian Noether theorem" are in general such $K$'s.

Given a Pfaffian system $\mathcal{P}$ on $V$, let $\mathcal{P}^\perp$ be the sub-vector bundle of $T^*V$ whose fibre over $x$ consists of those $\xi \in T^*_xV$ which vanish on $\mathcal{P}_x$. For each $a \in V$, there exist $r$ sections $\alpha_1, \dots, \alpha_r$ of $\mathcal{P}^\perp$ over an open subset $U \ni a$ such that $\alpha(x) := \big(\alpha_1(x), \dots, \alpha_r(x)\big)$ is a basis of $\mathcal{P}^\perp_x$ for every $x \in U$; in other words, $\alpha(x)$ induces an isomorphism of $\nu\mathcal{P}_x$ onto $\mathbb{R}^r$ that, as for $r = 1$, identifies $R_x$ to $-d\alpha(x)|_{\mathcal{P}^2_x} = -\big(d\alpha_1(x), \dots, d\alpha_r(x)\big)|_{\mathcal{P}^2_x} \in L_{\mathrm{alt}}(\mathcal{P}_x, \mathbb{R}^r)$.

It follows from Thom's transversality lemma that "almost every" Pfaffian form on a manifold of odd dimension is a contact form off a smooth hypersurface, see for example [9]; likewise, the exterior derivative of "almost every" Pfaffian form on a manifold $M$ of even dimension is symplectic off a hypersurface, necessarily nonempty if $M$ is compact without boundary.

In contrast, it is clear that, apart from those defined by a submersion and line fields, completely integrable Pfaffian systems *almost never* occur. Why devote so many efforts to such improbable objects? An answer is that they appear in a rather robust way (despite a certain loss of regularity under perturbations) in the case of the stable and unstable foliations of an Anosov diffeomorphism—hence, it seems, Novikov's initial interest in the subject; another answer, very present in Élie Cartan's work, is that the most symmetric objects often are the most beautiful and the most useful; here is an illustration, assuming some knowledge of de Rham cohomology:

**The "Gauss-Manin" connection associated to a proper submersion, and monodromy** One can associate to every proper submersion $\begin{matrix} E \\ \downarrow \pi \\ B \end{matrix}$ the vector bundle $H^\bullet E$ over $B$ on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ whose fibre over $b$ is the cohomology space $H^\bullet(E_b, \mathbb{K})$. To see that it is indeed a vector bundle endowed with a canonical flat linear connection $\mathcal{H}$, we are going to construct (assuming $E$ paracompact…) a vector bundle atlas $\{\tilde\varphi\}_{\varphi\in\Phi}$ such that, denoting by $\mathcal{H}_\varphi$ the linear flat connection on $\mathbf{dom}\,\tilde\varphi$ whose Christoffel map in the chart $\tilde\varphi$ is[4] $\Gamma = 0$, the connections $\mathcal{H}_\varphi$ and $\mathcal{H}_\psi$, coincide on $\mathbf{dom}\,\tilde\psi \cap \mathbf{dom}\,\tilde\varphi$ for $\varphi, \psi \in \Phi$; therefore, the local connections $\mathcal{H}_\varphi$ do define a global flat linear connection $\mathcal{H}$ on $H^\bullet E$.

In this construction, $\Phi$ is the atlas of $B$ consisting of those charts whose image is an open ball centred at 0 in $\mathbb{R}^n$. A connection on $\pi$ being chosen, the proof of Ehresmann's theorem shows that there exists for each $\varphi \in \Phi$ a trivialisation $h_\varphi : \pi^{-1}(\mathbf{dom}\,\varphi) \to \mathbf{dom}\,\varphi \times F_\varphi$ of $\pi$ over $\mathbf{dom}\,\varphi$; for $b \in \mathbf{dom}\,\varphi$, the canonical injection $i_b : E_b \hookrightarrow \pi^{-1}(\mathbf{dom}\,\varphi)$ induces an *isomorphism* $i^*_b$ of $H^\bullet(\pi^{-1}(\mathbf{dom}\,\varphi), \mathbb{K})$ onto $H^\bullet(E_b, \mathbb{K})$, which "reads" modulo $h_\varphi$ as the isomorphism $j^*_b$ of $H^\bullet(\mathbf{dom}\,\varphi \times F_\varphi, \mathbb{K})$ onto $H^\bullet(\{b\} \times F_\varphi, \mathbb{K}\big)$ associated to the inclusion $j_b : \{b\} \times F_\varphi \hookrightarrow \mathbf{dom}\,\varphi \times F_\varphi$ [the inverse isomorphism is $p^*_b$, where $p_b(x,y) := (b,y)$, as every closed differential form $\alpha$ on $\mathbf{dom}\,\varphi \times F_\varphi$ such that $j^*_b\alpha = 0$ is

[4]More simply, $\tilde\varphi$ is a plaque family of the foliation defined by $\mathcal{H}_\varphi$, which therefore is born "integrated"; by the way, Élie Cartan named *infinitesimal connection* what we call a connection; the problem is to "connect" two nearby fibres $E_b, E_{b'}$—for (infinitesimal) connections with nonzero curvature, however, the result depends, even locally, on the arc from $b$ to $b'$ along which parallel transport is taken.

exact: to see it, just apply our proof of Poincaré's lemma to the vector field $X$ on $\mathbf{dom}\,\varphi \times F_\varphi$ whose image by $\varphi \times \mathrm{id}_{F_\varphi}$ admits the flow $(x, y) \mapsto \big(b + e^t(x - b), y\big)$]. One can therefore associate to $\varphi$ the chart $\tilde{\varphi}$ of $H^\bullet E$ over $\varphi$, with image $\mathbf{im}\,\varphi \times H^\bullet(\pi^{-1}(\mathbf{dom}\,\varphi), \mathbb{K})$, given by $\tilde{\varphi}(b, c) := \big(\varphi(b), (i_b^*)^{-1}c\big)$, $c \in H^\bullet(E_b, \mathbb{K})$. Il is easy to check that one gets in this fashion the required vector bundle atlas and flat connection.

A subtle feature of the construction is that the fibre bundle $H^\bullet E$ and the connection are $\mathbb{K}$–analytic when $\pi$ is, whereas the local trivialisations $h_\varphi$ are *not*—they are obtained using partitions of unity.

For $b \in B$, parallel transport along each loop $\gamma$ in $B$, with base point $b$, defines an automorphism of $H^\bullet E_b$ since the connection is linear; as it is flat, this automorphism depends only on the homotopy class of $\gamma$; this defines a homomorphism of the fundamental group $\pi_1(B, b)$ into the group of automorphisms of $H^\bullet E_b$, called *monodromy.*

**Torsion, Levi-Civita connection and variants** The *torsion* $\tau_a \in L_{\mathrm{alt}}(T_aM, T_aM)$ at $a \in M$ of a linear connection on a manifold $M$ (i. e., on its tangent bundle) can be defined quickly as follows: for every parametrised surface $\sigma : (\mathbb{R}^2, 0) \to (M, a)$, one has $\tau_a(\partial_1\sigma(0), \partial_2\sigma(0)) = D_2\partial_1\sigma(0) - D_1\partial_2\sigma(0)$, where $D_1\partial_2\sigma(s,t) := \frac{D}{\partial s}\frac{\partial}{\partial t}\sigma(s,t)$ and $D_2\partial_1\sigma(s,t) := \frac{D}{\partial t}\frac{\partial}{\partial s}\sigma(s,t)$. For each Riemannian metric on $M$, there exists a unique linear connection *without torsion* ("symmetric") on $M$ that is *Riemannian*, i. e., such that the parallel transport from time $s$ at time $t$ along any path $\gamma$ in $M$ is an isometry of $T_{\gamma(s)}M$ onto $T_{\gamma(t)}M$: it is called the *Levi-Civita connection.* Its absence of torsion allows for example an intrinsic proof of the fact that the critical points of the action functional $\frac{1}{2}\int_0^1 \|\dot{\gamma}(t)\|^2\,dt$ on the space of paths $\gamma$ with fixed endpoints $\gamma(0), \gamma(1)$ in $M$ are the *geodesics*, solutions of the equation $\frac{D}{dt}\dot{\gamma}(t) = 0$.

Since parallel transport for the Levi-Civita connection preserves the scalar product, it induces a parallel transport of orthonormal frames of the tangent spaces of $M$, which is the parallel transport of a connection on the bundle of orthonormal frames; this connection is *principal*, meaning that parallel transport preserves the action of the structural group.

## As a conclusion

Of course, I have barely touched the subject, my only ambition being to provide some access to the ideas of Ehresmann and his master Élie Cartan. The work of the latter is not yet finished, as each generation tries to cast some light on it. A first rate contribution in that respect was Charles Ehresmann's introduction of fibre bundles, jets and connections, but also pseudogroups and groupoids, now again very popular [27, 28, 17] in spite of their ugly name (to say nothing of the horrible *algebroids*, direct from a bad science fiction film).

**Typical examples** The diffeomorphisms between open subsets of a manifold $M$ form a pseudogroup, and even a groupoid if it is forbidden to compose two of them when the domain of the second is not exactly the image of the first; the germs at points of $M$ of such local diffeomorphisms form a groupoid (one can compose a germ $f$ at $a$ and a germ $g$ at $b$ only if $b = f(a)$), and so do their jets of order $k$. When $M$ is endowed with an additional structure, for example a Riemannian metric or a symplectic form or a contact structure, the (jets or germs of) local diffeomorphisms preserving this structure form a sub-pseudogroup or a sub-groupoid of the previous one. The Riemannian example of an otherwise round sphere with a bump in the neighbourhood of a point shows that this pseudogroup or groupoid can be rather irregular, well apt to detect local symmetries ignored by the group of global isometries of our sphere onto itself, in general trivial. A fundamental object in foliation theory is the holonomy groupoid generalising monodromy.

In the works of Lie or Élie Cartan, "groups" were quite often pseudogroups—which appear already when one considers the flow of a non-complete vector field (similarly, what plays the role of a one-parameter group for time-dependent vector fields is a "groupoid with two parameters", which shows that many scientists manipulate groupoids without being aware of it!). The emphasis on abstract groups, which, according to the dogma, act only on themselves until they are represented, partially rejected into darkness Lie's original *groups*, i. e., pseudogroups of transformations that cannot always be abstracted from the space on which they act [3, 5].

To Élie Cartan, as I said, one goes back all the time: for example, the algorithmic "equivalence method" in [16] is a recent avatar of his "equivalence problem" [6, 19].

The equivalence problem is to find criteria for two structures to be locally equivalent up to local coordinate changes; of course, the langage of manifolds, used throughout this article, is coordinate-free, so that "coordinate change" means "diffeomorphism" (true problems cannot depend on the choice of coordinates).

Similarly, his theory of involution goes on inspiring Malgrange [22] after Kuranishi and many others [12, 1], such as Ehresmann, whose jets allow an intrinsic formulation of the prolongations of a differential system.

It should also be time to go back to Ehresmann before his beautifully concise texts become inaccessible; thus, my proof of his most famous theorem is the original one [13], so elliptic that many people replaced it by arguments far less elegant and natural. Science progresses to a large extent because its actors do not really understand the work of their predecessors and make it into something else, sometimes more interesting than the original, but there are limits. . .

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