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Thierry Paul, Mario Pulvirenti, Sergio Simonella

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THIERRY PAUL, MARIO PULVIRENTI, AND SERGIO SIMONELLA

Abstract. We consider the classical Kac's model for the approximation of the Boltzmann equation, and study the correlation error measuring the defect of propagation of chaos in the mean field limit. This contribution is inspired by a recent paper of the same authors [22] where a large class of models, including quantum systems, are considered. Here we outline the main ideas in the context of grand canonical measures, for which both the evolution equations and the proof simplify.

Contents

This note is dedicated to the memory of our colleague and friend Maria Conceicao Carvalho. Among her many important contributions to the theory of Kac's model, Sao's work [CCG99,CCG05] explained the long time behaviour of the homogeneous Boltzmann equation without using the H-theorem, but exploiting a graph expansion. Our contribution is, in spirit, close to this approach.

1. Introduction

The kinetic description of particle systems is strongly related to propagation of chaos. This property allows to substitute the complex dynamics of a huge number of particles by a single nonlinear partial differential equation for the probability density of a given particle. More precisely, one adopts a statistical description. At time zero, the N−particle system is assumed to be "chaotic" in the sense that each particle is distributed identically and independently from the others, at least up to an error which is vanishing when N diverges. The dynamics creates correlations and the independence property is lost at any positive time. However, under suitable scaling limits, the statistical independence of any finite group of particles can be recovered, after averaging, in the limit $N \to \infty$. As a consequence, a given particle evolves according to an effective kinetic equation. The nature of this dynamics is determined by the microscopic details of the system and by the regime of physical parameters. Such a mechanism works in the formal (in a few cases, rigorous) derivation of the most common kinetic equations.

If the system is described by a symmetric probability measure $W^N(Z_N, t)$ at time t, where $Z_N = (z_1, \cdots, z_N)$ is a configuration of the system, being $z_i = (x_i, v_i)$ position and velocity of the particle i, the probability density of finding the first j particles in the microscopic state Z_j is given by the marginal

$$f_j^N(Z_j, t) := \int dz_{j+1} \cdots dz_N W^N(Z_N, t) \ .$$

In case of a two-body interaction, we have that f_1^N depends on f_2^N, which depends on f_3^N and so on, and these marginals are not tensorized. Indeed, the dynamics creates correlations even though at time zero it is assumed that

$$W^N = (f_0)^{\otimes N} \ .$$

In many cases, the physical regime of interest can be expressed in terms of suitable scaling limits ($N \to \infty$, together with additional prescriptions), which one can hope to use to prove that

$$f_j^N(t) \to f(t)^{\otimes j}, \tag{1}$$

where $f(t)$ is a solution of the aforementioned kinetic equation.

In three dimensions, fundamental scaling limits for Hamiltonian systems are the following.

- Mean-field limit for classical systems: $N \to \infty$ and the coupling constant of the two-body interaction is scaled by $\frac{1}{N}$. Namely, a system of point particles with the mean-field Hamiltonian

$$H_N = \frac{1}{2} \sum_{i=1}^N v_i^2 + \frac{1}{2N} \sum_{i \neq j} \varphi(x_i - x_j) \ ,$$

  where φ is a two-body potential. The corresponding expected kinetic equation is, in this case, the Vlasov equation.

- Low-density or Boltzmann-Grad limit: $N \to \infty$ and the diameter of hard-sphere like particles is $\sim N^{-\frac{1}{2}}$. The corresponding kinetic equation is the Boltzmann equation.

- Weak-coupling limit: $N \to \infty$ and $\varepsilon \sim N^{-\frac{1}{3}}$ is a scale parameter coding the interaction length. The radial interaction potential - say, smooth and decaying - is scaled as $\varphi(\frac{r}{\varepsilon})$, so that the Hamiltonian of the system is
$$H_N = \frac{1}{2}\sum_{i=1}^{N} v_i^2 + \frac{\sqrt{\varepsilon}}{2}\sum_{i\neq j}\varphi\left(\frac{|x_i - x_j|}{\varepsilon}\right).$$
The corresponding kinetic equation is the Landau equation.

- High-density limit: $N \to \infty$ and $\varepsilon \sim N^{-\frac{1}{4}}$, where the Hamiltonian is
$$H_N = \frac{1}{2}\sum_{i=1}^{N} v_i^2 + \frac{\varepsilon}{2}\sum_{i\neq j}\varphi\left(\frac{|x_i - x_j|}{\varepsilon}\right).$$
The corresponding kinetic equation is the Lenard-Balescu equation.

For a recent analysis of these scalings we refer to [19, 20], where allowed classes of interaction potentials are also discussed. Scaling limits and propagation of chaos have a long history starting from the pioneering work of Bogoliubov [1], see e.g. [25, 24] for surveys.

The mechanism for which we expect that propagation of chaos actually holds is deeply different in the above cases. For the mean-field limit, the force exerted by particle j on particle i is $O(\frac{1}{N})$ so that the effect of one particle on the other is triviallynegligible. Hence z_i and z_j can be considered as independent random variables in the limit $N \to \infty$, if they were so at time zero. At variance, in the case of the Boltzmann-Grad limit, the forces are strong; but due to the low density regime, the probability of two *given* particles colliding in $(0, t)$ is small, although any given particle is subject to a collision with strictly positive rate. Finally in the weak-coupling limit, the forces are of order $\frac{1}{\sqrt{\varepsilon}} \sim N^{\frac{2}{3}}$, but the potential range is small and the interaction takes place in a time interval of order ε. Overall the variation of momentum of particle i due to the interaction with particle j is of order $\sqrt{\varepsilon}$, thus negligible. A similar argument holds in the high-density case.

Going further in this analysis, one may ask about quantitative estimates on the defect of chaos. In recent times, the authors have approached this problem in [22] by the method of "correlation errors" (also called "v-functions"), which goes back to previous work in the context of kinetic limits [2]-[11], [23]. Correlation errors are a tool to measure the tendency of f_j^N to factorize and converge in the mean field limit. We will give a precise definition in the next section (Definition 1 below; see also Def. 2.1 in [22]). In [22], they have been used to prove a result on the Kac's model, which we recall next.

We consider N particles following a stochastic dynamics: to each particle, say particle i, we associate a velocity $v_i \in \mathbb{R}^3$ and the vector

$$V_N = \{v_1, \cdots, v_N\}$$

changes by means of two-body collisions at random times, with random scattering angle. The probability density $W^N(V_N, t)$ evolves according to the master equation (forward Kolmogorov equation)

$$\partial_t W^N = \frac{1}{N} T_N W^N \tag{2}$$

where

$$T_N W^N = \sum_{i<j} T_{i,j} W^N,$$

$$T_{i,j} W^N(V_N) = \int d\omega B(\omega; v_i - v_j)\{W^N(V_N^{i,j}) - W^N(V_N)\}\ , \tag{3}$$

and where $V_N^{i,j} = \{v_1, \cdots, v_{i-1}, v_i', v_{i+1}, \cdots, v_{j-1}, v_j', v_{j+1}, \cdots, v_N\}$ and the pair v_i', v_j' gives the outgoing velocities after a collision with scattering (unit) vector ω and incoming velocities v_i, v_j. $\frac{B(\omega; v_i - v_j)}{|v_i - v_j|}$ is the differential cross-section of the two-body process and we shall assume here, for simplicity, that B is bounded. The resulting kinetic equation is the homogeneous Boltzmann equation

$$\begin{aligned} \partial_t f(v) &= \int dv_1 \int d\omega B(\omega; v - v_1)\{f(v')f(v_1') - f(v)f(v_1)\} \\ &=: Q(f,f)(v) \end{aligned} \tag{4}$$

Such a model has been introduced and studied by Kac in [15, 16] and has given rise to a wide literature later on (see e.g. [13, 26, 12, 14, 17, 18]). For instance in [22], we prove the following.

Theorem 1.1 ([22]). *For all $t > 0$ and all $j = 1, ..., N$, the marginals satisfy*

$$\| f_j^N(t) - f(t)^{\otimes j} \|_1 \le C_2\, e^{C_1 t}\, \frac{j^2}{N} \tag{5}$$

where $C_1, C_2 > 0$ *are explicitly computable constants.*

In the present paper, we are going to recover a similar result, albeit in a different setting which we refer to as *grand canonical ensemble.* This is a standard formalism in statistical mechanics, adapted here to the kinetic problem. With respect to the model introduced above, the novelty of the grand canonical framework is twofold: (i) we allow the total number of particles N to be a (Poisson) random variable with intensity $\mu \to \infty$, and consequently (ii) we do not label the particles from 1 to N (as done in the definition of marginal), but compute only averages of symmetrized quantities. This leads to the notion of *correlation functions*, denoted below $\{f_j^\mu\}_{j=1}^\infty$, which play the role of the marginals $\{f_j^N\}_{j=1}^\infty$ in this context.

The advantange of the grand-canonical formalism is that the correlations due to conservation of energy and mass are zero by assumption, and one can focus more easily on the correlations having a purely dynamical origin. In this spirit we will reformulate the method of [22] leading to a result analogous to (5), in terms of correlation functions (Theorem 3.2 below). We shall see that this bears simplifications of the corresponding proof in the canonical ensemble. Indeed, even if f_j^N and f_j^μ are asymptotically equivalent, the latter functions encode less correlations. This is reflected eventually in a smaller number of terms in the basic evolution equations for the error (see Remark 3.1 below).

The paper is organized as follows. In the next section, we introduce the Kac's model in the grand canonical ensemble, and define the correlation errors. In Section 3 we discuss the evolution equations, whose derivation is postponed to the appendix. In Section 3 we establish the main result, Theorem 3.2, and its proof is presented in the last section.

2. Kac model and grand canonical ensemble

We start by recalling the main ensembles which can be adopted to describe a collisional dynamics, in the spirit of the abovementioned Kac's model.

- *Microcanonical.* The binary collisions preserve energy, and the stochastic dynamics lives in the energy surface
$$E = \frac{1}{N} \sum_i v_i^2 \ .$$

- *Canonical.* E is random and only N is fixed, as in the model discussed in the Introduction.
- *Grand canonical.* N is random. We introduce a sequence of initial density distributions $\{W_0^n\}_{n\geq 0}$ on the phase space $\bigcup_{n\geq 0}\mathbb{R}^{3n}$. By definition, $\frac{1}{n!}W_0^n(V_n)$ is the symmetric probability density of the configuration V_n, being n the total number of particles. The average number of particles is then
$$\mu := \sum_{n\geq 0} n \int \frac{1}{n!} W_0^n(V_n) dV_n \ .$$
Here $\mu > 0$ is a free parameter (eventually $\mu \to \infty$) and the distributions do depend on μ, although this dependence is dropped in our notation ($W_0^n = W_0^{n,\mu}$). At time $t > 0$, the state of the system is described by the time-evolved measure $\{W^n(t)\}_{n\geq 0}$ solving (cf. (2))

$$\partial_t W^n = \frac{1}{\mu} T_n W^n \ . \tag{6}$$

Focusing now on the grand canonical case, let us introduce a natural description of correlations and propagation of chaos. In order to examine correlations due exclusively to the dynamics, we choose a perfectly tensorized and still symmetric initial state, namely

$$W_0^n = f_0^{\otimes n} e^{-\mu} \mu^n \tag{7}$$

(however most of our discussion could be extended to a larger class of initial measures).

Instead of computing probability densities of particles with labels $\{1, \cdots, j\}$, we define rescaled correlation functions by

$$f_j^\mu(V_j, t) = \mu^{-j} \sum_{k\geq 0} \frac{1}{k!} \int dv_{j+1} \cdots dv_{j+k} W^{j+k}(V_{j+k}, t) \ .$$

Loosely speaking, this corresponds to computing the amount of $j-$tuples of particles in the configuration V_j at time t. For a purely factorized state the average total number of $j-$tuples is μ^j, which explains the rescaling factor. In particular with the choice (7), one checks that $\|f_j^\mu\|_{L^1} = 1$ (initially and for all times).

Suppose that

$$\lim_{N\to\infty} f_j^\mu = f(t)^{\otimes j} \ , \tag{8}$$

say in L^1–norm. We may ask about convergence rates. We observe that, in general (e.g. for $j \sim \mu$) $f_j^\mu(t)$ is not expected to be asymptotically equivalent to $(f_1^\mu(t))^{\otimes j}$, even if it is assumed to be so at time zero. One can then pose the question: how large can be $j = j(\mu)$ in such a way that (8) holds true?

Having at disposal the estimate

$$\|f_j^\mu - f(t)^{\otimes j}\|_{L^1} \le C^j \left(\frac{1}{\mu}\right)^\alpha \tag{9}$$

for some $\alpha \in (0,1)$ and $C > 0$, we cannot make better predictions than $j(\mu) \sim \log \mu$. On the other hand if for a moderately large j, say $j \approx \mu^\gamma$, $\gamma \in (0,1)$, $f_j^\mu \approx (f_1^\mu(t))^{\otimes j}$, then it is natural to consider

$$(f_1^\mu(t) - f(t))^{\otimes j} =: E_j \; ; \tag{10}$$

namely the product of the differences rather than the difference of the products. Expanding (10) one finds that

$$E_j(t) := \sum_{K \subset J} (-1)^{|K|} f^\mu_{J\setminus K}(t) f(t)^{\otimes K}$$

where $J = \{1, 2, \cdots, j\}$ is the set of the first j indices, K is any subset of J, $J\setminus K$ is the relative complement of K in J and $|K|$ is the cardinality of K. $f_A^\mu(t)$ stands for the $|A|$–marginal $f^\mu_{|A|}(t)$ computed in the configuration $(z_i)_{i\in A}$. Similarly $f(t)^{\otimes K} = f(t)^{\otimes |K|}$ evaluated in $(z_i)_{i \in K}$. The formula (12) has an inverse formula, that is

$$f_j^\mu(t) = \sum_{K \subset J} E_{J\setminus K}(t) f(t)^{\otimes K} \tag{11}$$

where the notation $E_{J\setminus K}$ is the same one as for $F^\mu_{J\setminus K}$, and where we set $E_0 = E_\varnothing = 1$.

This motivates the following

Definition 1 (Correlation error (grand-canonical))**.** *For any $j \ge 1$, setting $J = \{1, 2, \cdots, j\}$, we define the "correlation error" of order j by*

$$E_j(t) := \sum_{K \subset J} (-1)^{|K|} f^\mu_{J\setminus K}(t) f(t)^{\otimes K} \; , \tag{12}$$

where the terms $K = \varnothing$ and $K = J$ have to be interpreted as $f^\mu_J = f^\mu_j$ and $(-1)^j f^{\otimes J} = (-1)^j f^{\otimes j}$, respectively.

For instance

$$\begin{aligned}
E_1(v_1) &= f_1^\mu(v_1) - f(v_1) \\
E_2(v_1,v_2) &= f_2^\mu(v_1,v_2) - f_1^\mu(v_1)f_1(v_2) - f(v_1)f_1^\mu(v_1) + f(v_1)f(v_2) \\
E_3(v_1,v_2,v_3) &= f_3^\mu(v_1,v_2,v_3) - f_2^\mu(v_1,v_2)f(v_3)\cdots + f_1^\mu(v_1)f(v_2)f(v_3)\cdots \\
&\quad - f(v_1)f(v_2)f(v_3)\,.
\end{aligned}$$

3. Dynamical equations

It is well known, and can be easily checked from (6), that the sequence $\{f_j^\mu\}$ satisfies the following hierarchy of equations (called BBKGY in analogy with the Hamiltonian case):

$$\partial_t f_j^\mu = \frac{T_j}{\mu} f_j^\mu + C_{j+1} f_{j+1}^\mu \tag{13}$$

for $j = 1,\cdots,\infty$, where

$$C_{j+1} f_{j+1}^\mu(V_j) = \sum_{i=1}^{j} C_{i,j+1} f_{j+1}^\mu(V_j) \tag{14}$$

and

$$C_{i,j+1} f_{j+1}^\mu(V_j) = \int dv_{j+1} \int d\omega B(\omega; v_i - v_j)\{f_{j+1}^\mu(V_{j+1}^{i,j+1}) - f_{j+1}^\mu(V_{j+1})\} \tag{15}$$

with the same notations as in (3).

This hierarchy must be compared with the limiting hierarchy (in the formal limit $\mu \to \infty$)

$$\partial_t f_j = C_{j+1} f_{j+1} \tag{16}$$

which is satisfied by products $f_j(t) = f(t)^{\otimes j}$, being $f(t)$ solution of the homogeneous Boltzmann equation (4).

Introducing correlation errors given by (12), we deduce from (13) the corresponding evolution equations. We find (see the Appendix)

$$\partial_t E_j = \frac{T_j}{\mu} E_j + D_j E_j + D_j^1(E_{j+1}) + D_j^{-1}(E_{j-1}) + D_j^{-2}(E_{j-2}) \tag{17}$$

where $D_j^1 = C_{j+1}$ and the operators D_j, D_j^{-1}, D_j^{-2} are defined by:

$$\begin{aligned}
D_j E_j &= \sum_{i\in J} C_{i,j+1}[f^{\otimes\{i\}} E_{J^i\cup\{j+1\}} + f^{\otimes\{j+1\}} E_J] \\
D_j^{-1} E_{j-1} &= \frac{1}{\mu} \sum_{i,s\in J} T_{i,s} f^{\otimes\{i\}} E_{J^i} \\
D_j^{-2} E_{j-2} &= \frac{1}{2\mu} \sum_{i,s\in J} T_{i,s} f^{\otimes\{i,s\}} E_{J^{i,s}}\ ,
\end{aligned} \tag{18}$$

where J is the set of the first j indices, $J^i = J\backslash\{i\}$, $J^{i,s} = J\backslash\{i,s\}$ and

$$f^{\otimes\{i\}} = f(v_i), \quad f^{\otimes\{i,s\}} = f(v_i)f(v_s) .$$

Here we use the convention the $D_1^{-2} = 0$. Moreover these operators may depend on time, although we will drop out this dependence from the notation.

Note that the first line contains operators which do not change the particle number, C_{j+1} is an operator decreasing by one the number of particles, while D_j^{-1} and D_j^{-2} are operators increasing the number of particles by one and two respectively.

Remark 3.1. *The marginals in the canonical setting satisfy the same hierarchy of equations as in* (13) *(with μ replaced by N), up to a combinatorial factor $(N-j)/N$ in front of the collision operator C_{j+1}. Such slight difference produces several spurious terms in the definitions of the operators D_j, D_j^{-1}, D_j^{-2} (see Eq.s (42)-(44) in* [22]*), which are absent in the definition* (18)*. Therefore the grand-canonical equations single out the contributions having a purely dynamical interpretation, in terms of variation of clusters (groups of particles) with j mutually correlated particles. Since the interactions are binary, such a cluster can be created in five ways only, corresponding respectively to the five terms in* (17)*, and depending on how many independent particles enter the game through the described collision.*

We conclude this section by establishing our main result.

Theorem 3.2. *Let us assume $f_j^\mu(0) = f_0^{\otimes j}$ being f_0 the initial datum for the Boltzmann equation. Then, for $\frac{j}{\sqrt{\mu}}$ sufficiently small, $G > 1$ depending on time and all $t > 0$*

$$\|f_j^\mu(t) - f^{\otimes j}(t)\|_1 \leq \frac{Gj^2}{\mu}. \tag{19}$$

Here we are assuming a full factorization at time zero. Looking at the proof it will be clear that this condition can be relaxed. Here we prefer to work in the simplest context to outline the main ideas.

4. Towards the Boltzmann equation

Starting from (13), by the Dyson (Duhamel) expansion we have:

$$f_j^\mu(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \tag{20}$$
$$U_j^\mu(t-t_1) C_{j+1} \dots U_{j+n-1}^\mu(t_{n-1}-t_n) C_{j+n} U_{j+n}^\mu(t_n) f_{0,n+j},$$

where $f_{0,j} = f_0^{\otimes j}$ and

$$U_j^\mu(t) = e^{\frac{T_j}{\mu} t}.$$

On the other hand we also have, for $f_j(t) = f(t)^{\otimes j}$, being $f(t)$ the solution to the Boltzmann equation

$$f_j(t) = \sum_{n=0}^{\infty} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n C_{j+1} \dots C_{j+n} f_{0,n+j} \tag{21}$$
$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} C_{j+1} \dots C_{j+n} f_{0,n+j}.$$

Assuming that B is bounded then we have

$$\|T_j\| \le C_1 \frac{j(j-1)}{2}, \qquad \|C_{j+1}\| \le C_1 j. \tag{22}$$

From now on we denote by $\|\cdot\|$ the L^1 -norm and by $C_i, i = 1, 2 \cdots$ any geometrical positive constant.

Then since $\|U_j^\mu(t)\| = 1$ both the right hand side of (20) and (21) are bounded by

$$\sum_{n\ge0} C_2^n \frac{t^n}{n!} j(j+1)\cdots(j+n-1) \le 2^j \sum_{n\ge0} (2C_2 t)^n,$$

which is converging for t small. Therefore the rest of the two series is negligible, uniformly in μ, and this allows us to perform the limit $\mu \to \infty$. Indeed realizing that, for all j,

$$\lim_{\mu\to\infty} U_j^\mu(t) = I$$

strongly, we have also the term by term convergence of (20) to (21) in L^1. Once having the convergence for $t < \frac{1}{2C_2}$, we can iterate the procedure since the new initial conditions, $f_j^\mu(t)$ satisfy $\|f_j^\mu(t)\| = 1$. Then we have convergence for any arbitrary time.

What is the *size of chaos* in this case? By size of chaos we mean the maximal α for which we have convergence for all j such that $\frac{j}{\mu^\alpha} \to 0$ as $\mu \to \infty$. Our main Theorem 3.2 shows that the size of chaos is $\frac{1}{2}$. On the other hand, as we shall see in a moment, we do not expect a size of chaos strictly greater than $\frac{1}{2}$ so we believe that our result can be considered as optimal as regards this feature.

We observe that

$$\lim_{\mu,j\to\infty} U_j^\mu(t) = I$$

only if $\frac{j^2}{\mu} \to 0$. Indeed expressing $U^\mu(t)$ in terms of the series expansion

$$U^\mu(t) f_{0,j} = \sum_n \frac{t^n}{n!}\frac{1}{\mu^n}\sum_{i_1<k_1}\cdots\sum_{i_n<k_n} T_{i_1,k_1}\cdots T_{i_n,k_n} f_{0,j}$$

we realize that the L^1 norm of the right hand side can be bounded by

$$\sum_n \frac{(Ct)^n}{n!}\left(\frac{j(j-1)}{2\mu}\right)^n.$$

Therefore we expect that an estimate of the size of chaos cannot be better than $\frac{1}{2}$ and this can be considered as optimal.

Coming back to the dynamical description given by (17), more detailed than (13), we find that size of the L^1 norms of the operators appearing in (17):

$$\|D_j\| = O(j),\ \|D_j^1\| = O(j),\ \|D_j^{-1}\| = O\left(\frac{j^2}{\mu}\right),\ \|D_j^{-2}\| = O\left(\frac{j^2}{\mu}\right).$$

Note that (17) is inhomogeneous so that it has nontrivial solutions even for initial data $E_j(0) = 0$, $j > 0$ (namely when the initial state is tensorized) which is the case we consider here.

The control of E passes trough a Dyson expansion around the two parameters semigroup $\bar{V}_j$ generated by $\frac{T_j}{\mu}E_j + D_jE_j$, namely the solution of

$$\partial_t \bar{V}_j(t,s) = \left(\frac{T_j}{\mu} + D_j\right)\bar{V}_j(t,s)$$
$$\bar{V}_j(s,s) = I.$$

We easily derive the following bound

$$\|\bar{V}_j(t,s)\| \le e^{C\left(\frac{j^2}{\mu}+j\right)(t-s)}. \tag{23}$$

The action of the other operators $C_{j+1}, D_j^{-1}, D_j^{-2}$ represents a positive, a negative and a negative double jump respectively in the space of indices.

We remind that initially the distribution factorizes, namely

$$E_j(0) = \delta_{j,0},$$

then

$$E_j(t) = \sum_{n=0}^{\infty} \sum_{k_1\cdots k_n} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \tag{24}$$
$$\bar{V}_{s_1}(t,t_1) D^{k_1} \dots \bar{V}_{s_n}(t_{n-1},t_n) D^{k_n} \bar{V}_{s_{n+1}}(t_n) E_{j_0}(0).$$

where $k_i \in \{-1,-2,1\}$. The sequence $\{s_1 \cdots s_n\}$ is defined recursively by

$$s_{r+1} = s_r + k_r, \qquad s_1 = j.$$

and

$$j_0 = j + \sum_{i=1}^{n} k_i$$

the initial index. Here $D^k = D^k_{s_k}$ and we drop the indices for notational simplicity.

We have to consider the case $j_0 = 0$ but in the following it will be convenient to consider general $j_0 \geq 0$.

Note also that $E_0 = 1$ and the only possible term involving E_0 is

$$D_2^{-2} E_0 = \frac{1}{\mu} T_2 f^{\otimes 2}$$

which is perfectly defined in $L^1(\mathbb{R}^{2d})$ being $f = f(t)$ the solution of the homogeneous Boltzmann equation.

For simplicity and without loss of generality we choose the parameters in such a way that

$$\|T_j\| \leq \frac{j^2}{2}, \quad \|D_j\| \leq \frac{j}{2}, \; \|D_j^1\| \leq j, \; \|D_j^{-1}\| \leq \frac{j^2}{\mu}, \; \|D_j^{-2}\| \leq \frac{j^2}{\mu}$$

and we can prove

Proposition 4.1. *Suppose that,*

$$\|E_j(0)\|_1 \leq \left(\frac{j^2}{\mu}\right)^{j/2} C_0^j, \quad \textit{for all } j \geq 1 = 1, \tag{25}$$

for some $C_0 \geq 1$. Then, there exists t_0 sufficiently small and $A > 0$ (both explicitily computable) such that for any $t \leq t_0$,

$$\|E_j(t)\|_1 \leq \left(\frac{j^2}{\mu}\right)^{j/2} (AC_0)^j, \quad \textit{for all } j \geq 1. \tag{26}$$

We observe preliminarily that we can assume $j \leq \frac{2}{C_0}\sqrt{\mu}$ because otherwise

$$\|E_j(t)\| \leq 2^j \leq 2^j \left(\frac{jC_0}{2\sqrt{\mu}}\right)^j = \left(\frac{jC_0}{\sqrt{\mu}}\right)^j .$$

Next from (24) we have

$$E_j(t) = \sum_{n=0}^{M-1} \sum_{k_1 \cdots k_n} \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n$$
$$\bar{V}_{s_1}(t,t_1) D^{k_1} \ldots \bar{V}_{s_n}(t_{n-1},t_n) D^{k_n} \bar{V}_{s_{n+1}}(t_n) E_{j_0}(0) +$$
$$\sum_{k_1 \cdots k_M} \int_0^t dt_1 \cdots \int_0^{t_{M-1}} dt_M \bar{V}_{s_1}(t,t_1) D^{k_1} \ldots \bar{V}_{s_M}(t_{M-1} - t_M) D^{k_M} E_{j_0}(t_M)$$

and denote by $\mathcal{T}_j^1(t)$ and $\mathcal{T}_j^2(t)$ respectively the two terms in the right hand side of the above expression.

Moreover we fix $M = \frac{2}{C_0}\sqrt{\mu}$. With this choice

$$j + s_m \le j + M \le \frac{4}{C_0}\sqrt{\mu}$$

so that all the operators D_j^{-1} and D_j^{-2} appearing in the definition of $\mathcal{T}_j^1$ and $\mathcal{T}_j^2$ satisfy the bound

$$\|T_j\| \le j, \ \ \|D_j^{-1}\| \le \frac{j^2}{\mu} \le j, \quad \|D_j^{-2}\| \le \frac{j^2}{\mu} \le j$$

provided that μ is large enough. As a consequence, by (23)

$$\|\bar{V}_j(t)\| \le e^{jt}. \tag{27}$$

We first estimate $\mathcal{T}_j^2(t)$. Observe that, for $t_0 \le 1$ for which $e^{jt_0} \le e^j$

$$\|\bar{V}_{s_1}(t,t_1) D^{k_1} \ldots \bar{V}_{s_M}(t_{M-1} - t_M) D^{k_M} E_{j_0}(t_M)\| \le e^j e^{Mt} (j+M)^M 2^{j+M}$$

as follows by the estimates

$$e^{s_1(t-t_1)} e^{s_2(t_1-t_2)} \cdots e^{s_M(t_{M-1}-t_M)} \le e^{(j+M)t} \le e^j e^{Mt}$$

$$(j+k_1)(j+k_1+k_2)\cdots(j+k_1+k_2+\cdot k_M) \le (j+M)^M \le (2M)^M$$

$$\|E_{j_0}(t_M)\| \le 2^{j+M}.$$

Therefore, setting $\tau = te^t$, using the bound $M^M \le C_1^M M!$ and choosing τ so small that $\tau 8 C_1 \le \frac{1}{2}$, we have

$$\|T_j^2(t)\| \le 4^M \frac{\tau^M}{M!} M^M e^j 2^{j+M} \le (2e)^j (\tfrac{1}{2})^M.$$

Finally since

$$\sup_M M^j (\tfrac{1}{2})^M \le (C_2 j)^j$$

we conclude that

$$\|\mathcal{T}_j^2(t)\| \le \left(\frac{C_3 j}{\sqrt{\mu}}\right)^j. \tag{28}$$

To estimate $\mathcal{T}_j^1(t)$ we split it into two contributions $\mathcal{T}_j^{1,<}(t)$ and $\mathcal{T}_j^{1,>}(t)$ relative to the events $j_0 \le j$ and $j_0 > j$ respectively. Then, proceeding as above

$$\|\mathcal{T}_j^{1,<}(t)\| \le \sum_{n\le M-1} \sum_{k_1\cdots k_n} e^j \frac{\tau^n}{n!}(j+n)^n \left(\frac{j^2}{\mu}\right)^{\frac{j-j_0}{2}} \|E_{j_0}\|.$$

The contribution $(\frac{j^2}{\mu})^{\frac{j-j_0}{2}}$ arises from the fact that to reach j_0 from j we need at least $\frac{j-j_0}{2}$ negative jumps, each of them produces a factor bounded by $\frac{j^2}{\mu}$.

Finally using (25) and the usual arguments we arrive at

$$\|\mathcal{T}_j^{1,<}(t)\| \le \sum_{n\ge 0}(C_4\tau)^n e^j \left(\frac{j^2}{\mu}\right)^{\frac{j-j_0}{2}} \left(\frac{C_0^2 j_0^2}{\mu}\right)^{\frac{j_0}{2}} \le \left(\frac{C_5 C_0^2 j^2}{\mu}\right)^{\frac{j}{2}}.$$

Finally

$$\|\mathcal{T}_j^{1,>}(t)\| \le \sum_{n\le M-1} \sum_{k_1\cdots k_n} e^j \frac{\tau^n}{n!}(j+n)^n \left(\frac{j_0^2}{\mu}\right)^{\frac{j_0}{2}} C_0^{j_0}.$$

Setting now $\ell = \ell(k_1, k_2\cdots k_n) = j_0 - j$ we have

$$\begin{aligned} (\frac{(j+\ell)^2}{\mu})^{\frac{j+\ell}{2}} C_0^\ell &= \left(\frac{j^2}{\mu}\right)^{j/2} \left(\frac{(j+\ell)^2}{\mu}\right)^{\frac{\ell}{2}} \left(\frac{j+\ell}{j}\right)^j C_0^\ell \\ &\le \left(\frac{j^2}{\mu}\right)^{j/2} (\frac{2}{C_0})^\ell e^\ell C_0^\ell \le \left(\frac{j^2}{\mu}\right)^{j/2} (2e)^n. \end{aligned} \tag{29}$$

where, in the last step we used that (remind that both j and M are bounded by $\frac{2\sqrt{\mu}}{C_0}$)

$$\frac{(j+\ell)}{\sqrt{\mu}} \le \frac{j+M}{\sqrt{\mu}} \le \frac{4}{C_0}, \quad \left(\frac{j+\ell}{j}\right)^j \le e^\ell, \quad \ell \le n.$$

Thus, for τ small enough:

$$\|\mathcal{T}_j^{1,>}(t)\| \le e^{2j} C_0^j \sum_{n\ge 0} C_5^n \tau^n \left(\frac{j^2}{\mu}\right)^{\frac{j}{2}} \le \left(\frac{C_6 C_0^2 j^2}{\mu}\right)^{\frac{j}{2}}$$

and the proof is achieved.

As a Corollary we can easily prove

Proposition 4.2. *Let us suppose that $E_j(0) = 0$ for $j \ge 1$ ($E_\varnothing(0) = 1$) corresponding to a fully factorized initial state. Then for all $t > 0$ and all*

$j \geq 1$, *one has*

$$\|E_j(t)\|_1 \leq \left(Ae^{Bt}\right)^j \left(\frac{j}{\sqrt{\mu}}\right)^j \tag{30}$$

where $B \geq 0$ *and* $A \geq 1$.

Note that from the evolution equation

$$\partial_t E_1 = D_1 E_1 + C_{1,2} E_2$$

and Proposition 4.2 we obtain the estimate

$$\|E_1(t)\| \leq tA^2 e^{2Bt} \frac{4}{\mu}$$

which will be useful in a moment.

Proof of Theorem 3.2

For some time-dependent function $\varphi(t)$, using Proposition 4.2 and the estimate for $E_1(t)$ above, we find that

$$\begin{aligned}
\|f_j^\mu(t) - f^{\otimes j}(t)\| \leq & j\|E_1(t)\| + \sum_{k=2}^{j} \frac{j!}{k!(j-k)!}\|E_K(t)\| \\
& \leq \frac{jC_8}{\mu} + \sum_{k=2}^{j} \frac{j!}{k!(j-k)!}\left(\frac{\varphi(t)k}{\sqrt{\mu}}\right)^k \\
& \leq \frac{jC_8}{\mu} + \sum_{k=2}^{j}\left(\frac{C_7 j}{\sqrt{\mu}}\right)^k = \frac{jC_8}{\mu} + \left(\frac{C_7 j}{\sqrt{\mu}}\right)^2 \quad \frac{1}{1 - \frac{C_7 j}{\sqrt{\mu}}} \\
& = \left(\frac{Gj}{\sqrt{\mu}}\right)^2
\end{aligned}$$

for μ large such that $\frac{C_7 j}{\sqrt{\mu}} < 1$.

Appendix. Derivation of (17)

In this appendix, we deduce the correlation error equations starting from the BBGKY hierarchy, (13), which we rewrite for the reader's convenience:

$$\partial_t f_j^\mu = \frac{T_j}{\mu} f_j^\mu + C_{j+1} f_{j+1}^\mu \qquad j \geq 1\,.$$

We shall see that the computation is drastically simpler with respect to the analogous derivation in the canonical ensemble reported in the Appendix of [22] (where additional terms appear due to the canonical constraint).

We recall the definition of correlation error

$$E_j := \sum_{K \subset J} (-1)^{|K|} f_{J\setminus K}^\mu f^{\otimes K} \tag{31}$$

together with the following notation. For any set of particle indices $K = \{i_1, \cdots, i_k\}$ of cardinality $k = |K|$, we write

$$\begin{aligned} f^\mu_K &= f^\mu_k\left(z_{i_1}, \cdots, z_{i_k}\right) \ , \\ E_K &= E_k\left(z_{i_1}, \cdots, z_{i_k}\right) \ , \\ f^{\otimes K} &= f(z_{i_1}) \cdots f(z_{i_k}) \end{aligned}$$

with the conventions $f^\mu_\varnothing = E_\varnothing = f^{\otimes\varnothing} = 1$. We also write $Q(f,f)_i = Q(f,f)(v_i)$ for the Boltzmann operator (4) evaluated in particle i.

We start by the simple computation for $j = 1$:

$$\begin{aligned} \partial_t E_{\{1\}} &= \partial_t\left(f^\mu_{\{1\}} - f^{\otimes\{1\}}\right) \\ &= C_2 f^\mu_{\{1,2\}} - Q(f,f)_1 \\ &= C_2\left(f^{\otimes\{1,2\}} + f^{\otimes\{1\}}E_{\{2\}} + E_{\{1\}}f^{\otimes\{2\}} + E_{\{1,2\}}\right) - Q(f,f)_1 \end{aligned}$$

where in the last step we used the inverse formula (cf. (11)) for $j = 2$. Noticing that, by (14)-(15), $C_2(f^{\otimes\{1,2\}}) = Q(f,f)_1$, we are left with

$$\partial_t E_{\{1\}} = C_2\left(f^{\otimes\{1\}}E_{\{2\}} + E_{\{1\}}f^{\otimes\{2\}}\right) + C_2\left(E_{\{1,2\}}\right)$$

and the two terms on the right hand side correspond to $D_1 E_{\{1\}}$ and $D^1_1(E_{\{1,2\}})$ respectively. Therefore (17) is verified for $j = 1$.

We assume now that $\{E_k\}_{k\le j-1}$ satisfy (17), and prove the same thing for $k = j$. Using definition (31) and the evolution equations for f^μ_J and f, we find that

$$\begin{aligned} \partial_t E_J &= \partial_t\left(f^\mu_J - \sum_{\substack{K\subset J\\ K\neq\varnothing}} f^{\otimes K}\, E_{J\setminus K}\right) \\ &= \frac{T_j}{\mu} f^\mu_J + \sum_{i\in J} C_{i,j+1} f^\mu_{J\cup\{j+1\}} - \sum_{\substack{K\subset J\\ K\neq\varnothing}} f^{\otimes K}\,\partial_t E_{J\setminus K} \\ &\quad - \sum_{\substack{K\subset J\\ K\neq\varnothing}} \sum_{i\in K} f^{\otimes K\setminus\{i\}} Q(f,f)_i E_{J\setminus K} \ . \end{aligned} \tag{32}$$

We compute the three terms in the second line, one by one. First, using the inverse formula and denoting $J^i = J\backslash\{i\}$, $J^{i,s} = J\backslash\{i,s\}$ etc.,

$$\begin{aligned}\frac{T_j}{\mu} f^\mu_J = \frac{T_j}{\mu} E_J + \sum_{\substack{i,s\in J\\ i\neq s}} \frac{T_{i,s}}{\mu} \left\{ f^{\otimes\{i\}} E_{J^i} + \frac{1}{2} f^{\otimes\{i,s\}} E_{J^{i,s}} \right\} \\ + \sum_{\substack{K\subset J\\ K\neq\varnothing}} f^{\otimes K} \frac{T_{j-k}}{\mu} E_{J\backslash K} \\ + \sum_{\substack{K\subset J\\ |K|\neq 0,1}} \sum_{\substack{i\in K\\ s\in J\backslash K}} \frac{T_{i,s}}{\mu} f^{\otimes\{i\}} f^{\otimes K^i} E_{J\backslash K} + \sum_{\substack{K\subset J\\ |K|\neq 0,1,2}} \sum_{\substack{i,s\in K\\ i\neq s}} \frac{T_{i,s}}{2\mu} f^{\otimes\{i,s\}} f^{\otimes K^{i,s}} E_{J\backslash K}\end{aligned} \tag{33}$$

and, similarly,

$$\begin{aligned}\sum_{i\in J} C_{i,j+1} f^\mu_{J\cup\{j+1\}} = \sum_{i\in J} C_{i,j+1} \left\{ E_J f^{\otimes\{j+1\}} + f^{\otimes\{i\}} E_{J^i\cup\{j+1\}} + E_{J\cup\{j+1\}} \right\} \\ + \sum_{\substack{K\subset J\\ |K|\neq 0}} \sum_{i\in J\backslash K} C_{i,j+1} f^{\otimes K} f^{\otimes\{j+1\}} E_{J\backslash K} \\ + \sum_{\substack{K\subset J\\ K\neq\varnothing}} \sum_{i\in J\backslash K} C_{i,j+1} f^{\otimes K} \left\{ f^{\otimes\{j+1\}} E_{J\backslash K} + f^{\otimes\{i\}} E_{(J\cup\{j+1\}\backslash K)^i} + E_{J\cup\{j+1\}\backslash K} \right\}.\end{aligned} \tag{34}$$

Secondly, using the inductive hypothesis and the explicit definition of the operators $D_{j-k}, D^1_{j-k}, D^{-1}_{j-k}, D^{-2}_{j-k}$, we obtain that

$$\begin{aligned}- \sum_{\substack{K\subset J\\ K\neq\varnothing}} f^{\otimes K}\, \partial_t E_{J\backslash K} = - \sum_{\substack{K\subset J\\ K\neq\varnothing}} f^{\otimes K} \frac{T_{j-k}}{\mu} E_{J\backslash K} \\ - \sum_{\substack{K\subset J\\ K\neq\varnothing}} \sum_{\substack{i,s\in J\backslash K\\ i\neq s}} \frac{T_{i,s}}{\mu} f^{\otimes K} \left\{ f^{\otimes\{i\}} E_{(J\backslash K)^i} + \frac{1}{2} f^{\otimes\{i,s\}} E_{(J\backslash K)^{i,s}} \right\} \\ - \sum_{\substack{K\subset J\\ K\neq\varnothing}} \sum_{i\in J\backslash K} C_{i,j+1} f^{\otimes K} \left\{ f^{\otimes\{j+1\}} E_{J\backslash K} + f^{\otimes\{i\}} E_{(J\cup\{j+1\}\backslash K)^i} + E_{J\cup\{j+1\}\backslash K} \right\}.\end{aligned} \tag{35}$$

We note that the first line in (33) and the first line in (34) reproduce the right hand side of the desired equation (17). Therefore we need to check that the remaining terms cancel out. This is straightforward:

- the second line in (33) cancels with the first term on the right hand side in (35);
- last line in (33) cancels with the second line in (35) (change variables $K \to K \cup \{i\}$ and $K \to K \cup \{i,s\}$);
- second line in (34) cancels with last line in (32);
- last line in (34) cancels with last line in (35).

This concludes the derivation of (17).

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(T.P) CNRS Laboratoire Ypatia des Sciences Mathématiques (LYSM), Roma, Italia & Laboratoire Jacques-Louis Lions (LJLL), Sorbonne Université, Paris, France

Email address: thierry.paul@sorbonne-universite.fr

(M.P.) Sapienza Università di Roma, Dipartimento di Matematica G. Castelnuovo, Piazzale A. Moro 5, 00185 Roma – Italia and International Research Center on the Mathematics and Mechanics of Complex Systems, ï¿œMeMoCS, University of L'Aquila, Italy

Email address: pulviren@mat.uniroma1.it

(S.S.) Sapienza Università di Roma, Dipartimento di Matematica G. Castelnuovo, Piazzale A. Moro 5, 00185 Roma – Italia

Email address: sergio.simonella@uniroma1.it