## A remark on stable foliations

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This is an elementary talk about the link between invariant manifolds, foliations and fibrations, and about their meaning.

Everything will be in the book
Generating maps, invariant manifolds and related topics

## Aknowledgements and some explanations

I thank the organizers for the brilliant idea of this conference and also for asking me to be part of its scientific committee.

Last week, I told Alain Chenciner I would attend it and also that I had proved a funny little thing while working at my book in July.
He answered that I should give a talk about it here.
Of course I said that members of the scientific committee are supposed to suggest names of speakers and not to speak.
He replied that rules are made to be broken.
The organizers and the scientific committee agreed.

I hope Alain will not be ashamed of me: in July, I was living among hares, partridges, one owl and many other wild animals, whose opinion I could not get even though they surely had one.

## Alain

Having known Alain Chenciner for forty-three years, I have the precious information that he was not born with a beard. When he decided to grow one, our friend Michel Herman did not like it: "You look like a pope", he said.

I was Alain's first student, probably not the best one, nor an easy one, as I worked only on problems I understood, which at the time meant very few problems.
Those problems were far from what he was doing and, I suspect, from what he liked.
Amazingly, he was able to understand at once what I was doing in such remote regions of mathematics and give me very good advice: he was a born advisor.
Since then, we have been friends and, after some time, colleagues. I am glad to be here and celebrate his birthday.

## How can they be so wrong?

I have sometimes heard talks in which an analytic (or $C^{\infty}$ ) dynamical system, restricted to a center manifold $W$, is studied via results about analytic (or $C^{\infty}$ ) dynamics.

This is wrong, as such center manifolds need not be $C^{\infty}$.

I have known it since my beginnings, see the appendix.

## How can they be so wrong? The Euler equation.

A standard example of an analytic dynamical system on $\mathbb{R}^{2}$ with no analytic center manifold at its equilibrium point is

$$
\dot{x}=-x+y^{2}, \quad \dot{y}=-y^{2} .
$$

The $x$-axis is the stable manifold of the system at 0 , tangent to the stable subspace of the linearized system and locally unique in that respect.
Every other invariant curve of the system passing through the origin must be a center manifold tangent to the $y$-axis and therefore locally a graph $x=\psi(y)$, where $\psi$ is a solution of the Euler equation $\frac{d x}{d y}=\frac{1}{y^{2}} x-1$.
Such solutions are easy to calculate and they are $C^{\infty}$, locally unique for $y \leq 0$ but definitely not for $y \geq 0$, where their graphs form a brush of curves having infinite contact at 0 .

## How can they be so wrong? The (unfolded) Euler equation.

Their common Taylor expansion $\hat{\psi}=\sum_{n \geq 0} \psi_{n} y^{n}$ at 0 satisfies $y^{2} \hat{\psi}^{\prime}=\hat{\psi}-y^{2}$, i.e., $\sum_{n \geq 2}(n-1) \psi_{n-1} y^{n}=\sum_{n \geq 0} \psi_{n} y^{n}-y^{2}$, hence $\psi_{0}=\psi_{1}=0, \psi_{2}=1$ and $\psi_{n}=(n-1) \psi_{n-1}$ for $n>2$, yielding the divergent series $\hat{\psi}=\sum_{n \geq 2}(n-1)!y^{n}$.

Having proved that our polynomial system has no analytic center manifold at 0 , we exhibit a polynomial system on $\mathbb{R}^{3}$ with no $C^{\infty}$ center manifold at its equilibrium point 0 , namely

$$
\dot{x}=-x+y^{2}, \quad \dot{y}=z-y^{2}, \quad \dot{z}=0
$$

whose equilibria form the parabola $x=z=y^{2}$.
By the center manifold theorem, for every finite $k$, this system admits a local $C^{k}$ invariant manifold $x=\psi(y, z)$, tangent to the $(y, z)$-plane at 0 ("center manifold").
We should show that no such $\psi$ can be $C^{\infty}$ near the origin.

## How can they be so wrong? The unfolded Euler equation.

Note that the system leaves invariant each horizontal slice $z=$ constant. For fixed positive $z$, the restricted system

$$
\dot{x}=-x+y^{2}, \quad \dot{y}=z-y^{2}
$$

has two equilibrium points, namely $a_{ \pm}^{z}:=(z, \pm \sqrt{z})$.
On the invariant line $y= \pm \sqrt{z}$, this system writes $\dot{x}=-x+z$, hence the solutions $x=c e^{-t}+z, c \in \mathbb{R}$; the corresponding solutions $t \mapsto\left(c e^{-t}+z, \pm \sqrt{z}\right)$ of the system tend exponentially fast to $a_{ \pm}^{z}$ and the eigenvalue of the linearized system at $a_{ \pm}^{z}$ in this "strongly stable" direction is -1 . As the second equation writes $\dot{y}=-(y \pm \sqrt{z})(y- \pm \sqrt{z})$, the other eigenvalue is $\mp 2 \sqrt{z}$, hence $a_{+}^{z}$ is a node and $a_{-}^{z}$ a saddle.

## How can they be so wrong? The unfolded Euler equation.

Since the vector field $(x, y) \mapsto\left(-x+y^{2}, z-y^{2}\right)$ is parallel to the $x$-direction only on the two invariant lines $y= \pm \sqrt{z}$, the orbits not contained in those lines are graphs of solutions of the differential equation $\frac{d x}{d y}=\frac{1}{y^{2}-z} x+\frac{y^{2}}{z-y^{2}}$.
On $\mathbb{R} \backslash\{-\sqrt{z}, \sqrt{z}\}$, every maximal solution of this affine equation is defined by an explicit formula on one of the three intervals $(-\infty,-\sqrt{z}),(-\sqrt{z}, \sqrt{z})$ and $(\sqrt{z},+\infty)$.
On each of the first two intervals, only one maximal solution does not tend to $\pm \infty$ when $y \rightarrow-\sqrt{z}$, and it tends to $z$. The corresponding two orbits form the complement of the saddle point $a_{-}^{z}$ in its unstable manifold $W^{u}\left(a_{-}^{z}\right)$.
On each of the last two intervals, every maximal solution tends to $z$ when $y \rightarrow \sqrt{z}$ : the corresponding orbits tend to the node $a_{+}^{z}$ when $t \rightarrow+\infty$.

## How can they be so wrong? The unfolded Euler equation.

Given a center manifold $x=\psi(y, z)$ of the initial equations at 0 , its invariance implies that, for small enough positive $z$,

- the (open) domain of $\psi^{z}: y \mapsto \psi(y, z)$ contains $[-\sqrt{z}, \sqrt{z}]$
- the graph of $\left.\psi^{z}\right|_{[-\sqrt{z}, \sqrt{z})}$ is contained in $W^{u}\left(a_{-}^{z}\right)$.

If $\alpha_{z}:=\frac{-1}{-2 \sqrt{z}}=\frac{1}{2 \sqrt{z}}$ denotes the ratio of the eigenvalues of the linearized system at $a_{+}^{z}$,
we claim that $\psi^{z}$ is not $k$ times derivable at $\sqrt{z}$ for $k \geq \alpha_{z}$, which does imply that $\psi$ is not $C^{\infty}$ in any neighborhood of 0 . Thus, it is at the node $a_{+}^{z}$ that the center manifold is not $C^{\infty}$. Assuming $2 \sqrt{z}<1$, two cases occur:

- For $\alpha_{z} \notin \mathbb{Z}$, by Poincaré's thesis, there is an analytic local coordinate system $(X, Y):\left(\mathbb{R}^{2}, a_{+}^{z}\right) \rightarrow\left(\mathbb{R}^{2}, 0\right)$ in which the system $\dot{x}=-x+y^{2}, \dot{y}=z-y^{2}$ reads $\dot{X}=-X, \dot{Y}=-2 \sqrt{z} Y$. Except the "strong stable manifold" $Y=0$ (i.e., $y=\sqrt{z}$ ), the invariant curves passing through 0 are of the form $X=c_{ \pm}|Y|^{\alpha_{z}}$ for $\pm Y \geq 0$; only $X=0$ is $C^{k}$ for $k>\alpha_{z}$.


## How can they be so wrong? The unfolded Euler equation.

There is no reason why $W^{u}\left(a_{-}^{z}\right)$ should be precisely the regular invariant curve $X=0$ near $a_{+}^{z}$, and it is not, which follows from the formula defining $W^{u}\left(a_{-}^{z}\right)$ as a graph for $y \in(-\sqrt{z}, \sqrt{z})$. This is a way to prove our claim.

An easier way, though less clear geometrically, is the following:

- For $\alpha_{z}=m \in \mathbb{Z}$ (hence $m \geq 2$ ), by a theorem of Dulac, there is an analytic local coordinate system

$$
\begin{aligned}
& (X, Y):\left(\mathbb{R}^{2}, a_{+}^{z}\right) \rightarrow\left(\mathbb{R}^{2}, 0\right) \text { in which the system } \\
& \dot{x}=-x+y^{2}, \dot{y}=z-y^{2} \text { reads either } \dot{X}=-X, \dot{Y}=-\frac{1}{m} Y \text { or } \\
& \dot{X}=-X+Y^{m}, \dot{Y}=-\frac{1}{m} Y .
\end{aligned}
$$

If we were in the first case then, near $a_{+}^{z}$, the curve $W^{u}\left(a_{-}^{z}\right)$ would be one of the analytic invariant curves $X=c Y^{m}$ but, in fact, we are in the second case, where the invariant curves through $a_{+}^{z}$ other than $y=\sqrt{z}$ are of the form $X=Y^{m}\left(c_{ \pm}-m \log |Y|\right)$ for $\pm Y \geq 0$ and therefore not $C^{m}$.

## Are they so wrong after all?

All right, the Euler system $\dot{x}=-x+y^{2}, \dot{y}=-y^{2}$ has no analytic center manifold at 0 , but the center manifolds are graphs $x=\psi(y)$, on which the dynamical system is determined by its $y$-component and therefore described by the perfectly analytic equation $\dot{y}=-y^{2}$.
In that case, restricting the dynamics to a center manifold is expressing a polynomial dynamical system in bad coordinates.
The horizontal lines $y=$ constant form an invariant foliation, the stable foliation of the Euler system. The projection $(x, y) \mapsto y$, whose fibers are the leaves of the stable foliation, semi-conjugates the Euler system to the equation $\dot{y}=-y^{2}$, which describes the induced dynamics on the space of leaves. Similar remarks can be made about the system $\dot{x}=-x+y^{2}, \dot{y}=z-y^{2}, \dot{z}=0$ : though its center manifolds at 0 are not $C^{\infty}$, the dynamics on any of them is the polynomial system $\dot{y}=z-y^{2}, \dot{z}=0$, viewed in bad coordinates.

## Strong stable foliations near attracting points: an example

Though "they" were mostly wrong, we shall show that the phenomenon just described is universal for attracting equilibria.

A simple example. Passing from continuous to discrete dynamical systems, it is quite easy to check that, for each integer $m \geq 2$ and each $\beta \in(-1,1)$, the polynomial map (Dulac normal form) $f:(x, y) \mapsto\left(\beta^{m} x+y^{m}, \beta y\right)$ has two kinds of local $C^{1}$ invariant curves through the origin:

- the $x$-axis, its strong stable manifold, which defines the only $f$-invariant germ of a curve tangent to the $x$-axis at 0
- the other ones, called pseudo-unstable manifolds, therefore are of the form $x=\psi(y)$ near 0 , and they can not be $C^{m}$ : otherwise, one would get a contradiction by derivating $m$ times at 0 the identity $\psi(\beta y)=\beta^{m} \psi(y)+y^{m}$ expressing invariance. As the function $\psi: y \mapsto y^{m}\left(c+\frac{\log |y|}{\beta^{m} \log |\beta|}\right)$ verifies this identity for each $c \in \mathbb{R}$, there is a bunch of $C^{m-1}$ pseudo-unstable curves, having $(m-1)^{t h}$ order contact with the $y$-axis at 0 .


## Strong stable foliations near attracting points: an example

As before, for each such pseudo-unstable manifold $W=\{x=\psi(y)\}$, the map $\left.f\right|_{W}$ is the perfectly analytic map $g: y \mapsto \beta y$, viewed in bad coordinates.
The projection $\Phi:(x, y) \mapsto y$ semi-conjugates $f$ to $g$, meaning that $\Phi \circ f=g \circ \Phi$.
The fibers $\Phi^{-1}(y)$ are the leaves of an $f$-invariant strong stable foliation, characterized by the fact that, for $|\beta|^{m}<\kappa<|\beta|$, they are the equivalence classes of the equivalence relation $z \sim z^{\prime}$ iff $\left|f^{n}(z)-f^{n}\left(z^{\prime}\right)\right| \rightarrow 0$ faster than $\kappa^{n}$ when $n \rightarrow+\infty$.
Of course, they are also the fibers of the "strong stable fibration" $\pi: \mathbb{R}^{2} \rightarrow W$ given by $\pi(x, y)=(\psi(y), y)$, which semi-conjugates $f$ to $f \mid w$ but is not $C^{m}$.

We will now see that all this is quite general.

## Hypotheses and notation

We denote by $f:(N, b) \rightarrow(N, b)$ a $C^{r}$ local map of a Banach manifold $N$ into itself (the notation means that $f(b)=b$ ). Here, $r \geq 1$ can be an integer or $\infty$, but also lie in $\mathbb{R} \backslash \mathbb{Z}$, in which case $C^{r}$ means $C^{[r]}$ with $(r-[r])$-Hölderian $[r]^{\text {th }}$ derivative. One can also have $r=\omega$ (resp. $\Omega$ ), in which case $C^{r}$ means real (resp. complex) analytic. We let $\infty<\omega<\Omega$. The differential $A:=\operatorname{Df}(b)$ is an endomorphism of $F:=T_{b} N$. We assume that its spectral radius $\rho(A)$ is less than 1 and that its spectrum $\operatorname{spec} A$ does not meet the circle $\{|z|=\kappa\}$ for some $\kappa \in(0, \rho(A))$; in other words, spec $A$ is the union of two compact subsets $\sigma^{s} \subset\{|z|<\kappa\}$ and $\sigma^{u} \subset\{|z|>\kappa\}$.
Therefore, $F$ is the direct sum $S \oplus U$ of two closed $A$-invariant vector subspaces, such that the endomorphisms $A_{s}: S \rightarrow S$ and $B: U \rightarrow U$ induced by $A$ satisfy spec $A_{s}=\sigma^{s}$ and spec $B=\sigma^{u}$; it follows that $B$ is invertible and $\rho\left(A_{s}\right)<\kappa<\rho\left(B^{-1}\right)^{-1} \leq \rho(B)<1$ hence $\ell:=\frac{\log \rho\left(B^{-1}\right)}{-\log \rho(B)} \geq 1$.

## Two results

> The previous example is a particular case of
> Theorem 1. If $r>\ell$ and $\ell<2$, then there exists a unique $C^{r}$ germ $\Phi:(N, b) \rightarrow(U, 0)$ semi-conjugating $f$ to the germ of $B$ at 0 and such that $D \Phi(b)$ is the projection $F=S \oplus U \rightarrow U$. Taking representatives of our germs, the fibres of the local fibration $\Phi$ are the leaves of the (local $\kappa$-)strong stable foliation of $f$ in the same sense as in the example.

For $\ell \geq 2$, things are slightly more complicated:
Theorem 2. If $r>\ell$, there exists a $C^{r}$ germ $\Phi:(N, b) \rightarrow(U, 0)$ semi-conjugating $f$ to the germ at 0 of a polynomial map $g:(U, 0) \rightarrow(U, 0)$ of degree at most $k:=[\ell]$, and such that $D \Phi(b)$ is the projection $F=S \oplus U \rightarrow U$.
Taking representatives, the fibres of any such local fibration $\Phi$ are the leaves of the (local $\kappa$-) strong stable foliation of $f$.

## Comments

When $f$ is invertible and $\operatorname{dim} N<\infty$, both results follow from the fact there is a system of (real and/or complex) $\mathrm{C}^{r}$ local coordinates $\left(x_{j}\right)_{1 \leq j \leq n}$ in which $f$ is in Dulac normal form $x_{j} \circ f=\alpha_{j} x_{j}+\sum_{\alpha^{p}=\alpha_{j}} a_{j p} x^{p} \quad$ with $\quad\left\{\alpha_{1}, \ldots, \alpha_{n}\right\}=\operatorname{spec} A$ :
indeed, one can assume $0<\left|\alpha_{1}\right| \leq \cdots \leq\left|\alpha_{n}\right|<1$; then, as each relation $\alpha^{p}=\alpha_{j}$ yields $\left|\alpha_{1}\right|^{p_{1}} \cdots\left|\alpha_{n}\right|^{p_{n}}=\left|\alpha_{j}\right|$,

- it satisfies $p_{1}+\cdots+p_{n} \leq \frac{\ln \left|\alpha_{j}\right|}{\ln \left|\alpha_{n}\right|} \leq \frac{\ln \left|\alpha_{1}\right|}{\ln \left|\alpha_{n}\right|}$
- for $\left|\alpha_{m-1}\right|<\kappa<\left|\alpha_{m}\right|$ and $m \leq j \leq n$, one has $p_{i}=0$ for

$$
i<m \text { and } p_{m}+\cdots+p_{n} \leq k=\left[\frac{\ln \left|\alpha_{m}\right|}{\ln \left|\alpha_{n}\right|}\right] .
$$

When $\operatorname{dim} U=\infty$, the polynomial $g$ in Theorem 2 cannot be specified further in general but, for $\operatorname{dim} U<\infty$, it can be put into Dulac normal form.
An interesting case is when $\operatorname{dim} S=\infty, \operatorname{dim} U<\infty$ and $f$ is not invertible: a parabolic semi-flow can be semi-conjugated locally to a finite dimensional flow in Dulac normal form.

## The situation in Theorems 1 and 2 is that of our example.

Indeed, in a $C^{r}$ local chart, $\Phi$ is the projection $S \times U \rightarrow U$, hence $f(x, y)=\left(f_{s}(x, y), g(y)\right)$ (in Theorem 1, $g=B$ ).
Choose norms on $S, U$ so that $\left|A_{s}\right|<\kappa<\left|B^{-1}\right|^{-1} \leq|B|<1$; then, for every small enough positive $\rho$,

- the ball $\mathbf{B}_{\rho}=\{\max \{|x|,|y|\} \leq \rho\}$ verifies $f\left(\mathbf{B}_{\rho}\right) \subset \mathbf{B}_{\rho}$
- Lip $\left.f_{s}\right|_{\mathbf{B}_{\rho}}<\kappa$ and $\left|g(y)-g\left(y^{\prime}\right)\right| \geq \kappa\left|y-y^{\prime}\right|$ for $y, y^{\prime} \in \mathbf{B}_{\rho}$,
hence the last assertion of Theorems 1 and 2: for $(x, y),\left(x^{\prime}, y^{\prime}\right)$ in $\mathrm{B}_{\rho}$ one has
$\lim _{n \rightarrow+\infty} \kappa^{-n}\left|f^{n}(x, y)-f^{n}\left(x^{\prime}, y^{\prime}\right)\right|=0$ if and only if $y=y^{\prime}$.
Again, each locally $f$-invariant $C^{1}$ submanifold $W \ni b$ of $N$ such that $T_{b} W=S$ ("pseudo-unstable" submanifold) is transversal to this strongly $\kappa$-stable foliation and $\left.f\right|_{W}$ is an altered version of the dynamics $g$ on the space of leaves.


## Proof of Theorem 1

Theorem 1 follows at once from the following result, extracted from my paper Invariant manifolds revisited, Proceedings of the Steklov Institute 236 (2002), 415-433:
Theorem 3. Let $f:(N, b) \rightarrow(N, b)$ and $g:(P, c) \rightarrow(P, c)$ be $C^{r}$ map germs between Banach manifolds, $A:=\operatorname{Df}(b)$ and $B:=D g(c)$. If $\rho(A)<1$ and $r>\ell:=\frac{\log \rho\left(B^{-1}\right)}{-\log \rho(A)}$ then, for every $C^{r}$ formal semi-conjugacy $\Psi:(N, b) \rightarrow(P, c)$ of $f$ to $g$ to order $k:=\max \{1,[\ell]\}$ (meaning that $g \circ \psi$ and $\psi \circ f$ have $k^{\text {th }}$ order contact at $b$ ), there exists a unique $C^{r}$ germ $\Phi:(N, b) \rightarrow(P, c)$ semi-conjugating $f$ to $g$ and having $k^{\text {th }}$ order contact with $\Psi$ at $b$.
Proof of Theorem 1. Via a local chart, one can assume $N=S \times U, b=0$ and $A=A_{s} \times B$, then apply Theorem 3 to the canonical projection $\Psi: S \times U \rightarrow U$ and $g=B$ : indeed, as $\rho(A)=\rho(B)$, the two $\ell$ 's coincide and $k=1$.

## Proof of Theorem 2

Theorem 2 also follows from Theorem 3, but for $\ell \geq 2$ one must find a polynomial map $g$ and a formal semi-conjugacy $\psi$. Via a chart, one can assume $N=S \times U, b=0$ and $A=A_{s} \times B$. The fact that $g \circ \psi$ and $\Psi \circ f$ have $k^{t h}$ order contact at 0 (denoted $g \circ \psi \sim_{k} \Psi \circ f$ ) depends only on the $k^{\text {th }}$ order Taylor polynomials of $\Psi, f$ (and $g$ ) at 0 , hence we can look for a polynomial $\psi$ of degree at most $k$, that must write $\Psi(x, y)=y+\sum_{2 \leq m+n \leq k} \Psi_{m, n} x^{m} y^{n}$, where
$x^{m}:=(\overbrace{x, \ldots, x}^{m \text { times }}), y^{n}:=(\overbrace{x, \ldots, x}^{n \text { times }})$ and each $\Psi_{m, n}$ is a continuous $(m+n)$-linear map $S^{m} \times U^{n} \rightarrow U$, symmetric with respect to both factors $S^{m}$ and $U^{n}$.
Likewise, $g(y)=B y+\sum_{2 \leq p \leq k} g_{p} y^{p}$, where each $g_{p}$ is a
continuous, symmetric $p$-linear map $U^{p} \rightarrow U$.

## Proof of Theorem 2

Lemma. The equation $g \circ \psi \sim_{k} \Psi \circ f$ has exactly one such polynomial solution $(g, \Psi)$ verifying $\Psi(0, y)=y$. Indeed, if $A_{s} x+\sum a_{m, n} x^{m} y^{n}, B y+\sum b_{m, n} x^{m} y^{n}$

$$
2 \leq m+n \leq k
$$

$$
2 \leq m+n \leq k
$$

are the components of the Taylor expansion of $f$ at 0 :

- As the identity $\Psi(0, y)=y$ means $\Psi_{0, n}=0$ for $2 \leq n \leq k$, the equation $g \circ \Psi \sim_{k} \Psi \circ f$ reads as follows for $x=0$ :
$\sum g_{p} y^{p} \sim_{k} \sum b_{0, p} y^{p}+\sum \Psi_{m, n}\left(\sum a_{0, q} y^{q}\right)^{m}\left(B y+\sum b_{0, s} y^{s}\right)^{n}$, hence $g_{p}$ is an affine function of the $\Psi_{m, n}$ 's with $m+n<p$.
- Injecting this into the equation $g \circ \Psi \sim_{k} \Psi \circ f$, one sees that each $\mathcal{A}_{m, n} \Psi_{m, n}:=B \circ \Psi_{m, n}-\Psi_{m, n} \circ\left(A_{s}^{m} \times B^{n}\right)$ with $m \geq 1$ is a polynomial function of the $\Psi_{p, q}$ 's with $p+q<m+n$.
We claim that the endomorphism $\mathcal{A}_{m, n}$ of $L_{s}^{m, n}(S, U ; U)$ so defined is an automorphism, which implies that each $\Psi_{m, n}$ with $m \geq 1$ is a polynomial function of the $\Psi_{p, q}$ 's with $p+q<m+n$, hence the lemma by induction.


## Proof of Theorem 2

To prove our claim, note that $\mathcal{A}_{m, n}$ is the composed map of the automorphism $\Phi_{m, n} \mapsto B \circ \Phi_{m, n}$ and the endomorphism $\Phi_{m, n} \mapsto \Phi_{m, n}-B^{-1} \circ \Phi_{m, n} \circ\left(A_{s}^{m} \times B^{n}\right)$, which is of the form "identity minus strict contraction" and therefore an automorphism.
Indeed, if $S, U$ are endowed with norms for which the norms of $B^{-1}, A_{s}$ and $B$ are close enough to their spectral radii to satisfy $\left|A_{s}\right|<\kappa<\left|B^{-1}\right|^{-1} \leq|B|<1$, one has $\left|B^{-1}\right|\left|A_{s}\right|<1$ and $\left|B^{-1} \circ \Phi_{m, n} \circ\left(A_{s}^{m} \times B^{n}\right)\right| \leq\left|B^{-1}\right|\left|\Phi_{m, n}\right|\left|A_{s}\right|^{m}|B|^{n} \leq$ $\left|B^{-1}\right|\left|A_{s}\right|\left|\Phi_{m, n}\right|$ since $m \geq 1$.

Note. This looks much like a brute-force proof of the Poincaré-Dulac formal normal form theorem in dimension 2, plus a little multilinear skill.

## On the proof of Theorem 3

A. Shoshitaishvili: $\Phi$ is a semi-conjugacy of $f$ to $g$ if and only if its graph is invariant by $f \times g:(x, y) \mapsto(f(x), g(y))$.
Thus, Theorem 3 follows at once from the following result (with $h=f \times g, V=\operatorname{graph} \psi$ and $W=\operatorname{graph} \Phi$ ), again from my 2002 Steklov paper, see also Variétés stables et formes normales, C. R. Acad. Sc. Paris 317, Série 1 (1993), 87-92:
Theorem 4. Let $M$ be a Banach manifold and $h:(M, a) \rightarrow(M, a)$ a $C^{r}$ map germ whose differential $L=D h(a)$ preserves a (factor) subspace $S$ of $E:=T_{a} M$. Assume the endomorphisms $A: S \rightarrow S$ and $B: E / S \rightarrow E / S$ induced by $L$ are such that $\rho(A)<1$, that $B$ is invertible and that $r>\ell:=\frac{\log \rho\left(B^{-1}\right)}{-\log \rho(A)}$. Then, for every $C^{r}$ submanifold germ $V$ of $M$ at a such that $T_{a} V=S$, formally $h$-invariant to order $k:=\max \{1,[\ell]\}$, there is a unique $h$-invariant $C^{r}$ submanifold germ $W$ having $k^{\text {th }}$ order contact with $V$ at a.

## On the proof of Theorem 3

Another proof of Theorem 3, in Sternberg's spirit, is to obtain $\Phi$ (taking local representatives of our germs) as the limit of $g^{-n} \circ \psi \circ f^{n}$ when $n \rightarrow+\infty$.

The original proof of Theorem 4 consisted in applying the implicit function theorem to the equation $z_{n+1}=h\left(z_{n}\right)$ in a suitable sequence space.

In the book, Theorem 4 and almost all the results known about invariant submanifolds and local semi-conjugacies are deduced from one theorem, whose rather simple and geometric proof requires no functional analysis whatsoever.

## Appendix: Thom, Michel, Alain and some others

> Thanks to René Thom, I had a research position at the Center of Mathematics of the École polytechnique for two years. This was in 1974, before I knew Alain. I was 24 and shared the office of a then rather obscure mathematician, Michel Herman. I was working on Thom's paper Sur les équations différentielles multiformes et leurs intégrales singulières and needed results about the smooth local classification of dynamical systems. Most people did not go beyond the topological classification but Michel knew better. He made me study first the analytic linearization theorem in Poincaré's thesis and its proof by majorant series (in the book by Nemitskii and Stepanov). When I learned about the existence of center manifolds, I tried to prove that, in the analytic case, one of them is analytic by using majorant series as in Poincaré's thesis.
> This did not work. I then tried to imitate Siegel's intricate proof of linearization in the presence of small denominators.

## Appendix: Thom, Michel, Alain and some others

It did not work either, and I finally found a counterexample.

During that period, though Thom was very kind and amazingly available, I felt the need for less intimidating an advisor. He suggested Alain, for whom he had the highest esteem.

In September 1975, a meeting on singularities and their applications was organized by Frédéric Pham at the Institute of Scientific Studies of Cargèse (Corsica).
The place is very beautiful, a metaphor of the Eden of Catastrophe Theory, in which many of us lived-and from which we would be expelled by Smale three years later. Jean-Pierre Ramis lectured on the local holomorphic foliations generated by holomorphic vector fields near their zeroes. He conjectured that if two such fields are formally conjugate, then the foliations they define are topologically conjugate.

## Appendix: Thom, Michel, Alain and some others

My little counterexample ruined this conjecture, as it provided two formally conjugate local holomorphic vector fields $\left(\mathbb{C}^{2}, 0\right) \rightarrow\left(\mathbb{C}^{2}, 0\right)$, one of which had two invariant holomorphic curves through the origin whereas the other had only one.
After his talk, Ramis did not pay much attention to my counterexample. He was walking on the beach with a very hairy pal of his (later largely responsible for the decline of mathematics in our secondary schools) and I had to follow them like a little poodle, which was a bit infuriating.

I published the counterexample as a one-page note: Une remarque sur les équations de Pfaff analytiques. C. R. Acad. Sci., Paris, Sér. A, 281 (1975), p. 641.

Then, I realized Leonhard Euler had done it better long before.

