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In order to account for rotation and anelastic effects in the normal modes of the Earth, a structure of the space of normal modes different from those generally used in the elastic self-adjoint case is necessary. This can be done with a duality relation between the eigenproblem and the one obtained simply by reversing the Earth's rotation. This leads to new biorthonormality relations between modes and dual modes. Seismograms can then be expressed in terms of a normal mode expansion. The normal modes of an anelastic rotating Earth can be computed with perturbation theory. In order to take into account the coupling terms between different dispersion curves, as well as between toroidal and spheroidal modes, the perturbations start from an anelastic, non-rotating Earth rather than from an elastic one. The secular terms of the perturbation series, due to coupling effects between modes of the same multiplet, can then be removed. This ensures that higher order perturbation theory converges to the anelastic modes with sufficient accuracy, and gives, up to third order, expressions for the eigenmodes and eigenfrequencies. These expressions can be used to compute modes and seismograms of an anelastic realistic Earth model, neglecting neither rotation, anelasticity, anisotropy or lateral heterogeneities.

INTRODUCTION

The exact formulation of the normal modes of an anelastic, dispersive, rotating and laterally heterogeneous Earth has been one of the "missing" theories in seismology. The normal modes and seismograms were always computed by neglecting the rotation or the physical dispersion somewhere in the theory, and anelasticity was generally treated using first-order perturbation theory. Examples of simplified theory were presented by *Dahlen and Smith* [1975] and *Valette* [1987] in the rotating elastic case, by *Dahlen* [1981] in the nonrotating anelastic and nondispersive case, and in the nonrotating, anelastic and dispersive case by *Lognonné* [1989]. All these theories were valid for a spherical or laterally heterogeneous Earth, but the two last ones were unable to incorporate the effect of the Coriolis operator in the biorthogonality relation between modes, which lead to singularities in eigenfunction normalization. This was also the case of the Galerkin theory, proposed by *Park and Gilbert* [1986], where the Coriolis operator was taken into account in the eigenequation, but neglected in the biorthogonality relation between modes. This theory was until now the most accurate in the computation of realistic Earth normal modes and seismograms. Unfortunately, this method requires large matrix diagonalizations.

However, normal modes observations have reached a state of the art which cannot be satisfied by these simplifications. Recent papers show that anelasticity greatly affects modes and seismogram observations, either by a direct coupling effect between modes, due to lateral variations in the anelasticity of the Earth [*Romanowicz*, 1990; *Roult et al.*, 1990], or by controlling the strength of coupling due to lateral heterogeneities and rotation, for the spherical averaged anelastic structure. This is the case for the coupling effects between modes belonging to different dispersion branches, which affect either the mean frequency of a given multiplet or, on the data, the amplitude of the associated resonant peak. Reports of such effects on the mean frequency of modes are given, for example, by *Masters et al.* [1982], *Romanowicz and Roult*, [1986], *Smith and Masters* [1989], *Roult et al.* [1990], and are relatively well explained by interactions between the fundamental spheroidal and toroidal branches, either due to the Coriolis coupling for very long periods [*Masters et al.*, 1983] or to the lateral heterogeneities for long periods [*Lognonné*, 1989]. Similarly, fundamental/overtones or overtones/overtones coupling effects due to the "roughness" [*Park*, 1989] of the Earth can produce great amplitude variations on some weakly excited normal modes, such as radial modes [*Park*, 1990] or core modes [*Lognonné and Romanowicz*, 1990b]. These coupling effects are strongly controlled by the Q ratio of the interacting modes [*Woodhouse*, 1980; *Park*, 1986]. A theory, which considers the anelasticity as a perturbation or neglects rotation, is unable to model these effects, and only a complete theory, including rotation, and anelasticity with physical dispersion should be used.

This theory is described in this paper. We start by recalling some properties of the gravito-anelastic operator of a laterally heterogeneous Earth. These properties show that the dual space is generated by the eigenfunctions of a dual-eigenproblem, obtained simply by reversing the Earth rotation velocity. This simple property comes from the fact that the spectrum of the eigenproblem and of its dual are identical, which means that the eigenfrequencies of modes do not depend on the sense of the Earth's rotation. Using the dual space, it is now possible to define a general biorthogonal relation between the modes and their duals, which allows expression of the seismograms in the time domain. Lateral variations in the elastic and anelastic structure, as well as rotation, can now be considered as perturbation. A perturbation procedure starting from a spherical non-rotating anelastic and isotropic Earth (SNRAI) is given. In a similar way as in the elastic, rotating case, as shown by *Lognonné and Romanowicz* [1990a], it is possible to choose the perturbation path in such a way that the secular terms cancel up to second order, which allows us to compute in a very fast and accurate way both eigenfrequencies and eigenmodes of any realistic Earth model, including anelasticity, physical dispersion, rotation and lateral variations.

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THE SEISMIC EQUATION IN THE ROTATING ANELASTIC CASE

Let us first recall the expression of the equation of motion of a rotating, elastic Earth, obtained, for example, by *Woodhouse and Dahlen* [1978]:

$$\partial_t^2|\mathbf{u}(t)\rangle - i\mathbf{B}\partial_t|\mathbf{u}(t)\rangle + \mathbf{A}|\mathbf{u}(t)\rangle = |\mathbf{f}(t)\rangle \quad , \qquad (1)$$

where $|\mathbf{u}(t)\rangle$ is the displacement field in braket notation, $|\mathbf{f}(t)\rangle$ the equivalent body force for sources and excitation terms, $\mathbf{A}$ the elasto-dynamic operator and $\mathbf{B}$ the Coriolis operator, both defined in the works by *Woodhouse and Dahlen* [1978] or *Valette* [1986]. In all relations which follow, as in relation (1), all brackets and operator will implicitely depend on space. The generalization to the anelastic case is, for instance, shown by *Liu et al.* [1976], *Dahlen* [1981] or *Tarantola* [1988] and leads to a substitution of the relationship between stress tensor τ_{ij} and strain ϵ^{kl} by a more general time convolution, which, in the case of local anelasticity, can be described using a kernel C_{ijkl}. If the properties of the medium do not depend on time, and as the tensor C_{ijkl} should be causal, the stress tensor can be expressed as

$$\tau_{ij}(t) = \int_{-\infty}^{t} C_{ijkl}(t-t')\, \epsilon^{kl}(t')\, dt' \ , \qquad (2)$$

which means that all products between the strain tensor ϵ^{kl} and the anelastic tensor C_{ijkl} appearing in the expression of the operator $\mathbf{A}$ have to be rewritten in terms of convolutions. Taking the Fourier-Laplace transformation, defined here, in bracket notation, as

$$|\mathbf{u}(\sigma)\rangle = \int_{0}^{+\infty} e^{-i\sigma t}\, |\mathbf{u}(t)\rangle\, dt \ , \qquad (3)$$

where $\sigma = \omega + i\alpha$ is a complex frequency in the upper part of the complex plane, we obtain the expression of the seismic equation (1) in the complex frequency domain:

$$-\sigma^2|\mathbf{u}(\sigma)\rangle + \sigma\mathbf{B}|\mathbf{u}(\sigma)\rangle + \mathbf{A}(\sigma)|\mathbf{u}(\sigma)\rangle = |\mathbf{f}(\sigma)\rangle \ . \qquad (4)$$

This is the general frequency expression of the seismic equation of an anelastic, physical dispersive and rotating Earth. Here the expression of the operator $\mathbf{A}$ is the same as in the elastic case, but with a complex and frequency dependent stiffness tensor. For example, it is possible to model attenuation with the isotropic absorption band model of *Liu et al.* [1976], where the elastic moduli μ and κ become complex and frequency dependent according to the relations

$$\mu(\sigma) = \mu\left(1 + Q_\mu^{-1}\left[\frac{2}{\pi}Ln\left(\frac{\sigma}{\omega_0}\right) - i\right]\right) \ ,$$
$$\kappa(\sigma) = \kappa\left(1 + Q_\kappa^{-1}\left[\frac{2}{\pi}Ln\left(\frac{\sigma}{\omega_0}\right) - i\right]\right) \ ,$$

However, it must be noted that the logarithmic branch at $\sigma = \infty$ of such a model is not physical. It is generally assumed that the intrinsic quality factor Q^{-1} goes to zero as $1/\omega$ and ω with the high and low frequencies, respectively [*Anderson and Minster*, 1979; *Anderson*, 1989]. The high frequency limits of the equation (4) is thus simply the elastic, nonrotating equation.

We will assume in what follows that the density and elastic structure of the Earth is such that no unstable normal modes exist, which means that all the eigenfrequencies occur in the upper complex σ plane and will address our description to the seismic normal modes. In this case the eventual singularities at the zero frequency in the definition of the attenuation can be neglected and the operator $A(\sigma)$ can be defined in the half-upper plane as well as in the half-lower plane by analytic continuation. Except for this assumption, the model can include laterally heterogeneous anelastic and anisotropic structure, as well as lateral variations of density. No assumptions are required either on the frequency dependence of the real and imaginary parts of the anelastic stifness tensor.

The eigenmodes $|\mathbf{u}_k\rangle$ and associated eigenfrequencies σ_k will be such that

$$-\sigma_k^2|\mathbf{u}_k\rangle + \sigma_k\mathbf{B}|\mathbf{u}_k\rangle + \mathbf{A}(\sigma_k)|\mathbf{u}_k\rangle = 0 \ . \qquad (5)$$

This is the problem to be solved, which is not only non self-adjoint, as long anelasticity is introduced, but also frequency dependent by the introduction of physical dispersion. However, even in the general case, some properties of the operator $\mathbf{B}$ and $\mathbf{A}(\sigma)$ remain. Let us now define the bilinear form:

$$\langle\mathbf{v}|\mathbf{u}\rangle = \int_V \rho\, dV\ \mathbf{u}(\mathbf{r})\ \mathbf{v}(\mathbf{r}) \ , \qquad (6)$$

where ρ is the density. It differs from the usual inner product by the fact that the left vector is not complex conjugated. It is then easy to show, using the expression of the operator $\mathbf{B}$ or $\mathbf{A}$ [*Valette*, 1986], more precisely the antisymmetry of the curl product for the Coriolis operator, or the symmetry of the stiffness and prestress tensor, that $\mathbf{B}$ and $\mathbf{A}$ are antisymmetric and symmetric, respectively, for the bilinear form (6), which means that

$$\langle\mathbf{v}|\mathbf{B}\mathbf{u}\rangle = -\langle\mathbf{B}\mathbf{v}|\mathbf{u}\rangle \ , \qquad (7)$$

$$\langle\mathbf{v}|\mathbf{A}\mathbf{u}\rangle = \langle\mathbf{A}\mathbf{v}|\mathbf{u}\rangle \ . \qquad (8)$$

THE DUAL SPACE AND THE BIORTHOGONAL RELATIONS

The normal mode equation (5) is not the only one required to solve the associated eigenproblem, especially if it is solved by a variational method, i.e., by minimizing a Rayleigh quotient. In the same manner, the computation of the response $|\mathbf{u}(\sigma)\rangle$ to a given external force $|\mathbf{f}(\sigma)\rangle$ cannot be determined with the eigensolutions of (5) only. In both cases, we need additional information, given by an associated space of adjoint eigenfunctions, if a Hilbert inner product can be defined, as in the elastic case, or, more generally, by an associated space of dual eigenfunctions with a dual relation or a biorthogonal product. In our case in order to deal with the antisymmetry of the Coriolis operator, we will define the dual space as the space mapped by the eigenfunctions $|\mathbf{v}_k\rangle$ of the eigenproblem obtained by reversing the Earth rotation. The eigenfunctions $|\mathbf{v}_k\rangle$ and their associated eigenfrequencies σ_k are thus solution of the spectral equation

$$-\sigma_k^2|\mathbf{v}_k\rangle - \sigma_k\mathbf{B}|\mathbf{v}_k\rangle + \mathbf{A}(\sigma_k)|\mathbf{v}_k\rangle = 0 \ . \qquad (9)$$

As shown in Appendix A, this mapping is possible as the spectrum of the two eigenproblems, with or without reversed rotation velocity, is the same. Physically, this means that the eigenfrequencies of the Earth do not depend on the sense of the Earth's rotation. However, for a given eigenfrequency, the two dual eigenmodes are different. As no degeneracy occurs in the rotating case, each eigenfrequency σ_k is associated with one and only one eigenmode $|\mathbf{u}_k\rangle$, and also one dual eigenmode $|\mathbf{v}_k\rangle$. In what follows, we shall note the space mapped by all eigenfunctions $|\mathbf{u}_k\rangle$ **U**, and its dual, mapped by the dual eigenfunctions $|\mathbf{v}_k\rangle$, **V**. Note that for each eigenfrequency, a linear form over **U** or **V** can be defined using the bilinear form (6).

Another important dual relation between modes is related to the fact that the tensor C_{ijkl} used in relation (2) is real. This relation can be used to show that [*Nowick and Berry*, 1972; *Dahlen*, 1981]

$$\mathbf{A}^*(\sigma) = \mathbf{A}(-\sigma^*) \ , \qquad (10)$$

where the asterisk means the complex conjugation. If we use this relation and the fact that the Coriolis operator anticommutes with complex conjugation, it it easy to show, by taking the complex conjugate of relations (5) and (9), that for each eigenfrequency σ_k, with the eigenmode and dual-eigenmode $|\mathbf{u}_k\rangle$ and $|\mathbf{v}_k\rangle$, a complex-dual eigenfrequency $-\sigma_k^*$ can be associated with the eigenmodes $|\mathbf{u}^*{}_k\rangle$ and $|\mathbf{v}^*{}_k\rangle$. Eigenfrequencies with positive real part ω_k will thus be noted with a positive index k, and the ones with a negative real part will be with a negative one. There is, however, no direct relation between the functions $|\mathbf{u}_k\rangle$ and $|\mathbf{v}_k\rangle$, except in the spherical case, for which, as showed in Appendix A, we have

$$|\mathbf{v}_k\rangle = |\mathbf{S}\mathbf{u}_k\rangle \ , \qquad (11)$$

where S is a orthogonal symmetry around a plane containing the rotation axis. More generally, we can write, in the general case,

$$\begin{aligned}-\sigma_k^2|\mathbf{u}^*{}_k\rangle \ - \ & \sigma_k\mathbf{B}|\mathbf{u}^*{}_k\rangle + \mathbf{A}(\sigma_k)|\mathbf{u}^*{}_k\rangle = \\ - \ & 4i\alpha_k\omega_k|\mathbf{u}^*{}_k\rangle \\ - \ & 2i\alpha_k\mathbf{B}|\mathbf{u}^*{}_k\rangle \\ - \ & [\mathbf{A}^*(\sigma_k) - \mathbf{A}(\sigma_k)]\ |\mathbf{u}^*{}_k\rangle \ . \qquad (12)\end{aligned}$$

For all seismic modes, rotation and anelasticity are perturbations compared to the elastic structure of the Earth. Equation (12) shows thus that the difference between $|\mathbf{v}_k\rangle$ and $|\mathbf{u}^*{}_k\rangle$ is to first order related to the attenuation, and to second order related to the cumulated effect of attenuation and rotation.

Let us now define more precisely the relation between the primal space and dual space. For this purpose, let us now take two eigenfrequencies σ_k and $\sigma_{k'}$, the associated mode $|\mathbf{u}_k\rangle$ and the dual mode $|\mathbf{v}_{k'}\rangle$, solution of

$$\mathcal{H}(\sigma_k)|\mathbf{u}_k\rangle = 0 \ , \qquad (13)$$

$$\widehat{\mathcal{H}}(\sigma_{k'})|\mathbf{v}_{k'}\rangle = 0 \ , \qquad (14)$$

where the integro differential operator $\mathcal{H}(\sigma)$ is given by

$$\mathcal{H}(\sigma) = -\sigma^2\mathcal{I} + \sigma\mathbf{B} + \mathbf{A}(\sigma) \ , \qquad (15)$$

and its dual operator was defined as

$$\widehat{\mathcal{H}}(\sigma) = -\sigma^2\mathcal{I} - \sigma\mathbf{B} + \mathbf{A}(\sigma) \ . \qquad (16)$$

Here $\mathcal{I}$ is the identity operator and the operators are such that

$$\langle\mathbf{v}|\mathcal{H}(\sigma)\mathbf{u}\rangle = \langle\widehat{\mathcal{H}}(\sigma)\mathbf{v}|\mathbf{u}\rangle \ . \qquad (17)$$

Let us now multiply the relations (13) and (14) by the bra $\langle\mathbf{v}_{k'}|$ and $\langle\mathbf{u}_k|$, respectively, make the difference of the two obtained relations using relations (7) and (8) and divide the result by $\sigma_{k'}^2 - \sigma_k^2$. We finally get the following biorthogonality relation:

$$\langle\mathbf{v}_{k'}|\mathbf{u}_k\rangle - \frac{1}{\sigma_k + \sigma_{k'}}[\langle\mathbf{v}_{k'}|\mathbf{B}\mathbf{u}_k\rangle + \frac{\langle\mathbf{v}_{k'}|\mathbf{A}(\sigma_{k'})\mathbf{u}_k\rangle - \langle\mathbf{v}_{k'}|\mathbf{A}(\sigma_k)\mathbf{u}_k\rangle}{\sigma_{k'} - \sigma_k}] = 0 \ . \qquad (18)$$

The relation (18) defines the general biorthogonality relation between the modes associated with two different eigenfrequencies. However, the quantity at the right remains defined when $\sigma_{k'} \rightarrow \sigma_k$ and its limit is simply $-\langle\mathbf{v}_k|\partial_\sigma\mathcal{H}(\sigma)\mathbf{u}_k\rangle/2\sigma_k$. As rotation and attenuation, as well as differences between $|\mathbf{v}_k\rangle$ and $|\mathbf{u}^*{}_k\rangle$ are typically small perturbations for realistic Earth model, this limit will not in general be zero for the seimic modes, and the pole of $\langle\mathbf{v}_k|\mathcal{H}(\sigma)\mathbf{u}_k\rangle$ for $\sigma = \sigma_k$ will thus be simple. This, however, may not be true for secular and subseismic modes, which need a more detailed study. We will thus use this limit for the normalization of the seismic modes, and we will choose the normalization so that

$$\langle\mathbf{v}_k|\mathbf{u}_k\rangle - \frac{1}{2\sigma_k}\left[\langle\mathbf{v}_k|\mathbf{B}\mathbf{u}_k\rangle + \langle\mathbf{v}_k|\partial_\sigma\mathbf{A}(\sigma_k)\mathbf{u}_k\rangle\right] = 1 \ . \qquad (19)$$

It is noteworthy that in the elastic limit, as shown in Appendix A, $\mathbf{v} \rightarrow \mathbf{u}^*$, $\sigma_k \rightarrow \omega_k$ so that relations (18) and (19) give the orthogonality relation obtained by *Dahlen and Smith* [1975]

$$\langle\mathbf{v}^*{}_{k'}|\mathbf{u}_k\rangle - \frac{1}{\omega_k + \omega_{k'}}\langle\mathbf{v}^*{}_{k'}|\mathbf{B}\mathbf{u}_k\rangle = \delta_{kk'} \ . \qquad (20)$$

We have defined the eigenequation (13), the dual one (14) and the biorthogonality relations (18) and (19). We can now compute the response of the Earth to a generalized body force and develop a perturbation theory in order to compute the modes of a laterally heterogeneous Earth.

Mode Summation and Seismograms

Let us now compute the response of the Earth produced by an equivalent body force $|\mathbf{f}(t)\rangle$. The displacement will be obtained by solving the seismic equation in the frequency domain, as written in relation (4)

$$\mathcal{H}(\sigma)|\mathbf{u}(\sigma)\rangle = |\mathbf{f}(\sigma)\rangle \ , \qquad (21)$$

and by computing the inverse Laplace transformation, defined as

$$|\mathbf{u}(t)\rangle = \frac{1}{2\pi}\int_{i\epsilon-\infty}^{i\epsilon+\infty}|\mathbf{u}(\sigma)\rangle\ e^{i\sigma t}\ d\sigma \ . \qquad (22)$$

To express this integral, we need to determine the pole of $|\mathbf{u}(\sigma)\rangle$. These poles are first the frequencies for which $\mathcal{H}(\sigma)$ is not inversible, by definition the discrete eigenfrequencies $\sigma_{k'}$ and eventually those corresponding to the dense or continuum spectrum induced by the fluid part of the Earth [*Valette*, 1989a], neglected in what follows. Note that as the attenuation is tending to zero when the frequency goes to infinity, $|\mathbf{u}(\sigma)\rangle$ decreases with the frequency as σ^{-2} and no singularities occur along the horizontal plane $\alpha = 0$. Let us now assume that the solution can be expressed in the form

$$|\mathbf{u}(\sigma)\rangle = \sum_{k'} c_{k'}(\sigma)|\mathbf{u}_{k'}\rangle \ . \qquad (23)$$

In principle, this summation is possible if the space of all modes is complete. However, such a completeness has been demonstrated only in the elastic and rotating case by *Valette*

[1989b], and we will assume that this is also the case in the anelastic case. Theoretically, all eigenfunctions must be included in this summation, that is, not only the modes of the seismic spectrum (e.g., with a period less than 1 hour) but also the secular modes with zero frequency and the subseismic modes, with period greater than 1 hour. However, as this paper is focused on the seismic modes, we will neglect in what follows the contribution of the secular modes in the seismograms and in the computation of the seismic normal modes of an aspherical Earth. The summation in relation (23) will thus be limited to the seismic modes.

Let us now insert the expression (23) in the relation (21),

$$\mathcal{H}(\sigma)\sum_{k'} c_{k'}(\sigma)|\mathbf{u}_{k'}\rangle = |\mathbf{f}(\sigma)\rangle\ , \tag{24}$$

and take an impulsive force, for which $|\mathbf{f}(\sigma)\rangle$ does not depend on the frequency and has no poles. For a given eigenfrequency σ_k, and as

$$\mathcal{H}(\sigma_k)|\mathbf{u}_k\rangle = 0\ ,$$

for the eigenmode $|\mathbf{u}_k\rangle$ only, the relation (24) shows that the pole for σ_k appears only for $c_k(\sigma)$, and is simple. We can thus, using the Cauchy theorem, express $|\mathbf{u}(t)\rangle$ as

$$|\mathbf{u}(t)\rangle = H(t)\sum_{k'} e^{i\sigma_{k'}t}\lim_{\sigma\to\sigma_{k'}} i(\sigma-\sigma_{k'})c_{k'}(\sigma)|\mathbf{u}_{k'}\rangle\ , \tag{25}$$

and have now to express the limits appearing in this relation. For this purpose, let us first multiply (24) by $\langle\mathbf{v}_k|$ and write

$$\begin{aligned} c_k(\sigma) &= \frac{\langle\mathbf{v}_k|\mathbf{f}\rangle}{\langle\mathbf{v}_k|\mathcal{H}(\sigma)\mathbf{u}_k\rangle} \\ &- \sum_{k'\neq k} c_{k'}(\sigma)\frac{\langle\mathbf{v}_k|\mathcal{H}(\sigma)\mathbf{u}_{k'}\rangle}{\langle\mathbf{v}_k|\mathcal{H}(\sigma)\mathbf{u}_k\rangle}\ . \end{aligned} \tag{26}$$

Using

$$\langle\mathbf{v}_k|\mathcal{H}(\sigma_{k'})\mathbf{u}_{k'}\rangle = \langle\widehat{\mathcal{H}}(\sigma_k)\mathbf{v}_k|\mathbf{u}_{k'}\rangle = 0\ ,$$

we get from relation (26)

$$\begin{aligned} c_k(\sigma) &= \frac{\langle\mathbf{v}_k|\mathbf{f}\rangle}{\langle\mathbf{v}_k|\mathcal{H}(\sigma)\mathbf{u}_k\rangle - \langle\mathbf{v}_k|\mathcal{H}(\sigma_k)\mathbf{u}_k\rangle} \\ &- \sum_{k'\neq k} c_{k'}(\sigma)\frac{\langle\widehat{\mathcal{H}}(\sigma)\mathbf{v}_k|\mathbf{u}_{k'}\rangle}{\langle\mathbf{v}_k|\mathcal{H}(\sigma)\mathbf{u}_k\rangle - \langle\mathbf{v}_k|\mathcal{H}(\sigma_k)\mathbf{u}_k\rangle}\ . \end{aligned} \tag{27}$$

Using the normalization relation (19), the relation (27) shows that the limit in (25) is given by

$$\lim_{\sigma\to\sigma_k} i(\sigma-\sigma_k)c_k(\sigma) = \frac{i\langle\mathbf{v}_k|\mathbf{f}\rangle}{\langle\mathbf{v}_k|\partial_\sigma\mathcal{H}(\sigma_k)\mathbf{u}_k\rangle} = \frac{\langle\mathbf{v}_k|\mathbf{f}\rangle}{2i\sigma_k}\ . \tag{28}$$

Making now the summations over all positive and negative indices, we obtain the response of the Earth to an impulsive equivalent body force:

$$|\mathbf{u}(t)\rangle = H(t)\sum_{k>0}\Re e\left(\frac{1}{i\sigma_k}\langle\mathbf{v}_k|\mathbf{f}\rangle e^{i\sigma_k t}|\mathbf{u}_k\rangle\right)\ . \tag{29}$$

The response to a tensor moment source,

$$|\mathbf{f}(t)\rangle = -\mathbf{M}(t)\nabla\delta(\mathbf{r}-\mathbf{r}_s)\ , \tag{30}$$

where $\mathbf{r}_s$ is the source location, can thus obtained by convolution and gives for $t>0$

$$|\mathbf{u}(t)\rangle = \sum_{k>0}\Re e\left(\frac{1}{i\sigma_k}\int_0^t dt'\mathbf{M}(t') : \nabla\mathbf{v}_k(\mathbf{r}_s)e^{i\sigma_k(t-t')}|\mathbf{u}_k\rangle\right)\ . \tag{31}$$

If the source function is a Heaviside $\mathbf{M}(t) = \mathbf{M}H(t)$, we finally obtain

$$|\mathbf{u}(t)\rangle = H(t)\sum_{k>0}\Re e\left(\frac{1}{\sigma_k^2}\mathbf{M} : \nabla\mathbf{v}_k(\mathbf{r}_s)\ \ (1-e^{i\sigma_k t})|\mathbf{u}_k\rangle\right)\ . \tag{32}$$

Let us note that this expression differs from others derived previously by *Gilbert* [1970] or *Dahlen* [1981]. An initial source phase shift due either to the imaginary part of the eigenfunctions or to σ_k^2 appears. Using Appendix A, it is easy to show that relation (32) becomes the same as Gilbert's in the elastic case, when $|\mathbf{v}_k\rangle \to |\mathbf{u}^*{}_k\rangle$. In order to use this expression, we need now to compute the normal modes and their duals for a laterally heterogeneous Earth. We will now show how this can be done by using perturbation theory.

PERTURBATION THEORY

Perturbation theory is a powerful method to compute the hybrid normal modes for the range of the lateral heterogeneities of the Earth. In the elastic case, *Lognonné and Romanowicz* [1990a] have shown that perturbation theory can be iterated and leads to the solution obtained with more timeconsuming methods, such as variational methods [*Morris et al.*, 1987]. Let us now start from a spherical non rotating anelastic and isotropic Earth (SNRAI) for which eigenfrequencies and eigenmodes can be easily computed [*Tromp and Dahlen*, 1990]. The spherical eigenmodes and their associated eigenfrequencies will be noted $|\mathbf{u}_j^{(0)}\rangle$ and σ_{0j}, the index j being associated with this spherical basis. Their corresponding displacement can be written in the form

$$\begin{aligned} {}_n\mathbf{u}_{\ell m}^{(0)} &= {}_nU_\ell(r)\ Y_\ell^m(\theta,\phi)\mathbf{e}_r \\ &+ {}_nV_\ell(r)\ \nabla_1 Y_\ell^m(\theta,\phi) \\ &+ {}_nW_\ell(r)\ \mathbf{e}_r\wedge\nabla_1 Y_\ell^m(\theta,\phi)\ , \end{aligned} \tag{33}$$

where U, V, W are complex functions of the radius r which depend on n and ℓ, the radial and angular order of the modes, $Y_\ell^m(\theta,\phi)$ is the fully normalized spherical harmonics, ∇_1 the gradient operator on the unit sphere, $\mathbf{e}_r$ the radial basis vector of the spherical Cartesian basis $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$, and θ, ϕ the colatitude and longitude, respectively. As a consequence of the spherical symmetries, the eigenfrequencies σ_{0j} will depend only on the index ℓ and n, each eigenspace being $2\ell+1$ degenerate. According to (11), the spherical associated dual eigenmodes will be taken proportional to

$$\begin{aligned} {}_n\mathbf{v}_{\ell m}^{(0)} &= \mathbf{S}_n\mathbf{u}_{\ell m}^{(0)} \\ &= {}_nU_\ell(r)\ (-1)^m\ Y_\ell^{-m}(\theta,\phi)\mathbf{e}_r \\ &+ {}_nV_\ell(r)\ (-1)^m\ \nabla_1 Y_\ell^{-m}(\theta,\phi) \\ &+ {}_nW_\ell(r)\ (-1)^m\ \mathbf{e}_r\wedge\nabla_1 Y_\ell^{-m}(\theta,\phi)\ . \end{aligned} \tag{34}$$

As there is no rotation in the spherical case, and as shown by (18) and (19), they will be normalized in such a way that

$$\langle\mathbf{v}_{j'}^{(0)}|\mathbf{u}_j^{(0)}\rangle - \frac{1}{2\sigma_{0j}}\langle\mathbf{v}_{j'}^{(0)}|\partial_\sigma\mathbf{A}_0(\sigma_{0j})\mathbf{u}_j^{(0)}\rangle = \delta_{jj'}\ . \tag{35}$$

where $\mathbf{A}_0(\sigma)$ is the spherical anelastic operator. This orthogonality relation will be verified only for the $2\ell+1$ modes associated with the eigenfrequency σ_{0_j}. In the relation (35) and in what follows, all dual products will be defined with respect to the spherical density and will be given by

$$\langle \mathbf{v}|\mathbf{u}\rangle = \int_V \rho_0 \, dV \, \mathbf{u}(\mathbf{r}).\mathbf{v}(\mathbf{r}) \; . \tag{36}$$

As a consequence of dispersion, the spherical modes with the same ℓ and m value and on the same type (e.g., spheroidals or toroidals) but with different n radial numbers are not orthogonal for the duality product (35). In contrast, spherical modes with different ℓ or m numbers, or modes of different types, will be biorthogonal for the duality relation (35).

Let us assume now that lateral heterogeneities in both density, elastic and anelastic structure are superimposed on the spherical Earth structure and that the rotation is now taken into account. The difference between the aspherical operator $\mathbf{A}$ and the one corresponding to the SNRAI model $\mathbf{A}_0$ is defined as

$$\langle \mathbf{v}|\delta\mathbf{A}(\sigma)\mathbf{u}\rangle = \langle \mathbf{v}|\mathbf{A}(\sigma)\mathbf{u}\rangle - \langle \mathbf{v}|\mathbf{A}_0(\sigma)\mathbf{u}\rangle \; , \tag{37}$$

and thus includes the lateral heterogeneities as well as the effect of the aspherical shape of the Earth's surface and discontinuities [*Woodhouse and Dahlen*, 1978; *Valette*, 1986].

Let us now consider a given eigenmode k, its eigenfrequency σ_k, its associated eigenfunction $|\mathbf{u}_k\rangle$ and and its dual $|\mathbf{v}_k\rangle$. All these terms can be expanded in terms of a power series of a small parameter ϵ, related to the perturbations due to the asphericity of the model. We shall thus write

$$\sigma_k = \sigma_{0k} + \delta_1\sigma_k + \delta_2\sigma_k + \dots \; , \tag{38}$$

$$|\mathbf{u}_k\rangle = |\mathbf{u}_k^{(0)}\rangle + |\mathbf{u}_k^{(1)}\rangle + |\mathbf{u}_k^{(2)}\rangle + \dots \; , \tag{39}$$

$$|\mathbf{v}_k\rangle = |\mathbf{v}_k^{(0)}\rangle + |\mathbf{v}_k^{(1)}\rangle + |\mathbf{v}_k^{(2)}\rangle + \dots \; , \tag{40}$$

where σ_{0k} is the spherical frequency of mode k, in what follows noted σ_0, and $\delta_n\sigma_k$, $|\mathbf{u}_k^{(n)}\rangle$, $|\mathbf{v}_k^{(n)}\rangle$ are the nth-order perturbations in the eigenfrequency, eigenmode and dual eigenmode of the singlet k, respectively. We will assume that all perturbations in the eigenmode can be expressed on the basis of the eigenmodes of the SNRAI model, which gives

$$|\mathbf{u}_k^{(n)}\rangle = \sum_j u_{jk}^{(n)} |\mathbf{u}_j^{(0)}\rangle \; . \tag{41}$$

In the same way, we will assume that all corresponding perturbations of dual modes will be expressed as

$$|\mathbf{v}_k^{(n)}\rangle = \sum_j v_{jk}^{(n)} |\mathbf{v}_j^{(0)}\rangle \; . \tag{42}$$

In these summations, $2\ell+1$ modes are associated with the spherical eigenfrequency σ_0. We will define in what follows the subspace $\mathcal{S}$ as the subspace mapped by these $2\ell+1$ modes, the index k being here also omitted for $\mathcal{S}$. The projector onto this subspace $\mathcal{P}$, will thus be defined for the modes $|\mathbf{u}\rangle$ and dual modes $|\mathbf{v}\rangle$, respectively, as

$$\mathcal{P}|\mathbf{u}\rangle = \sum_{j\in\mathcal{S}} |\mathbf{u}_j^{(0)}\rangle\langle \mathbf{v}_j^{(0)}| \left[\mathcal{I} - \frac{1}{2\sigma_0}\partial_\sigma \mathbf{A}_0(\sigma_0)\right] \mathbf{u}\rangle \; , \tag{43}$$

$$\mathcal{P}|\mathbf{v}\rangle = \sum_{j\in\mathcal{S}} |\mathbf{v}_j^{(0)}\rangle\langle \mathbf{u}_j^{(0)}| \left[\mathcal{I} - \frac{1}{2\sigma_0}\partial_\sigma \mathbf{A}_0(\sigma_0)\right] \mathbf{v}\rangle \; , \tag{44}$$

and verify obviously $\mathcal{P}^2 = \mathcal{P}$. The orthogonal of $\mathcal{S}$ is thus the image of $\mathcal{I} - \mathcal{P}$. As we will express all dual products using the reference spherical density, we first write equation (5) as

$$-\sigma_k^2\mathbf{K}|\mathbf{u}_k\rangle + \sigma_k\mathbf{B}|\mathbf{u}_k\rangle + \mathbf{A}(\sigma_k)|\mathbf{u}_k\rangle = 0 \; , \tag{45}$$

where the operators $\mathbf{K}$ and $\mathbf{B}$ are defined as

$$\begin{aligned} \mathbf{K} &= \frac{\rho}{\rho_0} \; , \\ \mathbf{B} &= \frac{\rho}{\rho_0} \, 2i\Omega\wedge \; , \end{aligned} \tag{46}$$

and make some modifications to this equation, in order to get simpler expressions for the perturbations. For this purpose, from (45) we get by using $\delta\sigma_k = \sigma_k - \sigma_0$ the expression

$$\begin{aligned} \big[(\mathbf{A}_0(\sigma_0) - \sigma_0^2\mathbf{K}_0) \;&+\; \delta\sigma_k(\mathbf{B} - 2\sigma_0\mathbf{K}) \\ &+\; \delta\mathbf{H}(\sigma_k) - \delta\sigma_k{}^2\mathbf{K}\;\big]|\mathbf{u}_k\rangle = 0 \end{aligned} \tag{47}$$

where

$$\delta\mathbf{H}(\sigma_k) = \left[\mathbf{A}(\sigma_k) - \mathbf{A}_0(\sigma_0) + \sigma_0\mathbf{B} - \sigma_0^2\delta\mathbf{K}\right] \; , \tag{48}$$

$$\delta\mathbf{K} = \frac{\delta\rho}{\rho_0} \; , \tag{49}$$

and where $\mathbf{K}_0$ is the identity operator. Let us now express $\mathbf{A}(\sigma_k)$ in terms of a Taylor series around the spherical frequency σ_0. Substituting the power series in equation (47), we end up with the following expression

$$\begin{aligned} \big[(\mathbf{A}_0(\sigma_0) - \sigma_0^2\mathbf{K}_0) \;&+\; \delta\sigma_k(\mathbf{B}' - 2\sigma_0\mathbf{K}') \\ &+\; \delta\mathbf{H}' - \delta\sigma_k{}^2\mathbf{K}' \\ &+\; \delta_3\mathbf{A}(\sigma_k)\;\big]|\mathbf{u}_k\rangle = 0 \; , \end{aligned} \tag{50}$$

where

$$\begin{aligned} \delta\mathbf{H}' &= \left[\delta\mathbf{A}(\sigma_0) + \sigma_0\mathbf{B} - \sigma_0^2\delta\mathbf{K}\right] \; , \\ \mathbf{B}' &= \mathbf{B} + \partial_\sigma\mathbf{A}(\sigma_0) - \sigma_0\partial_\sigma^2\mathbf{A}(\sigma_0) \; , \\ \mathbf{K}' &= \mathbf{K}_0 + \delta\mathbf{K} - \frac{1}{2}\partial_\sigma^2\mathbf{A}(\sigma_0) \; , \end{aligned} \tag{51}$$

$$\delta_3\mathbf{A}(\sigma_k) = \sum_{n=3}^{\infty} \frac{(\delta\sigma_k)^n}{n!}\partial_\sigma^n\mathbf{A}(\sigma_0) \; . \tag{52}$$

For common Earth models of anelasticity, the dispersion is relatively weak. The third-order term $\delta_3\mathbf{A}(\sigma_k)$ can thus be neglected, and only the two first derivatives of the operator $\mathbf{A}(\sigma)$ are significant. In this case one sees that the most important effect of the physical dispersion is to change slightly the apparent form of the density and Coriolis operators. Note also that the lateral variation in the elastic and anelastic structure appearing in the relation (51) are those observed at the spherical frequency σ_0. If the model displays a constant $Q(\omega)$, the derivatives of $\mathbf{A}$ are expressed only with the real part of the stiffness tensor, and the operators $\mathbf{B}'$ and $\mathbf{K}'$ remain self-adjoint. This is of course not the case for a strictly causal anelastic model.

Lognonné and Romanowicz [1990*a*] showed that the efficiency of the perturbation series was related to the interactions terms between modes of the same multiplet. These terms appear in the power series of the two expressions

$$\begin{aligned} \langle \mathbf{v}_{k'}|\mathbf{u}_k\rangle &- \frac{1}{\sigma_k + \sigma_{k'}}[\langle \mathbf{v}_{k'}|\mathbf{B}\mathbf{u}_k\rangle \\ &+ \frac{\langle \mathbf{v}_{k'}|\mathbf{A}(\sigma_{k'})\mathbf{u}_k\rangle - \langle \mathbf{v}_{k'}|\mathbf{A}(\sigma_k)\mathbf{u}_k\rangle}{\sigma_{k'} - \sigma_k}] = 0 \; , \end{aligned} \tag{53}$$

$$-\sigma_k^2\langle \mathbf{v}_{k'}|\mathbf{u}_k\rangle+\sigma_k\langle \mathbf{v}_{k'}|\mathbf{B}\mathbf{u}_k\rangle+\langle \mathbf{v}_{k'}|\mathbf{A}(\sigma_k)\mathbf{u}_k\rangle=0\ , \quad (54)$$

where $\langle \mathbf{v}_{k'}|$ and $|\mathbf{u}_k\rangle$ are the dual mode and mode of two different singlets k and k' of the same multiplet. Expanding $\mathbf{A}(\sigma)$ in terms of a Taylor expansion around σ_0, these two relations can be written as

$$\langle \mathbf{v}_{k'}|\left[\mathbf{K}'-\frac{1}{\sigma_k+\sigma_{k'}}\left(\mathbf{B}'+\delta\mathbf{D}(\delta\sigma_k,\delta\sigma_{k'})\right)\right]\mathbf{u}_k\rangle=\delta_{kk'}\ . \quad (55)$$

$$\begin{aligned}\langle \mathbf{v}_{k'}|\left[\left(\mathbf{A}_0(\sigma_0)-\sigma_0^2\mathbf{K}_0\right)\right. &+ \delta\sigma_k\left(\mathbf{B}'-(\sigma_k+\sigma_{k'})\mathbf{K}'\right)\\ &+ \delta\mathbf{H}'+\delta\sigma_k\delta\sigma_{k'}\mathbf{K}'\\ &+ \delta_3\mathbf{A}(\sigma_k)\left.\right]\ \mathbf{u}_k\rangle=0\ , \quad (56)\end{aligned}$$

where

$$\delta\mathbf{D}(\delta\sigma_k,\delta\sigma_{k'})=\sum_{n=3}^{\infty}\frac{1}{n!}\left(\frac{\delta\sigma_k^n-\delta\sigma_{k'}^n}{\delta\sigma_k-\delta\sigma_{k'}}\right)\partial_\sigma^n\mathbf{A}(\sigma_0)\ . \quad (57)$$

The zeroth-order terms of the expression (55) gives the constrains on the biorthogonality relations of the zeroth order modes $|\mathbf{u}_k^{(0)}\rangle$ and $|\mathbf{u}_k^{(0)}\rangle$, which gives

$$\langle \mathbf{v}_{k'}^{(0)}|\left[\mathbf{K}_0-\frac{1}{2\sigma_{0k}}\mathbf{B}_0\right]\mathbf{u}_k^{(0)}\rangle=\delta_{kk'}\ . \quad (58)$$

The other terms will give the projection of perturbations within $\mathcal{S}$. However, as soon as a ϵ^n term appears in the power series of the relations (55) and (56), a secular term appears in the expression of the nth-order perturbation of the mode, and the perturbation series practically stops to converge. To avoid this, we shall write the perturbations of the operators $\delta\mathbf{H}'$, $\mathbf{B}'$ and $\mathbf{K}'$ in the form of a power series:

$$\begin{aligned}\delta\mathbf{H}' &= \delta_1\mathbf{H}+\delta_2\mathbf{H}+...+\delta_n\mathbf{H}+...\ ,\\ \mathbf{B}' &= \mathbf{B}_0+\delta_1\mathbf{B}+\delta_2\mathbf{B}+...+\delta_n\mathbf{B}+...\ ,\\ \mathbf{K}' &= \mathbf{K}_0+\delta_1\mathbf{K}+\delta_2\mathbf{K}+...+\delta_n\mathbf{K}+...\ , \quad (59)\end{aligned}$$

and we shall choose the perturbations of these operators, i.e., the perturbation path of the procedure, in such a way that the power series of the relations (55) and (56) have the most of their first terms which cancel. Note that a zeroth-order operator $\mathbf{B}_0$ appears in relation (59), even if the Coriolis operator is generally seen as a perturbation. Indeed, as we choose to keep $\mathbf{K}_0$ always equal to the identity, and as shown by relation (58), this is the only way if one want to change the direction of projection within the subspace $\mathcal{S}$. Let us first recall the expressions of the perturbations in the eigenmode and eigenfrequency for perturbations in the operator such as (59). We shall describe here only the perturbation procedure for the mode $|\mathbf{u}_k\rangle$ of the space U. However, the same procedure must be used in order to determine the dual modes $|\mathbf{v}_k\rangle$ of the space V by substituting to all the operators their dual one, obtained by taking the opposite of $\mathbf{B}$. With the expansion of the relation (59) we now get

$$\begin{aligned}\left[\left(\mathbf{A}_0(\sigma_0)-\sigma_0^2\mathbf{K}_0\right)\right. &+ \delta\sigma_k\left(\mathbf{B}'-2\sigma_0\mathbf{K}'\right)\\ &+ \delta\mathbf{H}'-\delta\sigma_k{}^2\mathbf{K}'\\ &+ \delta_3\mathbf{A}(\sigma_k)\left.\right]|\mathbf{u}_k\rangle=\sum_{n=0}^{n=\infty}\mathbf{f}^{(n)}\ . \quad (60)\end{aligned}$$

Since each term of order ϵ^n must be equal to zero, all $\mathbf{f}^{(n)}$ terms must be zero. The zeroth-order term is

$$\mathbf{f}^{(0)}=[\mathbf{A}_0(\sigma_0)-\sigma_0^2\mathbf{K}_0]|\mathbf{u}_k^{(0)}\rangle\ , \quad (61)$$

and shows that the zeroth-order eigenfunctions belong to $\mathcal{S}$, which is the kernel of $[\mathbf{A}_0(\sigma_0)-\sigma_0^2\mathbf{K}_0]$. The other terms are, for $n\geq 1$,

$$\begin{aligned}\mathbf{f}^{(n)} &= [\mathbf{A}_0(\sigma_0)-\sigma_0^2\mathbf{K}_0]|\mathbf{u}_k^{(n)}\rangle\\ &+ |r^{(n-1)},\mathbf{u}_k\rangle-\delta_n\sigma_k[2\sigma_0\mathbf{K}_0-\mathbf{B}_0]|\mathbf{u}_k^{(0)}\rangle\ . \quad (62)\end{aligned}$$

Here $|r^{(n-1)},\mathbf{u}\rangle$ can be expressed in terms of the eigenfrequencies and eigenmodes perturbations up to order $n-1$. The first three terms are expressed in Appendix B. These higher order terms thus give the following relation

$$\begin{aligned}[\mathbf{A}_0(\sigma_0)-\sigma_0^2\mathbf{K}_0]|\mathbf{u}_k^{(n)}\rangle+|r^{(n-1)},\mathbf{u}_k\rangle&\\ -\delta_n\sigma_k[2\sigma_0\mathbf{K}_0-\mathbf{B}_0]|\mathbf{u}_k^{(0)}\rangle &= 0\ . \quad (63)\end{aligned}$$

For n=1, we obtain the first-order perturbation in the frequency as well as the zeroth-order perturbation of the eigenmode as a solution of an eigenproblem. This eigenproblem is obtained by projecting the first-order residual $\mathbf{f}^{(1)}$ on the subspace $\mathcal{S}$ associated to the SNRAI multiplet whose eigenfrequency is equal to σ_0 for the spherical model, and gives

$$\delta_1\sigma_k\mathcal{P}\left[2\sigma_0\mathbf{K}_0-\mathbf{B}_0\right]|\mathbf{u}_k^{(0)}\rangle=\mathcal{P}\delta_1\mathbf{H}|\mathbf{u}_k^{(0)}\rangle\ . \quad (64)$$

This equation is the generalization of the isolated multiplet perturbation theory [e.g., *Messiah*, 1962; *Backus and Gilbert*, 1961; *Dahlen*, 1968; *Madariaga*, 1972] in the SNRAI case. The associated $(2\ell+1)\times(2\ell+1)$ left and right matrices can be computed by using the basis of $\mathcal{S}$. It must be noted, however, that none of these matrices are hermitian in the general case and that the associated dual mode $|\mathbf{v}_k\rangle$ must be determined either by solving the associated dual eigenproblem (with reversed Earth rotation):

$$\delta_1\sigma_k\mathcal{P}\left[2\sigma_0\mathbf{K}_0-\widehat{\mathbf{B}}_0\right]|\mathbf{v}_k^{(0)}\rangle=\mathcal{P}\delta_1\widehat{\mathbf{H}}|\mathbf{v}_k^{(0)}\rangle\ , \quad (65)$$

or by inverting the matrix $[2\sigma_0\mathbf{K}_0-\mathbf{B}_0]$, as the kets $|\mathbf{u}_k^{(0)}\rangle$ and the dual bra $\langle\mathbf{v}_{k'}^{(0)}|$ verify the duality relation (58). We can now compute the higher order perturbations. However, the relation (63) gives us only the expression of the frequency perturbation and one of the projections of the perturbation on the orthogonal to $\mathcal{S}$. For the first one we obtain, multiplying expression (63) by $\langle\mathbf{v}_k^{(0)}|$, and using the relation (58):

$$2\sigma_0\delta_n\sigma_k=\langle\mathbf{v}_k^{(0)}|r^{(n-1)},\mathbf{u}_k\rangle\ . \quad (66)$$

For the computation of the orthogonal, let us first multiply expression (63) by $\mathcal{I}-\mathcal{P}$. We get

$$\begin{aligned}[\mathcal{I}-\mathcal{P}][\mathbf{A}_0(\sigma_0)-\sigma_0^2\mathbf{K}_0]|\mathbf{u}_k^{(n)}\rangle+[\mathcal{I}-\mathcal{P}]|r^{(n-1)},\mathbf{u}_k\rangle&\\ -\delta_n\sigma_k[\mathcal{I}-\mathcal{P}][2\sigma_0\mathbf{K}_0-\mathbf{B}_0]|\mathbf{u}_k^{(0)}\rangle &= 0 \quad (67)\end{aligned}$$

As $\mathbf{K}_0$ is the identity and as $|\mathbf{u}_k^{(0)}\rangle$ belongs to $\mathcal{S}$, the last term is zero as soon as the image of $\mathcal{S}$ by $\mathbf{B}_0$ is included in $\mathcal{S}$. We will take this constrain for $\mathbf{B}_0$. In this case let us remark that

$$[\mathcal{I}-\mathcal{P}][\mathbf{A}_0(\sigma_0)-\sigma_0^2\mathbf{K}_0][\mathcal{I}-\mathcal{P}]\ , \quad (68)$$

has a left and right inverse, and let us note its left one $\mathbf{\Delta}$. Using this inverse, we can now get the expression of an nth-order perturbation on the orthogonal of $\mathcal{S}$,

$$[\mathcal{I}-\mathcal{P}]|\mathbf{u}_k^{(n)}\rangle=\mathbf{\Delta}|r^{(n-1)},\mathbf{u}_k\rangle\ . \quad (69)$$

For the projection of the perturbations within the subspace $\mathcal{S}$, we must use the power series expression of the relations

(55) and (56). After some algebra developed in Appendix B, the projection is found and gives for $k \neq k'$,

$$\langle \mathbf{v}_{k'}^{(0)} | [2\sigma_0 \mathbf{K}_0 - \mathbf{B}_0] \mathbf{u}_k^{(n)} \rangle = \frac{\mathcal{A}_{kk'}^{(n)}}{\delta_1\sigma_k - \delta_1\sigma_{k'}} - \frac{2\sigma_0 \delta_1\sigma_k \mathcal{B}_{kk'}^{(n)}}{\delta_1\sigma_k - \delta_1\sigma_{k'}} . \quad (70)$$

These terms correspond to the interactions between singlets and are highly unstable. *Lognonné and Romanowicz* [1990a] have shown in the elastic case that they can practically stop the convergence of the perturbation procedure. As in the elastic case, we will thus modify the perturbation path in order to cancel the secular terms as far as possible for the perturbation series.

Constraining the Perturbation Path

Let us now determine the perturbation path which will cancel the secular terms up to the second order. In addition to the constraint given by relation (59), the first- and second-order perturbations for the operator $\mathbf{K}'$, the zero, first and second for $\mathbf{B}'$ and the first, second and third one for $\delta\mathbf{H}$ must be such that

$$\mathcal{A}_{kk'}^{(1)} = \mathcal{A}_{kk'}^{(2)} = \mathcal{B}_{kk'}^{(1)} = \mathcal{B}_{kk'}^{(2)} = 0 , \quad (71)$$

Let us start by cancelling the first-order $\mathcal{A}_{kk'}^{(1)}$ and $\mathcal{B}_{kk'}^{(1)}$. Using the result of Appendix C we end up with the constrains

$$\langle \widehat{\mathbf{N}}_0^{-1} \delta_1 \widehat{\mathbf{H}} \mathbf{v}_{k'}^{(0)} | \frac{\mathbf{B}_0}{2\sigma_0} \mathbf{u}_k^{(0)} \rangle + \langle \mathbf{v}_{k'}^{(0)} | \frac{\mathbf{B}_0}{2\sigma_0} \mathbf{N}_0^{-1} \delta_1 \mathbf{H} \mathbf{u}_k^{(0)} \rangle + \langle \mathbf{v}_{k'}^{(0)} | [2\sigma_0 \delta_1 \mathbf{K} - \delta_1 \mathbf{B}] \mathbf{u}_k^{(0)} \rangle = 0, \quad (72)$$

$$\langle \mathbf{v}_{k'}^{(0)} | \delta_2 \mathbf{H} \mathbf{u}_k^{(0)} \rangle + \langle \mathbf{v}_{k'}^{(0)} | \delta_1 \mathbf{H} \Delta \delta_1 \mathbf{H} \mathbf{u}_k^{(0)} \rangle + \langle \widehat{\mathbf{N}}_0^{-1} \delta_1 \widehat{\mathbf{H}} \mathbf{v}_{k'}^{(0)} | \mathbf{K}_0 \mathbf{N}_0^{-1} \delta_1 \mathbf{H} \mathbf{u}_k^{(0)} \rangle = 0 , \quad (73)$$

where

$$\mathbf{N}_0 = 2\sigma_0 \mathbf{K}_0 - \mathbf{B}_0 , \quad (74)$$

is the operator related to the norm within the subspace $\mathcal{S}$. This shows that only the projection of $\delta_1\mathbf{K}$, $\delta_1\mathbf{B}$ and $\delta_2\mathbf{H}$ within the subspace $\mathcal{S}$ are constrained and must be such that

$$\mathcal{P}[\delta_1\mathbf{B} - 2\sigma_0\delta_1\mathbf{K}]\mathcal{P} = \mathcal{P}\left[\delta_1 \mathbf{H} \mathbf{N}_0^{-1} \frac{\mathbf{B}_0}{2\sigma_0} + \frac{\mathbf{B}_0}{2\sigma_0} \mathbf{N}_0^{-1} \delta_1 \mathbf{H}\right]\mathcal{P} , \quad (75)$$

$$\mathcal{P}\delta_2\mathbf{H}\mathcal{P} = -\mathcal{P}\left[\delta_1\mathbf{H}\Delta\delta_1\mathbf{H} + \delta_1\mathbf{H}\mathbf{N}_0^{-1}\mathbf{K}_0\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right]\mathcal{P} . \quad (76)$$

The remaining part of $\delta_2\mathbf{H}$, related to the right and left projections within $\mathcal{S}$ will be taken equal to zero. This means that the first-order perturbations $\delta_1\mathbf{H}$ and $\delta_1\mathbf{K}$ will be taken in such a way that

$$[\mathcal{I} - \mathcal{P}]\delta_1\mathbf{H} = [\mathcal{I} - \mathcal{P}]\delta\mathbf{H}' , \ \delta_1\mathbf{H}[\mathcal{I} - \mathcal{P}] = \delta\mathbf{H}'[\mathcal{I} - \mathcal{P}] , \quad (77)$$

for $\delta_1\mathbf{H}$ and $\delta\mathbf{H}'$, but also for $\delta_1\mathbf{K}$, $\delta\mathbf{K}'$ and $\mathbf{B}_0$, $\mathbf{B}'$. This means that all terms like $\Delta\delta_1\mathbf{H}$ can be substituted by $\Delta\delta\mathbf{H}'$. In this case the first-order secular terms cancel, and the first-order perturbation is orthogonal to the subspace $\mathcal{S}$ and is given by

$$|\mathbf{u}_k^{(1)}\rangle = \Delta\delta_1\mathbf{H}|\mathbf{u}_k^{(0)}\rangle . \quad (78)$$

The second-order perturbation in the eigenfrequency can also be found by using relation (66), which, using the expressions of $\delta_1\mathbf{K}$, $\delta_1\mathbf{B}$ and $\delta_2\mathbf{H}$ gives, after some algebra

$$\delta_2\sigma_k = \langle \mathbf{v}_k^{(0)} | r^{(1)}, \mathbf{u}_k \rangle = -\frac{\delta_1\sigma_k^2}{\sigma_0} . \quad (79)$$

This correction is just due to the quadratic term of the normal mode equation. One must note that the usual second-order perturbation, due to the multiplet-multiplet coupling effect, is included in the first-order one, as it is, in the elastic case, for other nonsecular perturbation theories [*Park*, 1987; *Dahlen*, 1987, *Lognonné and Romanowicz*, 1990a].

The second-order terms can now be obtained in a similar way, and the corresponding expressions for the projections of operator perturbations $\delta_2\mathbf{K}$, $\delta_2\mathbf{B}$ and $\delta_3\mathbf{H}$ within $\mathcal{S}$ are given in Appendix D. These relations, as well as relations (75) and (76) are expressed with the zeroth-order perturbation $\mathbf{B}_0$, the first ones $\delta_1\mathbf{H}$, $\delta_1\mathbf{K}$. We must now use the relation (59) in order to find these perturbations. This can be done with the projection of the different perturbations on the subspace $\mathcal{S}$, which give, up to second order,

$$\mathcal{P}\delta_1\mathbf{H}\mathcal{P} = \mathcal{P}\left[\delta\mathbf{H}' + \delta\mathbf{H}'\Delta\delta\mathbf{H}' + \delta\mathbf{H}'\Delta\delta\mathbf{H}'\Delta\delta\mathbf{H}'\right]\mathcal{P} + \mathcal{F}_1(\mathbf{B}_0, \delta_1\mathbf{H}, \delta_1\mathbf{K}) , \quad (80)$$

$$2\sigma_0\mathcal{P}\delta_1\mathbf{K}\mathcal{P} = \mathcal{P}[\,2\sigma_0\delta\mathbf{K}' + \delta\mathbf{H}'\Delta[2\sigma_0\mathbf{K}_0]\Delta\delta\mathbf{H}' + \delta\mathbf{H}'\Delta[2\sigma_0\delta\mathbf{K}' - \mathbf{B}'] + [2\sigma_0\delta\mathbf{K}' - \mathbf{B}']\Delta\delta\mathbf{H}'\,]\mathcal{P} + \mathcal{F}_2(\mathbf{B}_0, \delta_1\mathbf{H}, \delta_1\mathbf{K}) , \quad (81)$$

$$\mathcal{P}\mathbf{B}_0\mathcal{P} = \mathcal{P}\left[\mathbf{B}' - 2\sigma_0\delta_1\mathbf{K}\right]\mathcal{P} + \mathcal{F}_3(\mathbf{B}_0, \delta_1\mathbf{H}, \delta_1\mathbf{K}) , \quad (82)$$

where we have assumed that $\delta_2\mathbf{B}$ is taken as zero. The functions $\mathcal{F}_{1,2,3}(\mathbf{B}_0, \delta_1\mathbf{H}, \delta_1\mathbf{K})$ are nonlinearly related to the projection within $\mathcal{S}$ of the operators $\mathbf{B}_0$, $\delta_1\mathbf{H}$ and $\delta_1\mathbf{K}$ and are given in Appendix D. The relations (80)-(82) give the extension to the anelastic rotating case of the renormalization procedure proposed by *Lognonné and Romanowicz* [1990a] in the elastic case. It differs essentially by the nonlinear functions $\mathcal{F}_{1,2,3}(\mathbf{B}_0, \delta_1\mathbf{H}, \delta_1\mathbf{K})$ which make the computations of $\mathbf{B}_0$, $\delta_1\mathbf{H}$ and $\delta_1\mathbf{K}$ more complex but take a better account of the effect due to the rotation and dispersion. It must be noted, however, that these functions are expressed only with the projection of $\mathbf{B}_0$, $\mathbf{K}_0$, $\delta_1\mathbf{H}$, $\delta_1\mathbf{K}$ and $\partial_\sigma^3\mathbf{A}_0(\sigma_0)$ within the subspace $\mathcal{S}$, whose dimension is $2\ell+1$. The other part, which appears in the right-hand side of equations (80)-(82), is the same as by *Lognonné and Romanowicz* [1990a]. It depends on the coupling effect between different multiplets and is much more complex to evaluate, especially for "rough" laterally heterogeneous Earth models, where the coupling can occur far along the same dispersion branch, and also between different dispersion branches. However, these last terms must be calculated only one time and $\mathbf{B}_0$, $\delta_1\mathbf{H}$, $\delta_1\mathbf{K}$ can be easily determined in a few iterations by a Gauss resolution of a $(2\ell+1)^2$ dimension problem.

We can now write the final expressions for the perturbations of the eigenmodes. The relation (78) can be used for the expression of the first-order perturbation and gives

$$|\mathbf{u}_k^{(1)}\rangle = \Delta\delta\mathbf{H}'|\mathbf{u}_k^{(0)}\rangle , \quad (83)$$

As the second-order perturbation now becomes also orthogonal to $\mathcal{S}$, we can express the second-order perturbation of the eigenmode, which gives

$$|u_k^{(2)}\rangle = \delta_1\sigma_k\Delta\left[B' - 2\sigma_0\delta K' - 2\sigma_0 K_0\Delta\delta H'\right]|u_k^{(0)}\rangle + \Delta\delta H'\Delta\delta H'|u_k^{(0)}\rangle \ . \quad (84)$$

In the same manner we obtain the third-order perturbations in the eigenfrequency, given by

$$2\sigma_0\delta_3\sigma_k = \langle v_k^{(0)}|r^{(2)}, u_k\rangle \ . \quad (85)$$

One must note that even if the starting model is a nonrotating model, a nonzero operator B_0 is needed. However, as shown in the relation (58), this operator only affects the biorthogonality relation between modes of the subspace $\mathcal{S}$. To zero order, it gives the same biorthogonality as in the spherical case. To first order, B_0 is related to the self-coupling due to the rotation and to lateral variations in density, if the operator A is not renormalized for the aspherical density [*Lognonné and Romanowicz*, 1990*a*]. To second order, it will be related to the escape of energy induced by coupling effects with the other multiplets. This operator thus just determines the initial direction of the perturbation $|u_k^{(0)}\rangle$, given by the eigenproblem of relation (64). In the same way the perturbations of the dual mode are obtained by taking in all the obtained expressions of the dual operators. Using expression (31), an approximation of the seismogram up to the third order for the eigenfrequency, and up to the second order for the eigenmode can finally be obtained.

CONCLUSION

The results shown in this paper give the eigenmode and eigenfrequency perturbations up to the second and third order, respectively. In the elastic case, the second-order approximation of the normal mode and third order of the eigenfrequency have been compared with the solution obtained with the variational method by *Lognonné and Romanowicz* [1990*a*]. They have found that the director cosine and the frequency deviation $\delta\sigma$ between the two solutions differ only by a relative error less than 10^{-3}, an accuracy much higher than the observation error. Of course, if a better precision is needed, the expressions can be generalized to the other higher orders. However, secular terms will remain for all perturbations terms in eigenmode higher than 3.

The calculations presented here do not require large matrix diagonalizations. Only small size matrices, of the same size as in the isolated multiplet approximation, must be diagonalized. Computations with much bigger matrices, related to the coupling between multiplets, are only matrix multiplications. The algorithm is thus well adapted to vectorial computers or to highly parallel computers. Of course, the modes and eigenfrequencies can be stored after being calculated for realistic three-dimensional models and can be used to compute seismograms. These seismograms will take a better account of the initial amplitude and phase of the the multiplets, which are affected either by the anelasticity or by the coupling effects. This may be used, for example, to perform more realistic seismic source inversions.

APPENDIX A

Let us show that the spectrum of an anelastic, dispersive and rotating Earth does not depend on the direction of rotation. For this purpose, let us assume that the eigenmodes $|u_k\rangle$ and associated eigenfrequency σ_k are solution of the eigenproblem

$$-\sigma_k^2|u_k\rangle + \sigma_k B|u_k\rangle + A(\sigma_k)|u_k\rangle = 0 \ . \quad (A1)$$

In equation (A1), lateral variations in density are taken into account by a renormalization of the operator A, as shown by *Lognonné and Romanowicz* [1990*a*]. Let us now use a complete basis of functions for the displacement fields within the Volume V of the Earth, for example, given by the eigenfunctions of SNREI model, in order to express (A1) in a matrix form. In a spherical Cartesian basis, these modes may be expressed with spherical harmonics and their displacement is given, in operator notation, by

$$ {}_n u_{\ell m}^{(0)} = {}_nU_\ell(r)\, Y_\ell^m(\theta,\phi) e_r + {}_nV_\ell(r)\, \nabla_1 Y_\ell^m(\theta,\phi) + {}_nW_\ell(r)\, e_r \wedge \nabla_1 Y_\ell^m(\theta,\phi) \ , \quad (A2)$$

where U, V, W are functions of r which depend on n and ℓ, the radial and angular order of the modes, $Y_\ell^m(\theta,\phi)$ is the fully normalized spherical harmonics, ∇_1 the gradient operator on the unit sphere, e_r the radial basis vector of the spherical cartesian basis e_r, e_θ, e_ϕ, and θ, ϕ the colatitude and longitude respectively. Let us now define the matrix element of the B and A operators as

$$B_{m'\ell'}^{m\ell} = \langle {}_n u_{\ell m}^* | B_{n'} u_{\ell' m'}\rangle \ , \qquad A_{m'\ell'}^{m\ell}(\sigma) = \langle {}_n u_{\ell m}^* | A(\sigma)_{n'} u_{\ell' m'}\rangle \ , \quad (A3)$$

where the asterisk is the complex conjugation, where index n is omitted and where the bracket product is defined in relation (36) in the text. Omitting also the index k, the mode u can be written in the form

$$u = \sum_{m\ell} u^{m\ell} u_{\ell m} \ ,$$

and equation (A1) can now be written as

$$-\sigma^2 u^{m\ell} + \sigma \sum_{m'\ell'} B_{m'\ell'}^{m\ell} u^{m'\ell'} + \sum_{m'\ell'} A_{m'\ell'}^{m\ell}(\sigma) u^{m'\ell'} = 0 \ . \quad (A4)$$

Let us now use the symmetry of the matrix elements of A noted by *Dahlen* [1981], which results from the property of the spherical harmonics, such that $Y_\ell^{m*} = (-1)^m Y_\ell^{-m}$, and from the symmetry of the operator A:

$$A_{m'\ell'}^{m\ell}(\sigma) = (-1)^{m+m'} A_{-m\ell}^{-m'\ell'}(\sigma) \ ,$$

and note that due to the antisymmetry of the Coriolis operator we have

$$B_{m'\ell'}^{m\ell} = -(-1)^{m+m'} B_{-m\ell}^{-m'\ell'} \ .$$

Taking the equation (A4) for the component $-m$, using these two symmetry relations and changing the index m' into $-m'$, we have then, after multiplication by $(-1)^m$:

$$-\sigma^2 \widetilde{u}^{m\ell} - \sigma \sum_{m'\ell'} B_{m\ell}^{m'\ell'} \widetilde{u}^{m'\ell'} + \sum_{m'\ell'} A_{m\ell}^{m'\ell'}(\sigma) \widetilde{u}^{m'\ell'} = 0 \ , \quad (A5)$$

where $\widetilde{u}$ is defined as

$$\widetilde{u}^m = (-1)^m u^{-m} \ .$$

This shows that σ and $\widetilde{u}$ are solution of the transposed eigenproblem of an anelastic Earth with an opposed rotation direction, which can be written in the form

$$-\sigma^2|\widetilde{u}\rangle - \sigma B^t|\widetilde{u}\rangle + A^t(\sigma)|\widetilde{u}\rangle = 0 \ , \quad (A6)$$

where the matrix terms of the transposed of the operator B and A are given by

$$[B^t]^{m\ell}_{m'\ell'} = B^{m'\ell'}_{m\ell}, \quad [A^t(\sigma)]^{m\ell}_{m'\ell'} = A^{m'\ell'}_{m\ell}(\sigma) .$$

The equation (A6) means also that σ is solution of the characteristic equation, formally written as

$$\det\left[-\sigma^2\mathcal{I} - \sigma\mathbf{B}^t + \mathbf{A}^t(\sigma)\right] = 0 , \quad \text{(A7)}$$

which shows by taking the transposed, that

$$\det\left[-\sigma^2\mathcal{I} - \sigma\mathbf{B} + \mathbf{A}(\sigma)\right] = 0 , \quad \text{(A8)}$$

i.e., that σ is an eigenfrequency of the eigenproblem of an anelastic Earth with a reversed rotation. Note, however, that there is no general way to find the associated eigenmode, noted $\mathbf{v}$, from $\mathbf{u}$ or $\tilde{\mathbf{u}}$ in the case of a laterally heterogeneous Earth. The relation between these two modes is simple in only two cases. The first one is the elastic, rotating case, where the operators $\mathbf{A}$ and $\mathbf{B}$ are self-adjoint and commute and anticommutes, respectively, with the complex conjugation, defined as a weak time reversal by *Lognonné and Romanowicz* [1990*a*]. If $\mathbf{u}$ and σ are eigensolutions of

$$-\sigma^2|\mathbf{u}\rangle + \sigma\mathbf{B}|\mathbf{u}\rangle + \mathbf{A}|\mathbf{u}\rangle = 0 , \quad \text{(A9)}$$

we have then by taking the complex conjugate of (A8), and as σ is real

$$-\sigma^2|\mathbf{u}^*\rangle - \sigma\mathbf{B}|\mathbf{u}^*\rangle + \mathbf{A}|\mathbf{u}^*\rangle = 0 ,$$

which shows that

$$\mathbf{v} = \mathbf{u}^* . \quad \text{(A10)}$$

The second one is the spherical, anelastic and rotating case, where the operators $\mathbf{A}$ and $\mathbf{B}$ commutes and anticommutes, respectively, with the symmetry operator $\mathbf{S}$ defined by *Lognonné and Romanowicz* [1990*a*] as an orthogonal symmetry around a plane containing the rotation axis, for example, the Greenwich meridian plane. In this case we have, by taking the $\mathbf{S}$ image of equation (A8),

$$-\sigma^2|\mathbf{S}\mathbf{u}\rangle - \sigma\mathbf{B}|\mathbf{S}\mathbf{u}\rangle + \mathbf{A}|\mathbf{S}\mathbf{u}\rangle = 0 , \quad \text{(A11)}$$

which shows that

$$\mathbf{v} = \mathbf{S}\mathbf{u} . \quad \text{(A12)}$$

Let us note that in this case using the definition of $\mathbf{S}$, the component of the vector $\mathbf{v}$ can be expressed as

$$v^m = (-1)^m u^{-m} , \quad \text{(A13)}$$

which is the relation found by *Dahlen* [1981] between a mode and its dual.

APPENDIX B

Starting from the relation (38), (39), (40), (50), and (59), we get easily the expressions of the three first remaining kets $|r^{(n)}, \mathbf{u}_k\rangle$ of equation(62)

$$|r^{(0)}, \mathbf{u}_k\rangle = \delta_1\mathbf{H}|\mathbf{u}_k^{(0)}\rangle , \quad \text{(B1)}$$

$$\begin{aligned}|r^{(1)}, \mathbf{u}_k\rangle =& \\ &-\delta_1\sigma_k\left([2\sigma_0\delta_1\mathbf{K} - \delta_1\mathbf{B}]|\mathbf{u}_k^{(0)}\rangle + [2\sigma_0\mathbf{K}_0 - \mathbf{B}_0]|\mathbf{u}_k^{(1)}\rangle\right) \\ &+\delta_2\mathbf{H}|\mathbf{u}_k^{(0)}\rangle + \delta_1\mathbf{H}|\mathbf{u}_k^{(1)}\rangle - (\delta_1\sigma_k)^2\mathbf{K}_0|\mathbf{u}_k^{(0)}\rangle , \quad \text{(B2)}\end{aligned}$$

$$\begin{aligned}|r^{(2)}, \mathbf{u}_k\rangle =& \\ &-\delta_2\sigma_k\left([2\sigma_0\delta_1\mathbf{K} - \delta_1\mathbf{B}]|\mathbf{u}_k^{(0)}\rangle + [2\sigma_0\mathbf{K}_0 - \mathbf{B}_0]|\mathbf{u}_k^{(1)}\rangle\right) \\ &-\delta_1\sigma_k\Big([2\sigma_0\delta_2\mathbf{K} - \delta_2\mathbf{B}]|\mathbf{u}_k^{(0)}\rangle \\ &\quad+[2\sigma_0\delta_1\mathbf{K} - \delta_1\mathbf{B}]|\mathbf{u}_k^{(1)}\rangle + [2\sigma_0\mathbf{K}_0 - \mathbf{B}_0]|\mathbf{u}_k^{(2)}\rangle\Big) \\ &+\delta_3\mathbf{H}|\mathbf{u}_k^{(0)}\rangle + \delta_2\mathbf{H}|\mathbf{u}_k^{(1)}\rangle + \delta_1\mathbf{H}|\mathbf{u}_k^{(2)}\rangle + \delta_3\mathbf{A}(\sigma_k)|\mathbf{u}_k^{(0)}\rangle \\ &-(\delta_1\sigma_k)^2\left(\mathbf{K}_0|\mathbf{u}_k^{(1)}\rangle + \delta_1\mathbf{K}|\mathbf{u}_k^{(0)}\rangle\right) \\ &-2\delta_1\sigma_k\delta_2\sigma_k\mathbf{K}_0|\mathbf{u}_k^{(0)}\rangle . \quad \text{(B3)}\end{aligned}$$

APPENDIX C

In order to determine the perturbation path which will cancel the secular terms and to determine the remaining projections of the perturbation on the starting multiplet, we must start with the two secular equations:

$$\langle\mathbf{v}_{k'}|\left[\mathbf{K} - \frac{1}{\sigma_k + \sigma_{k'}}(\mathbf{B} + \delta\mathbf{D}(\delta\sigma_k, \delta\sigma_{k'}))\right]\mathbf{u}_k\rangle = \delta_{kk'} . \quad \text{(C1)}$$

$$\begin{aligned}\langle\mathbf{v}_k|[\ (\mathbf{A}_0(\sigma_0) - \sigma_0^2\mathbf{K}_0) \ &+\ \delta\sigma_k(\mathbf{B} - (\sigma_k + \sigma_{k'})\mathbf{K}) \\ &+\ \delta\mathbf{H} + \delta\sigma_k\delta\sigma_{k'}\mathbf{K} \\ &+\ \delta_3\mathbf{A}(\sigma_k)\ ]\mathbf{u}_k\rangle = 0 , \quad \text{(C2)}\end{aligned}$$

where the indexes $'$ on $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{K}$ have been omitted. Let us now substitute the mode and dual mode, singlets for the same multiplet, by the power series (38)-(40) in text and expand (C1) into a power series of ϵ. We then have

$$\begin{aligned}&\langle\mathbf{v}_{k'}|\left[\mathbf{K} - \frac{1}{\sigma_k + \sigma_{k'}}(\mathbf{B} + \delta\mathbf{D}(\delta\sigma_k, \delta\sigma_{k'}))\right]\mathbf{u}_k\rangle = \\ &\langle\mathbf{v}_{k'}^{(0)}|\frac{\mathbf{N}_0}{2\sigma_0}\mathbf{u}_k^{(0)}\rangle \\ &+\langle\mathbf{v}_{k'}^{(0)}|\frac{\mathbf{N}_0}{2\sigma_0}\mathbf{u}_k^{(1)}\rangle + \langle\mathbf{v}_{k'}^{(1)}|\frac{\mathbf{N}_0}{2\sigma_0}\mathbf{u}_k^{(0)}\rangle + \mathcal{B}^{(1)}_{kk'} \\ &+\langle\mathbf{v}_{k'}^{(0)}|\frac{\mathbf{N}_0}{2\sigma_0}\mathbf{u}_k^{(2)}\rangle + \langle\mathbf{v}_{k'}^{(2)}|\frac{\mathbf{N}_0}{2\sigma_0}\mathbf{u}_k^{(0)}\rangle + \mathcal{B}^{(2)}_{kk'} + \ldots\end{aligned}$$

where

$$\mathbf{N}_0 = 2\sigma_0\mathbf{K}_0 - \mathbf{B}_0 , \quad \text{(C3)}$$

and where the first $\mathcal{B}^{(n)}_{kk'}$ terms are given by

$$\begin{aligned}\mathcal{B}^{(1)}_{kk'} =& \langle\mathbf{v}_{k'}^{(0)}|\left[\delta_1\mathbf{K} - \frac{\delta_1\mathbf{B}}{2\sigma_0}\right]\mathbf{u}_k^{(0)}\rangle \\ &+\frac{\delta_1\sigma_k + \delta_1\sigma_{k'}}{2\sigma_0}\langle\mathbf{v}_{k'}^{(0)}|\frac{\mathbf{B}_0}{2\sigma_0}\mathbf{u}_k^{(0)}\rangle , \quad \text{(C4)}\end{aligned}$$

$$\begin{aligned}\mathcal{B}^{(2)}_{kk'} =& \langle\mathbf{v}_{k'}^{(1)}|\frac{\mathbf{N}_0}{2\sigma_0}\mathbf{u}_k^{(1)}\rangle \\ &+\langle\mathbf{v}_{k'}^{(0)}|\left[\delta_1\mathbf{K} - \frac{\delta_1\mathbf{B}}{2\sigma_0}\right]\mathbf{u}_k^{(1)}\rangle \\ &+\langle\mathbf{v}_{k'}^{(1)}|\left[\delta_1\mathbf{K} - \frac{\delta_1\mathbf{B}}{2\sigma_0}\right]\mathbf{u}_k^{(0)}\rangle \\ &+\langle\mathbf{v}_{k'}^{(0)}|\left[\delta_2\mathbf{K} - \frac{\delta_2\mathbf{B}}{2\sigma_0}\right]\mathbf{u}_k^{(0)}\rangle \\ &+\frac{\delta_1\sigma_k + \delta_1\sigma_{k'}}{2\sigma_0}\Big(\langle\mathbf{v}_{k'}^{(0)}|\frac{\mathbf{B}_0}{2\sigma_0}\mathbf{u}_k^{(1)}\rangle \\ &\quad+\langle\mathbf{v}_{k'}^{(1)}|\frac{\mathbf{B}_0}{2\sigma_0}\mathbf{u}_k^{(0)}\rangle + \langle\mathbf{v}_{k'}^{(0)}|\frac{\delta_1\mathbf{B}}{2\sigma_0}\mathbf{u}_k^{(0)}\rangle\Big) \\ &-\left(\left(\frac{\delta_1\sigma_k + \delta_1\sigma_{k'}}{2\sigma_0}\right)^2 - \frac{\delta_2\sigma_k + \delta_2\sigma_{k'}}{2\sigma_0}\right)\langle\mathbf{v}_{k'}^{(0)}|\frac{\mathbf{B}_0}{2\sigma_0}\mathbf{u}_k^{(0)}\rangle \\ &-\langle\mathbf{v}_{k'}^{(0)}|\frac{\delta_2\mathbf{D}}{2\sigma_0}\mathbf{u}_k^{(0)}\rangle , \quad \text{(C5)}\end{aligned}$$

where

$$\delta_2\mathbf{D}(\delta\sigma_k,\delta\sigma_{k'})=\frac{1}{6}\left(\delta_1\sigma_k^2+\delta_1\sigma_k\delta_1\sigma_{k'}+\delta_1\sigma_{k'}^2\right)\partial_\sigma^3\mathbf{A}(\sigma_0)\ .$$

In the same way we substitute in (C2) the mode and frequencies by their power series and, using relations (C1), get the development into the power of ϵ,

$$\begin{aligned}&\langle\mathbf{v}_{k'}^{(0)}|\delta_1\mathbf{H}\mathbf{u}_k^{(0)}\rangle\\&+\langle\mathbf{v}_{k'}^{(0)}|\delta_1\mathbf{H}\mathcal{P}\mathbf{u}_k^{(1)}\rangle+\langle\mathbf{v}_{k'}^{(1)}|\mathcal{P}\delta_1\mathbf{H}\mathbf{u}_k^{(0)}\rangle+\mathcal{A}_{kk'}^{(1)}\\&+\langle\mathbf{v}_{k'}^{(0)}|\delta_1\mathbf{H}\mathcal{P}\mathbf{u}_k^{(2)}\rangle+\langle\mathbf{v}_{k'}^{(2)}|\mathcal{P}\delta_1\mathbf{H}\mathbf{u}_k^{(0)}\rangle+\mathcal{A}_{kk'}^{(2)}\\&+\ldots\end{aligned}\qquad\text{(C6)}$$

where the $\mathcal{A}_{kk'}^{(n)}$ are found using

$$\langle\mathbf{v}_{k'}|\left(\mathbf{A}_0(\sigma_0)-\sigma_0^2\mathbf{K}_0\right)\mathbf{u}_k^{(1)}\rangle+\langle\mathbf{v}_{k'}|(\mathcal{I}-\mathcal{P})\delta_1\mathbf{H}\mathbf{u}_k^{(0)}\rangle=0\ ,\qquad\text{(C7)}$$

where $\mathcal{I}$ is the identity operator and $\mathcal{P}$ is the projector into the subspace associated to the singlets k and k'. This gives thus

$$\begin{aligned}\mathcal{A}_{kk'}^{(1)}&=\langle\mathbf{v}_{k'}^{(0)}|\delta_1\mathbf{H}(\mathcal{I}-\mathcal{P})\mathbf{u}_k^{(1)}\rangle+\langle\mathbf{v}_{k'}^{(0)}|\delta_2\mathbf{H}\mathbf{u}_k^{(0)}\rangle\\&+\ \delta_1\sigma_k\delta_1\sigma_{k'}\langle\mathbf{v}_{k'}^{(0)}|\mathbf{K}_0\mathbf{u}_k^{(0)}\rangle\ ,\qquad\text{(C8)}\\\mathcal{A}_{kk'}^{(2)}&=\langle\mathbf{v}_{k'}^{(0)}|\delta_3\mathbf{H}\mathbf{u}_k^{(0)}\rangle+\langle\mathbf{v}_{k'}^{(1)}|\delta_2\mathbf{H}\mathbf{u}_k^{(0)}\rangle\\&+\ \langle\mathbf{v}_{k'}^{(0)}|\delta_2\mathbf{H}\mathbf{u}_k^{(1)}\rangle+\langle\mathbf{v}_{k'}^{(1)}|\delta_1\mathbf{H}\mathbf{u}_k^{(1)}\rangle\\&+\ \delta_1\sigma_k\delta_1\sigma_{k'}\ (\ \langle\mathbf{v}_{k'}^{(1)}|\mathbf{K}_0\mathbf{u}_k^{(0)}\rangle\\&\qquad+\langle\mathbf{v}_{k'}^{(0)}|\mathbf{K}_0\mathbf{u}_k^{(1)}\rangle+\langle\mathbf{v}_{k'}^{(0)}|\delta_1\mathbf{K}\mathbf{u}_k^{(0)}\rangle\ )\\&\quad+\ (\delta_1\sigma_k\delta_2\sigma_{k'}+\delta_2\sigma_k\delta_1\sigma_{k'})\langle\mathbf{v}_{k'}^{(0)}|\mathbf{K}_0\mathbf{u}_k^{(0)}\rangle\\&\quad+\ \delta_1\sigma_k\langle\mathbf{v}_{k'}^{(0)}|\delta_2\mathbf{D}\mathbf{u}_k^{(0)}\rangle\\&\quad+\ \langle\mathbf{v}_{k'}^{(0)}|\delta_3\mathbf{A}\mathbf{u}_k^{(0)}\rangle\ .\qquad\text{(C9)}\end{aligned}$$

Equating each term of order ϵ^{n+1} for (C6) and each term of order ϵ^n for (C3) we obtain, using

$$\mathcal{P}\delta_1\mathbf{H}|\mathbf{u}_k^{(0)}\rangle=2\sigma_0[\mathbf{K}_0-\frac{\mathbf{B}_0}{2\sigma_0}]|\mathbf{u}_k^{(0)}\rangle\ ,\qquad\text{(C10)}$$

$$\mathcal{P}\delta_1\widehat{\mathbf{H}}|\mathbf{v}_{k'}^{(0)}\rangle=2\sigma_0[\mathbf{K}_0-\frac{\widehat{\mathbf{B}}_0}{2\sigma_0}]|\mathbf{v}_{k'}^{(0)}\rangle\ ,\qquad\text{(C11)}$$

the expressions of the projection of the perturbations within the multiplet:

$$\begin{aligned}\langle\mathbf{v}_{k'}^{(0)}|\left[2\sigma_0\mathbf{K}_0-\mathbf{B}_0\right]\mathbf{u}_k^{(n)}\rangle&=\frac{\mathcal{A}_{kk'}^{(n)}}{\delta_1\sigma_k-\delta_1\sigma_{k'}}\\&-\ \frac{2\sigma_0\delta_1\sigma_k\mathcal{B}_{kk'}^{(n)}}{\delta_1\sigma_k-\delta_1\sigma_{k'}}\ .\end{aligned}\qquad\text{(C12)}$$

These terms are the perturbations due to the interactions between the singlets and are highly unstable, as the difference in the denominator is very small.

APPENDIX D

In order to compute the second-order terms, let us substitute the second-order perturbation by its expression (79). From relations (C4) and (C8), and using the relations (C3), (C5), (C7) and (C9), we obtain the following expressions for the operator perturbations $\delta_2\mathbf{K}$, $\delta_2\mathbf{B}$ and $\delta_3\mathbf{H}$:

$$\begin{aligned}\mathcal{P}[\delta_2\mathbf{B}-2\sigma_0\delta_2\mathbf{K}]\mathcal{P}=\mathcal{P}[\ &\delta_1\mathbf{H}\Delta\mathbf{N}_0\Delta\delta_1\mathbf{H}\\&+\delta_1\mathbf{H}\Delta[2\sigma_0\delta_1\mathbf{K}-\delta_1\mathbf{B}]+[2\sigma_0\delta_1\mathbf{K}-\delta_1\mathbf{B}]\Delta\delta_1\mathbf{H}\\&+\delta_1\mathbf{H}\mathbf{N}_0^{-1}\delta_1\mathbf{K}+\delta_1\mathbf{K}\mathbf{N}_0^{-1}\delta_1\mathbf{H}\\&-\frac{1}{\sigma_0}\left(\left[\delta_1\mathbf{H}\mathbf{N}_0^{-1}\right]^2\frac{\mathbf{B}_0}{2\sigma_0}+\frac{\mathbf{B}_0}{2\sigma_0}\left[\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right]^2\right)\\&-\frac{1}{6}\left(\left[\delta_1\mathbf{H}\mathbf{N}_0^{-1}\right]^2\partial_\sigma^3\mathbf{A}(\sigma_0)\right.\\&\quad+\delta_1\mathbf{H}\mathbf{N}_0^{-1}\partial_\sigma^3\mathbf{A}(\sigma_0)\mathbf{N}_0^{-1}\delta_1\mathbf{H}\\&\quad\left.\left.+\partial_\sigma^3\mathbf{A}(\sigma_0)\left[\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right]^2\right)\right]\mathcal{P}\ ,\end{aligned}\qquad\text{(D1)}$$

$$\begin{aligned}\mathcal{P}\delta_3\mathbf{H}\mathcal{P}=-\mathcal{P}[\ &\delta_1\mathbf{H}\Delta\delta_1\mathbf{H}\Delta\delta_1\mathbf{H}\\&+\delta_1\mathbf{H}\mathbf{N}_0^{-1}\delta_1\mathbf{K}\mathbf{N}_0^{-1}\delta_1\mathbf{H}\\&-\ \frac{1}{\sigma_0}\left(\left[\delta_1\mathbf{H}\mathbf{N}_0^{-1}\right]^2\mathbf{K}_0\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right.\\&\qquad\left.+\ \delta_1\mathbf{H}\mathbf{N}_0^{-1}\mathbf{K}_0\left[\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right]^2\right)\\&+\ \frac{1}{6}\left(\left[\delta_1\mathbf{H}\mathbf{N}_0^{-1}\right]^2\partial_\sigma^3\mathbf{A}(\sigma_0)\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right.\\&\qquad+\delta_1\mathbf{H}\mathbf{N}_0^{-1}\partial_\sigma^3\mathbf{A}(\sigma_0)\left[\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right]^2\\&\qquad\left.+2\partial_\sigma^3\mathbf{A}(\sigma_0)\left[\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right]^3\right)\ ]\mathcal{P}\ .\end{aligned}\qquad\text{(D2)}$$

Note here that terms like $\langle\mathbf{v}_k^{(0)}|\mathbf{K}_0|\mathbf{u}\rangle$, $\langle\mathbf{v}_k^{(0)}|\mathbf{B}_0|\mathbf{u}\rangle$ and $\langle\mathbf{v}_k^{(0)}|\delta_2\mathbf{H}|\mathbf{u}\rangle$ cancel as soon as $|\mathbf{u}\rangle$ is orthogonal to $\mathcal{S}$. Let us now substitute in these relations terms like $\Delta\delta_1\mathbf{H}$ by $\Delta\delta\mathbf{H}'$, and terms like $\Delta\delta_1\mathbf{B}$, $\Delta\delta_1\mathbf{K}$ by $\Delta\mathbf{B}'$, $\Delta\delta\mathbf{K}'$. Making the summation of the first and second perturbations, and using the constrain (59), we obtain the contrains for the projection of the different perturbations on the subspace $\mathcal{S}$, which give

$$\begin{aligned}\mathcal{P}\delta_1\mathbf{H}\mathcal{P}&=\mathcal{P}\left[\delta\mathbf{H}'+\delta\mathbf{H}'\Delta\delta\mathbf{H}'+\delta\mathbf{H}'\Delta\delta\mathbf{H}'\Delta\delta\mathbf{H}'\right]\mathcal{P}\\&+\ \mathcal{F}_1(\mathbf{B}_0,\delta_1\mathbf{H},\delta_1\mathbf{K})\ ,\qquad\text{(D3)}\\2\sigma_0\mathcal{P}\delta_1\mathbf{K}\mathcal{P}&=\mathcal{P}[2\sigma_0\delta\mathbf{K}'+\delta\mathbf{H}'\Delta[2\sigma_0\mathbf{K}_0]\Delta\delta\mathbf{H}'\\&+\ \delta\mathbf{H}'\Delta[2\sigma_0\delta\mathbf{K}'-\mathbf{B}']\\&+\ [2\sigma_0\delta\mathbf{K}'-\mathbf{B}']\Delta\delta\mathbf{H}']\mathcal{P}\\&+\ \mathcal{F}_2(\mathbf{B}_0,\delta_1\mathbf{H},\delta_1\mathbf{K})\ ,\qquad\text{(D4)}\\\mathcal{P}\mathbf{B}_0\mathcal{P}&=\mathcal{P}\left[\mathbf{B}'-2\sigma_0\delta_1\mathbf{K}\right]\mathcal{P}\\&+\ \mathcal{F}_3(\mathbf{B}_0,\delta_1\mathbf{H},\delta_1\mathbf{K})\ ,\qquad\text{(D5)}\end{aligned}$$

where we have assumed that $\delta_2\mathbf{B}$ is taken as zero. The functions $\mathcal{F}_{1,2,3}(\mathbf{B}_0,\delta_1\mathbf{H},\delta_1\mathbf{K})$ are nonlinearly related to the projection within $\mathcal{S}$ of the operators $\mathbf{B}_0$, $\delta_1\mathbf{H}$ and $\delta_1\mathbf{K}$. Using the relations (75) and (76) of the main text and (D1) and (D2) of this appendix, we see easily that they are given by

$$\begin{aligned}\mathcal{F}_1(\mathbf{B}_0,\delta_1\mathbf{H},\delta_1\mathbf{K})=\mathcal{P}&\\[\quad&\delta_1\mathbf{H}\mathbf{N}_0^{-1}[\mathbf{K}_0+\delta_1\mathbf{K}]\mathbf{N}_0^{-1}\delta_1\mathbf{H}\\-\quad&\frac{1}{\sigma_0}\left(\left[\delta_1\mathbf{H}\mathbf{N}_0^{-1}\right]^2\mathbf{K}_0\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right.\\&\qquad\left.+\ \delta_1\mathbf{H}\mathbf{N}_0^{-1}\mathbf{K}_0\left[\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right]^2\right)\\+\quad&\frac{1}{6}\left(\left[\delta_1\mathbf{H}\mathbf{N}_0^{-1}\right]^2\partial_\sigma^3\mathbf{A}(\sigma_0)\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right.\\&\qquad+\delta_1\mathbf{H}\mathbf{N}_0^{-1}\partial_\sigma^3\mathbf{A}(\sigma_0)\left[\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right]^2\\&\qquad\left.+\ 2\partial_\sigma^3\mathbf{A}(\sigma_0)\left[\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right]^3\right)\ ]\mathcal{P}\ ,\end{aligned}\qquad\text{(D6)}$$

$$\begin{aligned}\mathcal{F}_2(\mathbf{B}_0,\delta_1\mathbf{H},\delta_1\mathbf{K})=\mathcal{P}&\\[\quad&\delta_1\mathbf{H}\mathbf{N}_0^{-1}\delta_1\mathbf{K}+\delta_1\mathbf{K}\mathbf{N}_0^{-1}\delta_1\mathbf{H}\\-\quad&\frac{1}{\sigma_0}\left(\left[\delta_1\mathbf{H}\mathbf{N}_0^{-1}\right]^2\frac{\mathbf{B}_0}{2\sigma_0}+\frac{\mathbf{B}_0}{2\sigma_0}\left[\mathbf{N}_0^{-1}\delta_1\mathbf{H}\right]^2\right)\end{aligned}$$

$$
\begin{aligned}
- \quad & \frac{1}{6}\Big(\left[\delta_1 \mathbf{H}\mathbf{N}_0^{-1}\right]^2 \partial_\sigma^3 \mathbf{A}(\sigma_0) \\
& +\delta_1 \mathbf{H}\mathbf{N}_0^{-1}\partial_\sigma^3 \mathbf{A}(\sigma_0)\mathbf{N}_0^{-1}\delta_1 \mathbf{H} \\
& + \partial_\sigma^3 \mathbf{A}(\sigma_0)\left[\mathbf{N}_0^{-1}\delta_1 \mathbf{H}\right]^2\Big)\ ]\mathcal{P}\ , \qquad \text{(D7)}
\end{aligned}
$$

$$
\begin{aligned}
& \mathcal{F}_3(\mathbf{B}_0, \delta_1 \mathbf{H}, \delta_1 \mathbf{K}) = -\mathcal{P} \\
& [\quad \delta_1 \mathbf{H}\mathbf{N}_0^{-1}\frac{\mathbf{B}_0}{2\sigma_0} + \frac{\mathbf{B}_0}{2\sigma_0}\mathbf{N}_0^{-1}\delta_1 \mathbf{H}\ ]\mathcal{P}\ . \qquad \text{(D8)}
\end{aligned}
$$

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