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RATIONALITY OF NORMAL FORMS OF ISOTROPY STRATA OF A REPRESENTATION OF A COMPACT LIE GROUP

P. AZZI, R. DESMORAT, J. GRIVAUX, AND B. KOLEV

Abstract. In this article we study the isotropy stratification of a linear representation V of a compact Lie group G. We prove that the closed isotropy strata are real algebraic manifolds and that for each isotropy subgroup H, every rational invariant of the induced representation $(V^H, N(H))$ can be obtained as the restriction of a global invariant of (V, G), where $N(H)$ is the normalizer of H and V^H is the fixed point set of H.

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1. Introduction

It was observed in [2], on an example from solid mechanics, that the coefficients of the normal form for some orbit types (also called *isotropy classes* or *symmetry classes* in mechanics) of a tensorial representation $(V, \mathrm{SO}(3))$ could be expressed rationally using some polynomial invariants of $(V, \mathrm{SO}(3))$. Let us start by giving an example of what we mean by this assertion. Consider the natural representation of the rotation group $\mathrm{SO}(3,\mathbb{R})$ on the vector space $\mathbb{H}^2(\mathbb{R}^3)$ of traceless symmetric 3×3 matrices with real coefficients

$$\rho(g)A = gAg^{-1}, \qquad A \in \mathbb{H}^2(\mathbb{R}^3), \quad g \in \mathrm{SO}(3,\mathbb{R}).$$

For this representation, there are exactly three orbit types corresponding to the symmetry groups $\mathbb{D}_2$ (three distinct eigenvalues, generic case), $\mathrm{O}(2)$ (two distinct eigenvalues) and $\mathrm{SO}(3)$ (only one eigenvalue). A normal form for the second orbit type represented by the symmetry group $\mathrm{O}(2)$ is provided by the linear subspace

$$V^{\mathrm{O}(2)} = \left\{A \in \mathbb{H}^2(\mathbb{R}^3);\ gAg^{-1} = A, \quad \forall g \in \mathrm{O}(2)\right\} = \left\{\mathrm{diag}(-\lambda,-\lambda,2\lambda);\ \lambda\in\mathbb{R}\right\}.$$

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Then, on the dense open subset $\lambda \neq 0$ of $(\mathbb{H}^2(\mathbb{R}^3))^{\mathrm{O}(2)}$, one has $\lambda = \dfrac{I_3}{I_2}$, where $I_2 = \operatorname{tr} A^2$ and $I_3 = \operatorname{tr} A^3$ are a generating set of the invariant algebra

$$\mathbb{R}[\mathbb{H}^2(\mathbb{R}^3)]^{\mathrm{SO}(3,\mathbb{R})}.$$

More generally, consider a linear representation $\rho\colon G \to \mathrm{GL}(V)$ of a compact Lie group G on a real vector space V. Given an orbit type represented by a symmetry group $H \subset G$, each orbit of this type intersects the fix point set V^H, which may be considered as a normal form for these orbits. The normalizer $N(H)$ of H in G stabilizes the linear subspace V^H and ρ induces a faithful representation of $N(H)/H$ on V^H. It was conjectured, after the observations in [2], that every polynomial invariant of this representation is indeed the restriction on V^H of a rational invariant of the initial representation ρ of G. In a subsequent paper [33], it was moreover conjectured that the closed strata of any real linear representation V of a compact Lie group were moreover real algebraic subset of V. In this paper, we will prove the following theorem:

Theorem 1.1. *Let $\rho\colon G \to \mathrm{GL}(V)$ be a linear representation of compact Lie group G on a real vector space V, let H be an isotropy group of ρ, let $N(H)$ be the normalizer of H in G, and let V^H be the fixed locus of H.*

1. *Every rational invariant of the induced representation $(V^H, N(H))$ is the restriction of an invariant of (V, G) which is rationally defined on V^H. In other words, the restriction map $\mathbb{R}[V] \to \mathbb{R}[V^H]$ induces a morphism*

$$S : \mathbb{R}(V)^G \dashrightarrow \mathbb{R}(V^H)^{N(H)},$$

   which is surjective when restricted to its domain of definition.
2. *The closed stratum $\overline{\Sigma}_{[H]}$ is a closed real algebraic subset of V.*

Related results can be found in the literature but under different hypotheses. In almost all papers, the problem is considered in the complex case and for generic orbit types. The most classical result fitting in this framework is Chevalley's restriction theorem that corresponds to a connected semisimple complex Lie group acting by the adjoint action on its Lie algebra $\mathfrak{g}$. More generally, Luna in [29] and then Luna and Richardson in [30, theorem 4.2] proved that, for a linear action of a reductive group G on an affine variety V (hence in particular for a vector space), the restriction map $K[V^H] \to K[V]$ maps $K[V]^G$ isomorphically onto $K[V^H]^\Gamma$ when K is an algebraically closed field, and H is the generic isotropy, and $\Gamma = N(H)/H$ (see also [29],[36]). Further results can be found in [15, 21, 31, 34] where the same isomorphism is proved on the fields of invariants of (V, G) and $(V(H), N(H))$ (that is $K(V)^G \simeq K(V^H)^\Gamma$) under the hypothesis that K is an algebraically closed field. To illustrate our theorem we provide below some explicit examples.

Example 1.2. We start with the real version of Chevalley's restriction theorem. If G is a connected compact Lie group, all maximal tori are conjugate (see [8, Chap. IV Thm 1.6]) and they are maximal abelian subgroups. Let r denote the rank of G, which is the dimension of any maximal torus T in G. Recall that an element x of $\mathfrak{g}$ is *regular* if the closure of the 1-parameter subgroup of G generated by x is a maximal torus in G. Regular points form a dense open subset $\mathfrak{g}^{\mathrm{reg}}$ of $\mathfrak{g}$, and for any x in $\mathfrak{g}^{\mathrm{reg}}$, G_x is the centralizer of the maximal torus of G containing the 1-parameter subgroup generated by x, which is the torus itself. Hence, if T is a maximal torus, if $W = N(T)/T$ is the corresponding Weil group, and if $\mathfrak{t}$ is the Lie algebra of T, there is an isomorphism $\mathbb{R}[V]^G \simeq \mathbb{R}[\mathfrak{t}]^W$ (see [8, Chap. VI Prop. 2.1]). Our result implies a weak form of this statement, namely the same isomorphism at the level of fields of invariants.

Example 1.3. For the standard representation of the symmetric group $\mathfrak{S}_3$ on $V = \mathbb{R}^3$, there are three isotropy classes $[\mathbb{1}]$, $[\mathfrak{S}_2]$, $[\mathfrak{S}_3]$. The invariant algebra $\mathbb{R}[\mathbb{R}^3]^{\mathfrak{S}_3}$ is generated by the three elementary symmetric functions σ_1, σ_2 and σ_3. The fixed point set $V^{\mathfrak{S}_2}$ is the plane $x = y$ and

$N(\mathfrak{S}_2) = \mathfrak{S}_2$. Hence, $\mathbb{R}[V^{\mathfrak{S}_2}]^{N(\mathfrak{S}_2)} = \mathbb{R}[x, y]$ and we have

$$x = \frac{\sigma_1\sigma_2 - 9\sigma_3}{2\sigma_1^2 - 6\sigma_2} \quad \text{and} \quad z = \frac{\sigma_1^3 - 4\sigma_1\sigma_2 + 9\sigma_3}{\sigma_1^2 - 3\sigma_2}.$$

Example 1.4. Here, we provide an example where $N(H)/H$ is not trivial. We consider the representation of the rotation group $\mathrm{SO}(3,\mathbb{R})$ on the vector space $V = \mathbb{H}^4(\mathbb{R}^3)$ of harmonic symmetric tensors of order 4 (or homogeneous harmonic polynomials in three variables of degree four). It is known, see [14, 16], that there are eight orbit types, and among them the symmetry class $[\mathbb{D}_2]$, where $\mathbb{D}_2$ is the dihedral group of order 2. The invariant algebra $\mathbb{R}[\mathbb{H}^4(\mathbb{R}^3)]^{\mathrm{SO}(3,\mathbb{R})}$ is generated by 9 polynomials J_k which have been obtained in [3] (see also [2]). The normalizer $N(\mathbb{D}_2)$ is the octahedral group $\mathbb{O}$ and $\Gamma = N(\mathbb{D}_2)/\mathbb{D}_2 \simeq \mathfrak{S}_3$. The fixed point set $V^{\mathbb{D}_2}$ is the three-dimensional vector space spanned by the harmonic polynomials

$$\begin{cases} p_1 = -z^4 + 6y^2z^2 - y^4, \\ p_2 = -z^4 + 6x^2z^2 - x^4, \\ p_3 = -y^4 + 6x^2y^2 - x^4. \end{cases}$$

In this basis, the action of the monodromy group $\Gamma \simeq \mathfrak{S}_3$ is just the standard action of the permutation group $\mathfrak{S}_3$ on the triple $(\lambda_1, \lambda_2, \lambda_3)$ and the invariant algebra $\mathbb{R}[V^{\mathbb{D}_2}]^{\mathfrak{S}_3}$ is generated by the elementary symmetric polynomials $(\sigma_1, \sigma_2, \sigma_3)$ in the λ_i. It was shown in [2] that σ_1, σ_2 and σ_3 can be expressed rationally in terms of the J_k by

$$\begin{cases} \sigma_1 = -\dfrac{9\,\left(3\,J_7 - 3\,J_2\,J_5 + 3\,J_3\,J_4 - {J_2}^2\,J_3\right)}{2\,\left(6\,J_6 - 9\,J_2\,J_4 - 20\,{J_3}^2 + 3\,{J_2}^3\right)}, \\ \sigma_2 = \dfrac{4}{7}\,{\sigma_1}^2 - \dfrac{1}{14}\,J_2, \\ \sigma_3 = \dfrac{1}{24}\,J_3 + \dfrac{1}{7}\,{\sigma_1}^3 - \dfrac{1}{56}\,\sigma_1\,J_2. \end{cases}$$

Outline. The article is organized as follows. In section 2, we recall basic material on representations of compact Lie groups and invariant theory. In section 3, we recall an explicit linear model for the complexification of a compact Lie group and provide several useful results on the complexification of a real continuous representation of a compact Lie groups. Besides, we prove one key technical tool, namely that for compact group, normalizers commute with complexification. The section 4 is devoted to the proof of theorem 1.1. In addition, two appendices are provided. In Appendix A, we recall some results about Lie groups, real analytic functions, and totally real manifolds. Finally, in Appendix B, we provide for the reader's convenience a proof of the Principal Orbit Theorem in the linear case.

2. Preliminaries

2.1. **Linear representations of compact Lie groups.** Let G be a compact (real) Lie group and let $\rho : G \to \mathrm{GL}(V)$ be a continuous linear representation of G on a finite dimensional vector space V. It is well-known that ρ is real-analytic (see corollary A.2). In particular, we will be able to apply all the tools of differential topology concerning smooth action of Lie groups on manifolds. We write $g \cdot v := \rho(g)v$ to lighten the notation.

For any vector v in V, we denote by $G \cdot v$ its orbit under G. The quotient set V/G of V under the action of G is the orbit space of V.

The *isotropy subgroup* (or *symmetry group*) of a vector v in V is defined by

$$G_v := \{g \in G;\ g \cdot v = v\}.$$

It is a closed subgroup of G, hence a Lie subgroup of G. The orbit $G \cdot v$ is a smooth compact submanifold of V, which is diffeomorphic to the homogeneous space G/G_v.

Two vectors in the same orbit have conjugate symmetry groups. Indeed,

$$\forall v \in V,\ \forall g \in G, \quad G_{g\cdot v} = g\ G_v\ g^{-1}.$$

Of course, the converse is false but we say that v_1 and v_2 are in the same *isotropy class* (or have the same *orbit type* in the terminology of [1]) if their symmetry groups are *conjugate* in G, that is if there exists $g \in G$ such that

$$G_{v_2} = g\, G_{v_1} g^{-1}.$$

Given a closed subgroup H of G, we say that the conjugacy class $[H]$ of H in G is an *isotropy class* (or an *orbit type*) if there exists a vector v such that $[H] = [G_v]$.

The finiteness of isotropy classes for a continuous action of a compact Lie group on a compact manifold was initially conjectured by Montgomery and solved in [13, 32] (in the case of a smooth action it is way more easy, see also [4, Chapter VII] and [38, Theorem 5.11]). This implies the finiteness of isotropy classes for a linear representation of a compact Lie group: we can see it by extending a representation on V to an action on the projective compactification $\mathbb{P}(V \oplus \mathbb{R})$, or on the sphere $S(V)$ if we take an invariant inner product.

Given a compact Lie group G, the inclusion relation on the set of closed subgroups induces a partial order on the set of their conjugacy classes [7, 1], called by some authors *containment relation* [9]. It is defined as follows:

$$[H_1] \preceq [H_2], \qquad \text{if } H_1 \text{ is conjugate to a subgroup of } H_2.$$

To understand the geometry of conjugacy classes, a very useful notion is that of a *slice*. We recall here the general definition, but in our setting not all properties are necessary (see 2.2).

Definition 2.1. For any v in V, a local slice of the G-action at v is an embedded disc $S \subset V$ passing through v such that:

- S is transverse to the orbit $G \cdot v$.
- S is stable under G_v.
- If s_1, s_2 are two points in S and there exists g in G with $g \cdot s_1 = s_2$, then g belongs to G_v. In particular, $G_s \subset G_v$ for each point $s \in S$.
- $G \cdot S$ is an open neighborhood of the orbit $G \cdot v$.

Remark 2.2. We have furthermore the following properties.

- Given a slice S, the open G-stable neighborhood $G \cdot S$ is diffeomorphic to the quotient $G \times_{G_v} S$, and after linearizing the action of G_v on S, $G \cdot S$ is locally diffeomorphic to $G \times_{G_v} N_v$ where N_v is the normal space of the orbit at v (viewed as a representation of the isotropy group G_v).
- Slices exist in general for proper actions of Lie groups on manifolds [10, Theorem 2.3.1], and in particular for actions of compact Lie groups. However, for linear representations of compact Lie groups, it is possible to produce slices easily (see lemma B.3 below).
- In our settings, where G is compact, the two last properties are automatic (of course up to shrinking S) as soon as the two first ones are granted.
- The last property can be made slightly stronger: up to shrinking S, we can even assume that the orbit map from $G \times S$ to V is a submersion, in particular it is an open map.

The existence of slices has the following corollary.

Corollary 2.3. *Let v in V. Then there exists a neighborhood U of v such that for all w in U, $[G_w] \preceq [G_v]$.*

Proof. Let S be a slice at v. If $w \in G \cdot S$ then w is in the orbit of a point w' in S. It follows that $G_{w'} \subset G_v$, so $[G_w] = [G_{w'}] \preceq [G_v]$. □

Definition 2.4. A vector $v \in V$ (or its orbit) is called *principal* if there exists a neighborhood U of v such that for all w in U, $[G_w] = [G_v]$. Equivalently, v is principal if it has locally minimal isotropy.

Remark 2.5.

- By definition, the set of principal points is open.

- The definition of principal points is not totally standard. Some authors, like [1, section 6] have adopted another definition: A vector $v \in V$ is principal if there exists a neighborhood U of v such that for all w in U, $[G_v] \preceq [G_w]$. Anyway, for a compact Lie group representation, the two definitions are equivalent.

Let us denote by π the canonical projection from V to V/G. The principal orbit type theorem can be stated (in the case of linear representations) as follows (see [7, Theorem 3.1], [1, section VI], [10, Theorem 2.8.5]).

Theorem 2.6 (Principal orbit type theorem)**.** *For any open subset Ω of V such that $\pi(\Omega)$ is connected, the set U_Ω of principal points in Ω is open, dense in Ω, and $\pi(U_\Omega)$ is connected.*

For the interested reader, we present a proof in Appendix B. It follows from this result that there is a unique minimal orbit type.

Definition 2.7. Let G be a compact Lie group acting linearly on V and H be an isotropy subgroup.

- The *fixed locus* V^H is defined by
$$V^H := \{v \in V;\ h \cdot v = v \text{ for all } h \in H\},$$
it is the vector subspace of elements of V fixed by H.
- The *open fixed locus* $\overset{\circ}{V}{}^H$ is the subset of V^H defined by
$$\overset{\circ}{V}{}^H := \left\{v \in V^H;\ G_v = H\right\}.$$
- The *stratum* $\Sigma_{[H]}$ is defined by
$$\Sigma_{[H]} := \{v \in V;\ [G_v] = [H]\}.$$
It is the set of vectors in V having isotropy class $[H]$ and it is the orbit of $\overset{\circ}{V}{}^H$.
- The *closed stratum*
$$\overline{\Sigma}_{[H]} = \left\{v \in V;\ gHg^{-1} \subset G_v, \text{ for some } g \in G\right\}$$
consists of vectors V having isotropy at least $[H]$. It is the orbit of V^H.

Remark 2.8.

- The terminology we use here is not totally standard, but it varies according to different authors (for instance, what we call stratum is called orbit bundle in [38]).
- The fixed locus V^H is defined for arbitrary closed subgroups of G (not only isotropy subgroups) but we will generally avoid to do so.
- The terminology "closed stratum" and the corresponding notation will be justified by corollary 2.11. The same holds for the opened fixed locus.
- The partial order relation on isotropy classes induces a (reverse) partial order relation on the strata.
$$[H_1] \preceq [H_2] \iff \Sigma_{[H_2]} \preceq \Sigma_{[H_1]}$$

The *normalizer* of H is
$$N(H) := \left\{g \in G \mid gHg^{-1} = H\right\}.$$
It is the maximal subgroup of G in which H is a normal subgroup. We have the following result (see [2], [16]):

Lemma 2.9. *For each closed subgroup H of G, V^H is $N(H)$-stable. Moreover, if $H = G_{v_0}$ is the* isotropy group *of some point $v_0 \in V$, then $N(H) = \left\{g \in G;\ g \cdot V^H \subset V^H\right\}$.*

Proof. Let v in V^H. Then, for any g in $N(H)$ and any h in H, we have
$$h \cdot (g \cdot v) = g \cdot (g^{-1}hg) \cdot v = (gh') \cdot v = g \cdot v.$$

If moreover $H = G_{v_0}$, and $g \in G$ is such that $g \cdot V^H \subset V^H$, then, $g \cdot v_0$ is in V^H and thus $H \subset gHg^{-1}$, because $G_{g\cdot v_0} = gG_{v_0}g^{-1}$. Since H is compact, $H = gHg^{-1}$ by [38, Proposition 3.7] and g belongs to $N(H)$, which ends the proof. □

The linear representation $\rho : G \to \mathrm{GL}(V)$ induces, by restriction, a linear representation of $N(H)$ on V^H

$$\rho_{N(H)} \colon N(H) \longrightarrow \mathrm{GL}(V^H).$$

This induced representation is however not faithful in general but, *when H is an isotropy group*, its kernel is exactly H and we get a *faithful* linear representation

$$\rho_{\Gamma^H} : \Gamma^H \longrightarrow \mathrm{GL}(V^H),$$

where $\Gamma^H := N(H)/H$ is called the *monodromy group* in [2]. Note that, in that case, two vectors v_1, v_2 in $V^H \cap \Sigma_{[H]}$ are in the same G-orbit if and only if they are in the same Γ^H-orbit.

Proposition 2.10. *Let H be an isotropy subgroup. Then $\overset{\circ}{V}{}^H$ is open and dense in V^H, more precisely it contains a Zariski dense open subset of V^H.*

Proof. Let x be in $\overset{\circ}{V}{}^H$. Then thanks to corollary 2.3 there exists a neighborhood U of x such that for y in U, $[G_y] \preceq [H]$. If y in $U \cap V^H$, $[H] \preceq [G_y] \preceq [H]$ so $[G_y] = [H]$. This means that $H \subset G_y = gHg^{-1}$ for some g in G so by [38, Proposition 3.7] again, $G_y = H$. This proves that $\overset{\circ}{V}{}^H$ is open. The second part is difficult and will be proven later on (see corollary 3.17). □

Corollary 2.11. *$\overline{\Sigma}_{[H]}$ is the closure of $\Sigma_{[H]}$.*

Proof. First we prove that $\overline{\Sigma}_{[H]}$ is closed. If v_n is a sequence in $\overline{\Sigma}_{[H]}$ that converges to v then there exists elements g_n in G such that $g_n H g_n^{-1} \subset G_{v_n}$. After extracting a subsequence, we can assume that g_n converges to an element g in G. Then $gHg^{-1} \subset G_v$ so v belongs to $\overline{\Sigma}_{[H]}$. Now, let v be in $\overline{\Sigma}_{[H]}$. By definition, there exists g in G such that $g^{-1} \cdot v$ belongs to V^H. Thanks to proposition 2.10, $g^{-1} \cdot v$ is the limit of a sequence of elements in $\overset{\circ}{V}{}^H$. □

For a compact Lie group representation (V, ρ), the partition into (non empty) isotropy strata

$$V = \Sigma_{[H_0]} \cup \cdots \cup \Sigma_{[H_n]}$$

is called its *isotropy stratification* or *orbit type stratification*. It can be shown that it is a real stratification, and even a Whitney stratification (see [10, §2.7 and Theorem 2.7.4]).

Example 2.12. Consider, for instance, the representation of the rotation group $G = \mathrm{SO}(3)$ on $V = S^2\mathbb{R}^3$. Then there are exactly three orbit types

$$[\mathbb{D}_2] \preceq [\mathrm{O}(2)] \preceq [\mathrm{SO}(3)],$$

where $[\mathbb{D}_2]$ is the principal orbit type. In that case, $V^{\mathbb{D}_2}$, the subspace of diagonal matrices, intersects all the orbits but since $N(\mathbb{D}_2)$ is the octahedral group $\mathbb{O}$ and the monodromy $\Gamma^{\mathbb{D}_2}$ is isomorphic to the symmetric group $\mathfrak{S}_3$, the set $V^{\mathbb{D}_2}$ is not (formally speaking) a slice, but it can be seen as a *slice with finite monodromy*.

2.2. **Invariant theory and orbits.** In the material we will present, we will constantly deal with two types of groups:

- **Compact Lie groups:** they admit automatically a faithful representation [8, Theorem III.4.1], and thanks to Weyl's unitary trick, this representation can be chosen unitary. Besides, every continuous finite-dimensional representation of G is fully irreducible, i.e. splits as a direct sum of irreducible representations.
- **Complex reductive groups:** they are complex Lie groups admitting a faithful complex analytic representations and such that every finite-dimensional analytic representation splits as a direct sum of irreducible representations.

It turns out that both type of groups are closely linked through the process of complexification, which will be detailed in 3.1. More precisely, complex reductive groups are exactly complexifications of compact Lie groups [25, Theorem 4.31]. The idea behind this link is that Weyl's unitary trick allows to prove that a complex Lie group is reductive as soon as it contains a Zariski dense compact subgroup (see for instance [24, Lemma 6.2.7]). Lastly, if $G \subset \mathrm{GL}_n(\mathbb{C})$ is reductive, then, G is automatically an algebraic subgroup of $\mathrm{GL}(n,\mathbb{C})$ [25, Theorem 5.11].

Let G be a real (resp. complex) Lie group, and let $K = \mathbb{R}$ (resp. $K = \mathbb{C}$). The linear action of G on a K-vector space V extends naturally to the polynomial algebra $K[V]$ via the formula

$$(g \cdot p)(v) := p(g^{-1} \cdot v)$$

for every polynomial $p \in K[V]$ and every vector $v \in V$. The set of all polynomials that are invariants under the action of G is a subalgebra of $K[V]$ denoted by $K[V]^G$ and called the *invariant algebra* of V.

The foundational result of invariant theory, due initially to Hilbert [20] in the case of the action of $\mathrm{GL}(n,\mathbb{C})$ on $\mathrm{Sym}^d \mathbb{C}^n$, runs as follows.

Theorem 2.13. [19, Theorem X.5.6], [24, Theorem 6.3.1] *Let G be a compact (resp. complex reductive) Lie group, let $K = \mathbb{R}$ (resp. $K = \mathbb{C}$), let V be a finite dimensional K-vector space, and let $\rho\colon G \to \mathrm{GL}(V)$ be a continuous (resp. analytic) representation of G. Then, the invariant algebra $K[V]^G$ is finitely generated. This means that there exists a finite set of invariant polynomials $p_1, \dots, p_N$ such that*

$$K[V]^G = K[p_1, \dots, p_N].$$

Remark 2.14. Although this theorem is stated most of the time for complex reductive groups (or even reductive groups over an algebraically closed field of characteristic zero), the proof in the compact case works in the same way since it relies only on the Noetherianity of $K[V]$ and the existence of a Reynolds operator (which is simply in this case obtained by averaging on the group).

It is clear that any invariant is constant on G-orbits. The geometry of orbits can be understood via the invariants, but the situation is different for the real and the complex cases.

$K = \mathbb{R}$, G compact:

- The G-orbits are compact and smooth.
- The invariants separate the G-orbits. In other terms, given two vectors $v_1, v_2 \in V$ belonging to different G-orbits, it is always possible to find a function $p \in \mathbb{R}[V]^G$ such that $p(v_1) \neq p(v_2)$ (see [1, Appendix C]).
- The orbit space V/G can be described as a semialgebraic subset of $\mathbb{R}^N$. Indeed, if $\{p_1, \dots, p_N\}$ denotes a generating set for $\mathbb{R}[V]^G$, then the mapping

$$P\colon v \mapsto \big(p_1(v), p_2(v), \dots, p_N(v)\big)$$

  induces an homeomorphism between V/G and $P(V) \subset \mathbb{R}^N$ which is a semialgebraic subset of $\mathbb{R}^N$.

$K = \mathbb{C}$, G reductive:

- The G-orbits are constructible [1], and their closure are the same for the usual or for the Zariski topology.
- Two Zariski-closed G-stable sets of V can be separated by invariants.
- Each G-orbit is adherent to a unique closed G-orbit.
- The complex scheme $V/G = \mathrm{spec}\, \mathbb{C}[V]^G$ parameterizes closed G-orbits.

In the same way one defines the algebra of polynomial invariants $K[V]^G$, one can define the *invariant field* $K(V)^G$ of *rational invariants*, which is a subfield of the field $K(V)$ of all rational functions on V. Note that this field is always finitely generated since it is contained in the finitely generated field $K(V)$ (see for instance [23]).

[1] A set is constructible if it is Zariski open in its Zariski closure.

Following Popov and Vinberg [34], we say that a rational invariant f separates the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ if it is defined at points of both orbits and assumes different values at these points. A subset F of $K(V)^G$ separates the orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ if it contains an element that separates these orbits. Finally, we say that a subset F of $K(V)^G$ separates orbits in general position if there exists a nonempty Zariski open subset $U \subset V$ such that F separates the orbits of any two inequivalent points of U.

Theorem 2.15. [34, Lem. 2.1] *Let G be a reductive complex group and $\rho : G \to \mathrm{GL}(V)$ a complex representation of G. If a finite set $F \subset \mathbb{C}(V)^G$ separates orbits in general position, then it generates the field $\mathbb{C}(V)^G$.*

Remark 2.16.

- The converse of theorem 2.15 is also true (see [34, Theorem 2.3]) and is originally due to Rosenlicht [35].
- Theorem 2.15 is wrong in the real setting. Consider the real representation of the trivial group $G = \{e\}$ on the real space $V = \mathbb{R}$. Then the invariant field is given by $\mathbb{R}(V)^G = \mathbb{R}(x)$. The polynomial $p(x) = x^3$ separates all the orbits but is not a generator of $\mathbb{R}(V)^G$.

2.3. **The algebraic case.** The geometry of the linear action of a compact Lie group (or more generally the action of a Lie group on a manifold) has been investigated in the framework of algebraic geometry. The theory is delicate, we will present a quick overview of the situation as well as results that we need. The setting is as follows: let G be a complex reductive group acting algebraically on an complex affine algebraic variety (the theory works for any closed field of characteristic zero, but we will use it essentially in the complex case). The aim is to provide a local model for the action near an orbit of G. The most simple example happens when the action is free. If we pursue the analogy with differential geometry, we would expect a local trivialisation near the orbit. However, the following example shows that it is not possible to expect such a result in the Zariski topology.

Example 2.17. Let us consider the group μ_n of n-th roots of unity, acting naturally on $\mathbb{C}^*$. In this case, the space of orbits is also isomorphic by $\mathbb{C}^*$, the quotient map π being given by $z \mapsto z^n$. We see that it is impossible to trivialize π in the Zariski topology on the base. Indeed, a Zariski open subset of $\mathbb{C}^*$ is simply obtained by removing a finite number of points, and the projection remains nontrivial on any such open subset.

The problem comes from the fact that a smooth and surjective morphism between algebraic varieties does not always have a section in the Zariski topology. However, it has a section in the etale topology [18, 17.16.3 (ii)]. Concretely this means the following: if $\varphi\colon X \to Y$ is smooth and surjective, then for any y in Y there is a neighborhood U_y of y and an etale morphism $\varphi\colon V \to U_y$ such that the pull-back morphism $\widetilde{f}\colon V \times_{U_y} X \to V$ has a section. For the interested reader, let us mention that the problem of trivializing principal G-bundles in the Zariski topology was studied in depth by Grothendieck in [17]: in fact he proved that for a given algebraic group G, then all etale locally trivial principal G-bundles are Zariski locally trivial if and only if G is affine, connected, and without torsion [17, Theorem 3].

Let us provide a very interesting example, due to Richardson, that illustrates another difficulty related to the topology of the orbits.

Example 2.18. [28, Remark 4° page 98] Let us consider the natural action of $\mathrm{SL}(2,\mathbb{C})$ on the set V_3 of cubic binary forms, *i.e.*, homogeneous complex polynomials of degree 3 in two complex variables. Now, $\mathbb{P}(V_3)$ is isomorphic to $\mathrm{S}^3\,\mathbb{P}^1$, the isomorphism being induced by the map which assigns to a cubic binary form its (unordered) three roots. The action of $\mathrm{SL}(2,\mathbb{C})$ is the natural action of $\mathrm{PGL}(2,\mathbb{C})$ on triplets of points in $\mathbb{P}^1$ after quotienting by ± 1. Hence there are only 3 possible projective orbits: $[w^3], [zw^2]$ and $[z(w-z)w]$, whose respective stabilizers in $\mathrm{PGL}(2,\mathbb{C})$

are

$$\mathrm{PGL}(2,\mathbb{C})_{[w^3]} = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}, a \in \mathbb{C}^*, b \in \mathbb{C} \right\},$$
$$\mathrm{PGL}(2,\mathbb{C})_{[zw^2]} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}, a \in \mathbb{C}^* \right\}$$
$$\mathrm{PGL}(2,\mathbb{C})_{[z(1-z)w]} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\}.$$

We can now look at the pre-images of these stabilizers in $\mathrm{SL}(2,\mathbb{C})$. They act by characters on binary forms. More precisely, for ε in $\{-1,+1\}$, if we denote by $\gamma \star f$, the right action of $\gamma \in \mathrm{SL}(2,\mathbb{C})$ on $f \in V_3$, given by $(\gamma \star f)(\xi) = f(\gamma \cdot \xi)$, where $\xi = (z,w)$, we get

$$\begin{pmatrix} \alpha & \beta \\ 0 & 1/\alpha \end{pmatrix} \star w^3 = \frac{w^3}{\alpha^3}, \qquad \begin{pmatrix} \alpha & 0 \\ 0 & 1/\alpha \end{pmatrix} \star zw^2 = \frac{zw^2}{\alpha}$$
$$i\varepsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \star z(w-z)w = -i\varepsilon z(w-z), \qquad i\varepsilon \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \star z(w-z)w = -i\varepsilon z(w-z)$$
$$\varepsilon \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \star z(w-z)w = -\varepsilon z(w-z)w, \qquad i\varepsilon \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \star z(w-z)w = i\varepsilon z(w-z)w$$
$$\varepsilon \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \star z(w-z)w = -\varepsilon z(w-z)w.$$

Hence we have 4 orbit types:

- The orbit of 0 has stabilizer $\mathrm{SL}(2,\mathbb{C})$.
- The orbit of w^3 has stabilizer $\mathbb{C} \rtimes \mathbb{Z}/3\mathbb{Z}$.
- The orbit of zw^2 has trivial isotropy.
- The orbit of $z(w-z)w$ has isotropy

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \right\} \simeq \mathbb{Z}/3\mathbb{Z}.$$

Now, the set U of binary forms which have three distinct roots is a Zariski open subset of V_3 defined by the nonvanishing of the discriminant. It is dense, and all points in U have isotropy $\mathbb{Z}/3\mathbb{Z}$. However, $\mathrm{SL}(2;\mathbb{C})_{zw^2}$ is trivial. Hence the generic orbit has isotropy $\mathbb{Z}/3\mathbb{Z}$, but another orbit type has trivial stabilizer. This example shows that we cannot expect corollary 2.3 to be true in whole generality in the algebraic setting.

In the preceding example, the reason why this problem occurs is that the orbit of zw^2 is not closed. In the case of a closed orbit, Luna's slice theorem gives a good description of the situation.

Theorem 2.19. [28] *Let G be a reductive group acting on an affine algebraic variety X. Let x be a point of X and assume that the orbit $G \cdot x$ is closed in X. Then there exists an etale slice at x, that is an affine subvariety V of X passing through x satisfying the following properties:*

- *V is G_x-stable*
- *The natural G-morphism $\varphi\colon G \times_{G_x} V \to U$ is etale, and its image U is an affine neighborhood of the orbit $G \cdot x$.*

Using this theorem, Luna was able to recover results that were holding in the case of compact Lie group actions, like finiteness of orbit types. The result that will be of crucial importance for us is the algebraic version of corollary 2.3.

Corollary 2.20. [28, Remark 4° pp. 98] *Let G be a reductive group acting on an affine algebraic variety X. Let x be a point of X and assume that the orbit $G \cdot x$ is closed in X. Then there exists a Zariski open neighborhood of x such that for any y in X, G_y is conjugate to a subgroup of G_x.*

3. Complexification

3.1. **Complexification of a compact Lie group.** The complexification of a real Lie group is formally defined as the solution of a universal problem, which always exists and is unique up to a complex analytic isomorphism ([6, Chapter 3]).

If G is compact, it is possible to define $G^{\mathbb{C}}$ as follows (see [8, III.8]): let $\mathcal{A}$ be the algebra of representative functions on G, that is functions that generate a finite-dimensional representation inside $\mathcal{C}^0(A,\mathbb{R})$. Tannaka-Krein duality [8, III.7] guarantees that G identifies with real characters of $\mathcal{A}$, that is every character of $\mathcal{A}$ is of the form $f \to f(g)$ for g in G. Then $G^{\mathbb{C}}$ is defined as the complex characters of $\mathcal{A}^{\mathbb{C}}$.

Example 3.1. If $G = \mathrm{U}(1)$, $\mathcal{A}$ is the algebra of trigonometric polynomials $\mathbb{R}[\cos(\theta),\sin(\theta)]$. Its complexification is the algebra $\mathbb{C}[\cos(\theta),\sin(\theta)]$. Given a complex polynomial $P(\cos\theta,\sin\theta)$, we can associate the Laurent Polynomial $P\left(\frac{z+1/z}{2},\frac{z-1/z}{2i}\right)$. In this way we see that

$$\mathbb{C}[\cos(\theta),\sin(\theta)] \simeq \mathbb{C}[z,1/z],$$

since every Laurent polynomial is uniquely determined by its restriction on $\mathrm{U}(1)$. Hence, the complex characters of $\mathbb{C}[\cos(\theta),\sin(\theta)]$ are exactly the points of $\mathbb{C}^*$.

There is an explicit way to describe this complexification, which is pretty useful to understand more precisely the geometry of $G^{\mathbb{C}}$. To achieve this, we use the fact that every compact Lie group admits a faithful representation [8, Thm. III.4.1], and thanks to Weyl's unitary trick, it admits a faithful unitary representation. Before going further, let us recall some elementary facts of linear algebra.

Lemma 3.2.

(1) *Polar decomposition: if* $\mathrm{P}(n)$ *denotes the set of hermitian positive definite matrices, then the product map*
$$\mathrm{U}(n)\times\mathrm{P}(n) \xrightarrow{\sim} \mathrm{GL}(n,\mathbb{C})$$
is a diffeomorphism.

(2) *If* ι *denotes the Cartan involution* $M \to (M^*)^{-1}$ *of* $\mathrm{GL}(n,\mathbb{C})$*, then for any* M *in* $\mathrm{GL}(m,\mathbb{C})$ *with polar decomposition* gh*,* $h^2 = \iota(M)^{-1}M$*.*

(3) $\mathrm{U}(n) = \{M \in \mathrm{GL}(n,\mathbb{C}), \iota(M) = M\}$ *and* $\mathrm{P}(n) = \{M \in \mathrm{GL}(n,\mathbb{C}), \iota(M) = M^{-1}\}$*.*

(4) *The map* $Z \to e^{iZ}$ *from* $\mathfrak{u}(n)$ *to* $\mathrm{P}(n)$ *is a diffeomorphism, where* $\mathfrak{u}(n) = \mathrm{Lie}(\mathrm{U}(n))$ *is the Lie algebra of skew-hermitian matrices.*

This being done, the complexification of a compact Lie group can be described as follows.

Proposition 3.3. [8, Prop. III.8.3] *Let* $G \subset \mathrm{U}(n)$ *be a compact Lie group and let* $\mathfrak{g}$ *be its Lie algebra. Then*
$$G^{\mathbb{C}} = \left\{ge^{iZ}\ ;\ g \in G \text{ and } Z \in \mathfrak{g}\right\}$$
In particular, $G^{\mathbb{C}}$ *is diffeomorphic to* $G\times\mathfrak{g}$*.*

Remark 3.4. This result might be surprising at first glance because it is not clear at all that the right hand side is a group. Let us briefly explain by hand why it is the case. Given $g_1e^{iZ_1}$ and $g_2e^{iZ_2}$, where $g_1,g_2 \in G$ and $Z_1,Z_2 \in \mathfrak{g}$, we have

$$e^{iZ_1}g_2 = g_2\mathrm{Ad}(g_2^{-1})e^{iZ_1} = g_2e^{i\mathrm{Ad}(g_2^{-1})(Z_1)} = g_2e^{iZ_3},$$

where $Z_3 \in \mathfrak{g}$. We get thus $g_1e^{iZ_1}g_2e^{iZ_2} = g_1g_2e^{iZ_2}e^{iZ_3}$. Writing

$$e^{iZ_2}e^{iZ_3} = ge^{iZ},$$

with $g \in \mathrm{U}(n)$ and $Z \in \mathfrak{u}(n)$, we have thanks to lemma 3.2 (2),

$$e^{2iZ} = \iota(e^{iZ_2}e^{iZ_3})^{-1}e^{iZ_2}e^{iZ_3} = e^{iZ_3}e^{2iZ_2}e^{iZ_3}.$$

Consider now the real analytic function

$$\varphi\colon \mathfrak{g}\times\mathfrak{g} \to \mathfrak{u}(n), \qquad (Z_2,Z_3) \mapsto \frac{1}{2i}\log(e^{iZ_3}e^{2iZ_2}e^{iZ_3}).$$

By applying lemma A.1 two times, we get that $\varphi(Z_2, Z_3) \in \mathfrak{g}^{\mathbb{C}}$ for sufficiently small Z_2, Z_3 and by corollary A.5, we conclude that for all Z_2, Z_3 in $\mathfrak{g}$, $\varphi(Z_2, Z_3) \in \mathfrak{g}^{\mathbb{C}} \cap \mathfrak{u}(n) = \mathfrak{g}$. Hence this proves that for all Z_2, Z_3 in $\mathfrak{g}$, Z is in $\mathfrak{g}$.

We can apply again the same trick: consider the real analytic map

$$\Psi \colon \mathfrak{g} \times \mathfrak{g} \to \mathrm{U}(n), \qquad (Z_2, Z_3) \mapsto e^{iZ_2} e^{iZ_3} e^{-i\varphi(Z_2,Z_3)}.$$

For Z_2, Z_3 close to 0, $\log \Psi$ takes values in $\mathfrak{g}^{\mathbb{C}} \cap \mathfrak{u}(n) = \mathfrak{g}$, so Ψ takes values in G. As G is a closed real analytic submanifold of $\mathrm{U}(n)$ by corollary A.2, we deduce using again corollary A.5 that for all Z_2, Z_3 in $\mathfrak{g}$, $\Psi(Z_2, Z_3) \in G$. Since $\Psi(Z_2, Z_3) = g$, we deduce that g belongs to G and we are done.

Remark 3.5. The group G is a closed subgroup of $\mathrm{GL}(n;\mathbb{C})$, and it is straightforward to check that $\mathrm{Lie}(G^{\mathbb{C}}) = \mathfrak{g} + i\mathfrak{g} = \mathfrak{g}^{\mathbb{C}}$ since $\mathfrak{u}(n)$ is totally real in $\mathfrak{gl}(n;\mathbb{C})$. Hence $\mathrm{Lie}(G^{\mathbb{C}})$ carries a natural complex structure for which the bracket is complex linear. It defines on $G^{\mathbb{C}}$ a complex Lie group structure for which the exponential map is a local biholomorphism around 0. Besides, G is totally real in $G^{\mathbb{C}}$.

We have an even stronger property :

Proposition 3.6. *The complexified group $G^{\mathbb{C}}$ is an affine algebraic subvariety of $\mathrm{GL}(n, \mathbb{C})$.*

Proof. This is [8, Prop. III.8.2], the key point being the fact that the algebra of representative functions on G is finitely generated. □

Corollary 3.7. *If G is a compact Lie group, then G is analytically Zariski dense in $G^{\mathbb{C}}$.*

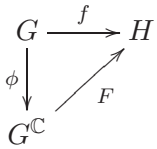
Proof. The group G is a totally real analytic submanifold of $G^{\mathbb{C}}$. Hence, thanks to corollary A.2, the Zariski closure of G in $G^{\mathbb{C}}$ is the union of connected components of $G^{\mathbb{C}}$ that contain G. By Proposition 3.3, this union is $G^{\mathbb{C}}$ itself. □

Corollary 3.8. *Let G be a compact Lie group, and assume that $G^{\mathbb{C}}$ acts holomorphically on a (non-necessarily finite dimensional) complex vector space W. Then $W^{G^{\mathbb{C}}} = W^G$.*

Proof. For w in W^G, let us consider a linear form u on W. Let $\phi \colon G^{\mathbb{C}} \to \mathbb{C}$ given by $\phi(g) = u(g \cdot w - w)$. Then ϕ is holomorphic and ϕ vanishes on G. Thanks to corollary 3.7, ϕ vanishes on $G^{\mathbb{C}}$. This means that for all g in $G^{\mathbb{C}}$, all linear forms take the same values on w and $g \cdot w$, so $g \cdot w = w$. Hence w is fixed by $G^{\mathbb{C}}$. □

Finally, let us prove that $G^{\mathbb{C}}$ satisfies a universal property (which is more general than [8, Proposition III.8.6] which concerns only linear representations).

Proposition 3.9. *Let G be a compact Lie group, $G^{\mathbb{C}}$ its complexification, and $\phi \colon G \to G^{\mathbb{C}}$ the natural morphism. For any complex Lie group H and any morphism $f \colon G \to H$ of Lie groups, there exists a unique homomorphism $F : G^{\mathbb{C}} \to H$ of complex Lie groups such that $f = F \circ \phi$.*



Proof. We use the notation of proposition 3.3. Let σ denote the differential of f at the origin and let $\sigma^{\mathbb{C}} \colon \mathfrak{g}^{\mathbb{C}} \to \mathfrak{h}$ be its complexification. We define a map $F \colon G^{\mathbb{C}} \to H$ by $F(ge^{iZ}) = f(g)e^{i\sigma(Z)}$.

First we claim that F is holomorphic. By Lie's third theorem (see [37, §II.8, Thm 1]), the morphism $\sigma^{\mathbb{C}}$ can be uniquely integrated to a local holomorphic group morphism $\widetilde{F} \colon G^{\mathbb{C}} \dashrightarrow H$ defined in a neighborhood of the identity, whose differential at the identity element is $\sigma^{\mathbb{C}}$. Hence, $\widetilde{F}_{|G}$ integrates σ, so $\widetilde{F}_{|G} = f$. Hence, for g close to the identity and Z close to 0, we get

$$\widetilde{F}(ge^{iZ}) = \widetilde{F}(g)\widetilde{F}(e^{iZ}) = f(g)e^{\sigma^{\mathbb{C}}(iZ)} = f(g)e^{i\sigma(Z)} = F(ge^{iZ}),$$

so $F = \widetilde{F}$ near the identity. It follows that F is holomorphic near the identity. Since F is real analytic, F is holomorphic on $(G^{\mathbb{C}})^e$. Since for g in G and ℓ in $G^{\mathbb{C}}$, $F(g\ell) = f(g)F(\ell)$, it follows that F is holomorphic on $G^{\mathbb{C}}$. Lastly, $F_{|G} = f$ is a group morphism on G, so by corollary 3.7, applied two times, F is a group morphism. □

3.2. **Complexification of a real representation.** Let $\rho : G \to \mathrm{GL}(V)$ be a real continuous representation of a compact Lie group G. By proposition 3.9, there exists a unique analytic extension

$$\rho^{\mathbb{C}} : G^{\mathbb{C}} \to \mathrm{GL}(V^{\mathbb{C}})$$

of ρ. This complexified representation enjoys some very specific properties that we will list below.

Proposition 3.10. *If we see $G^{\mathbb{C}}$ as an affine algebraic group (via proposition 3.6), then $\rho^{\mathbb{C}}$ is an algebraic map.*

Proof. This is proved in [8, Proposition III.8.6]. □

Remark 3.11. The link between compacity and algebraicity is a beautiful result that relies on the Peter-Weyl theorem and on the finiteness of the algebra of representative functions. Outside of this context, such results become immediately wrong. For instance, $\mathbb{C}$ and $\mathbb{C}^\times$ are both complex affine algebraic groups, but $\exp\colon \mathbb{C} \to \mathbb{C}^\times$ is a holomorphic Lie group morphism that is no longer algebraic. This comes from the fact that $\mathbb{C}$ is not reductive.

Proposition 3.12. *Given (G, V), there exists a $G^{\mathbb{C}}$-stable and nonempty Zariski open subset U of $V^{\mathbb{C}}$ such that all orbits of points in U are closed.*

Proof. Since G is compact, we can endow V with an inner product. Hence we have a chain of maps

$$G \xrightarrow{\rho} O(n) \subset \mathrm{U}(n) \subset GL(n;\mathbb{C})$$

Thanks to proposition 3.3, it is possible to check that $O(n)^{\mathbb{C}} = O(n;\mathbb{C})$ so we get that $\rho^{\mathbb{C}}$ takes values in $O(n;\mathbb{C})$. Hence the action preserves a symmetric nondegenerate complex bilinear form, and the result follows from [27]. □

Lemma 3.13. *For any v in V, $(G^{\mathbb{C}})_v = (G_v)^{\mathbb{C}}$.*

Proof. By corollary 3.7, G_v is Zariski dense in $(G_v)^{\mathbb{C}}$ so $(G_v)^{\mathbb{C}}$ fixes v for the complexified representation. This yields the inclusion $(G_v)^{\mathbb{C}} \subset (G^{\mathbb{C}})_v$.

For the converse implication, we argue as in the proof of proposition 3.12. Let us denote by $d\rho$ the differential of the G-action at the origin. Let $\ell = ge^{iZ}$ be an element of $(G^{\mathbb{C}})_v$. Then $\iota(\ell)$ belongs to $(G^{\mathbb{C}})_v$ too, so $e^{2iZ} = \iota(\ell)^{-1}\ell$ belongs to $(G^{\mathbb{C}})_v$. It means that $\exp(2i\, d\rho(Z)) \cdot v = v$. Now $2i\, d\rho(Z)$ is a complex hermitian endomorphism of V, so it is diagonalizable with real eigenvalues. It follows that $d\rho(Z)(\ell) = 0$, so Z belongs to $\mathrm{Lie}(G_v)$. Hence, e^{iZ} fixes v so g fixes v too. It follows that g belongs to G_v, which proves that ℓ belongs to $(G_v)^{\mathbb{C}}$. □

Proposition 3.14. *For any v in V, $G^{\mathbb{C}} \cdot v$ is closed.*

Proof. It follows directly from [11, Lemma 2.2] since G is compact. □

Remark 3.15. For the reader's convenience, we can provide a sketch of the proof of the above proposition. It relies heavily on the beautiful theory of Kempf and Ness [22]. Let us fix an inner product on V such that G acts by orthogonal transformations, and extend it to an hermitian product on $V^{\mathbb{C}}$. The moment map $\mu\colon V^{\mathbb{C}} \to \mathfrak{g}^*$ is given by

$$\mu(v)(\xi) = \frac{1}{2i}\langle \xi \cdot v | v \rangle$$

This definition makes sense because ξ belongs to $\mathfrak{u}(V^{\mathbb{C}})$ so it is skew-hermitian. Then Kempf-Ness theorem states that an orbit $G^{\mathbb{C}} \cdot v$ is closed if and only $G^{\mathbb{C}} \cdot v \cap \mu^{-1}(0) \neq \emptyset$ (see for instance [26, Theorem 4]). However, since G acts by orthogonal transformations, the moment map μ vanishes on V. The result follows.

Proposition 3.16. *For any isotropy H, the complex open fixed locus in $(V^{\mathbb{C}})^{H^{\mathbb{C}}}$ contains a nonempty Zariski open set.*

Proof. Let v be a vector in V such that $G_v = H$. Thanks to proposition 3.14, $G^{\mathbb{C}} \cdot v$ is closed, and thanks to corollary 2.20, there exists a Zariski open set U containing x such that for any w in U, G_w is conjugate to a subgroup of G_v. Let us now look at $\widetilde{U} = U \cap (V^{\mathbb{C}})^{H^{\mathbb{C}}}$, which is a nonempty Zariski open subset of $(V^{\mathbb{C}})^{H^{\mathbb{C}}}$. Then for w in $\widetilde{U}$, $(G^{\mathbb{C}})_w$ is conjugate to a subgroup of $H^{\mathbb{C}}$ so there exists g in $G^{\mathbb{C}}$ such that

$$H^{\mathbb{C}} \subset (G^{\mathbb{C}})_w \subset g^{-1} H^{\mathbb{C}} g.$$

It follows from lemma A.3 that g normalizes $H^{\mathbb{C}}$, which implies that $(G^{\mathbb{C}})_w = H^{\mathbb{C}}$. □

Corollary 3.17. *For any isotropy H, the open fixed locus $\overset{\circ}{V}{}^{H}$ contains a nonempty (real) Zariski open set.*

Proof. Thanks to corollary 3.8 we have $(V^H)^{\mathbb{C}} = (V^{\mathbb{C}})^H = (V^{\mathbb{C}})^{H^{\mathbb{C}}}$. Then the result follows from proposition 3.16 and corollary A.7. □

We can also compare real and complex orbits. This result won't be strictly necessary in the sequel but it can help to understand the geometry of the situation.

Proposition 3.18. *For any vector v in V, $(G^{\mathbb{C}} \cdot v) \cap V = G \cdot v$.*

Proof. One inclusion is obvious. For the other inclusion, let H be the isotropy of v. Thanks to lemma 3.13, the orbit $G^{\mathbb{C}} \cdot v$ is isomorphic to $G^{\mathbb{C}}/H^{\mathbb{C}}$. Let ι denotes the Cartan involution on $G^{\mathbb{C}}$. Assume that $\ell \cdot v$ is real for some element ℓ in $G^{\mathbb{C}}$. Then $\iota(\ell) \cdot v = \overline{\ell \cdot v} = v$ so $\iota(\ell)^{-1}\ell$ belongs to $H^{\mathbb{C}}$. If $\ell = ge^{iZ}$, then e^{2iZ} belongs to $H^{\mathbb{C}}$. It implies that Z is in $\mathfrak{h}$, so e^{iZ} is in $H^{\mathbb{C}}$. Hence $\ell \cdot v = g \cdot v$. □

Remark 3.19. The result [5, Proposition 2.3] predicts in our case that the intersection of a complex orbit with real points is a *finite union* of real orbits. Hence this result is weaker, but holds for more general groups.

Example 3.20. Let us give an example where this intersection is strictly bigger than the real orbit. For this we take again Example 2.18, but with the action of $\mathrm{GL}(2;\mathbb{C})$ instead of $\mathrm{SL}(2;\mathbb{C})$ acting on the vector space of degree 3 binary forms. Then the $\mathrm{GL}(2,\mathbb{R})$-orbit of $z(w-z)w$ consists of all binary forms of degree 3 with real coefficients and 3 distinct roots. However, real binary forms that are in the $\mathrm{GL}(2,\mathbb{C})$-orbit of $z(w-z)w$ consist of real binary forms with 3 distinct roots (not necessarily real). This locus is a union of two $\mathrm{GL}(2;\mathbb{R})$ orbits: the orbit of $z(w-z)w$ and the orbit of $z(z^2+w^2)$. There is no contradiction with the result we proved, because $\mathrm{GL}(2,\mathbb{R})$ is not compact. As a matter of fact, $\mathrm{GL}(2,\mathbb{R})$ is the split real form of $\mathrm{GL}(2,\mathbb{C})$ and not the compact one, which is $\mathrm{U}(2)$.

Finally, we go back to the algebra of invariants.

Lemma 3.21. *The three complex algebras $(\mathbb{R}[V]^G)^{\mathbb{C}}$, $\mathbb{C}[V^{\mathbb{C}}]^G$ and $\mathbb{C}[V^{\mathbb{C}}]^{G^{\mathbb{C}}}$ are naturally isomorphic.*

Proof. We have $\mathbb{R}[V]^{\mathbb{C}} \simeq \mathbb{C}[V^{\mathbb{C}}]$. Then the result follows from corollary 3.8. □

Corollary 3.22. *If $\{J_1, \dots, J_N\}$ is a set of generators for $\mathbb{R}[V]^G$, then, it is also a generating set for $\mathbb{C}[V^{\mathbb{C}}]^{G^{\mathbb{C}}}$.*

3.3. **Complexification of normalizers.** The aim of this section is to prove that the complexification of the normalizer of a compact subgroup of a compact Lie group is naturally isomorphic to the normalizer of the corresponding complexified subgroup (proposition 3.26). This result is in accordance with Luna's result [30, Lemma 1.1]. In the following three lemmas, G denotes a compact Lie group acting on a vector space V and H is a Lie subgroup of G. We denote by $G^{\mathbb{C}}$ and $H^{\mathbb{C}}$ their respective complexifications.

Lemma 3.23. *Let H be a closed subgroup of a Lie group G and let Γ be a set of representatives in H of the finite group H/H^e, where H^e is the identity component of H. Then,*

$$\mathrm{Lie}(N(H)) = N(\mathfrak{h}) \cap \{X \in \mathfrak{g},\ \forall \gamma \in \Gamma,\ \mathrm{Ad}(\gamma)(X) - X \in \mathfrak{h}\}.$$

Proof. Let $X \in \mathrm{Lie}(N(H))$ then for all real t, e^{tX} is in $N(H)$. Let $Y \in \mathfrak{h}$, then e^{sY} is in H and therefore $\mathrm{Ad}(e^{tX})(e^{sY})$ is in H. Taking the derivatives with respect to s and then to t, we get $\mathrm{Ad}(X)(Y) \in \mathfrak{h}$ which implies that $X \in N(\mathfrak{h})$. Now, let γ be in Γ. Then $\mathrm{Ad}(e^{tX})(\gamma)$ belongs to the connected component $H^e\gamma$ of H containing γ. Consequently, $e^{tX}\gamma e^{-tX}\gamma^{-1} \in H$. Deriving at $t = 0$ we get that $X - \mathrm{Ad}(\gamma)(X)$ is in $\mathfrak{h}$.

Conversely, let $X \in N(\mathfrak{h}) \cap \{X \in \mathfrak{g},\ \forall \gamma \in \Gamma,\ \mathrm{Ad}(\gamma)(X) - X \in \mathfrak{h}\}$, and let Y be in $\mathfrak{h}$. Then,

$$e^{tX}e^{sY}e^{-tX} = \mathrm{Ad}(e^{tX})(e^{sY}) = e^{s\,\mathrm{Ad}(e^{tX})(Y)} = e^{s\exp(t\mathrm{ad}(X))(Y)} \in H.$$

It follows that $e^{tX}H^e e^{-tX} \subset H^e$. We must now deal with the other connected components. Let γ be in Γ. Then

$$e^{tX}\gamma e^{-tX}\gamma^{-1} = e^{tX}\,\mathrm{Ad}(\gamma)(e^{-tX}) = e^{tX}e^{-\mathrm{Ad}(\gamma)(tX)} = e^{tX}e^{t(X-\mathrm{Ad}(\gamma)(X))-tX}.$$

Set $Y = X - \mathrm{Ad}(\gamma)(X)$, then, by assumption, $Y \in \mathfrak{h}$ and $tY - tX = -\mathrm{Ad}(\gamma)(X) \in \mathfrak{h}$. By applying Baker-Campbell-Hausdorff formula (lemma A.1) on $e^{tX}\gamma e^{-tX}\gamma^{-1} = e^{tX}e^{tY-tX}$, we get that $e^{tX}\gamma e^{-tX}\gamma^{-1} = e^{tZ}$ where Z is an iteration of Lie brackets in terms of X and Y hence $Z \in \mathfrak{h}$. Therefore, we have

$$e^{tX}(\gamma e^{sY})e^{-tX} = \underbrace{(e^{tX}\gamma e^{-tX})\gamma^{-1}}_{\in H}\,\gamma\,\underbrace{(e^{tX}e^{sY}e^{-tX})}_{\in H^e} \in H.$$

Consequently, e^{tX} normalizes H, which implies that X belongs to $\mathrm{Lie}(N(H))$. □

Lemma 3.24. *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{u}(n)$, $Z \in \mathfrak{h}$, $B \in N(\mathfrak{h})$ and $A \in \mathfrak{u}(n)$.*

$$\text{If } e^{2iA} = e^{iB}e^{iZ}e^{iB} \quad \text{then} \quad A - B \in \mathfrak{h}.$$

Proof. Consider the following real analytic function

$$f\colon \mathfrak{u}(n) \times N(\mathfrak{h}) \to \mathfrak{u}(n), \qquad (Z, B) \mapsto \frac{1}{i}\ln(e^{iB}e^{iZ}e^{iB}) - 2B.$$

Note that f is well-defined because $e^{iB}e^{iZ}e^{iB}$ is an element of $\mathrm{P}(n)$, thanks to lemma 3.2 (2). Indeed,

$$\iota(e^{iB}e^{iZ}e^{iB}) = e^{-iB}e^{-iZ}e^{-iB} = (e^{iB}e^{iZ}e^{iB})^{-1}.$$

Then, we can take its logarithm in $\mathfrak{u}(n)$ thanks to lemma 3.2 (3). Next, we apply Baker-Campbell-Hausdorff formula (lemma A.1) two times on $e^{iB}e^{iZ}e^{iB}$ and we deduce that $f(B, Z)$ is in $\mathfrak{h}$ for B and Z sufficiently close to zero. Thanks to corollary A.5, this is valid everywhere since f is real analytic. Hence $A - B$ belongs to $\mathfrak{h}$. □

Finally, we need the following classical result.

Lemma 3.25. *Let u be a diagonalizable endomorphism of a vector space V with real eigenvalues. If $W \subset V$ is stable by $\exp(u)$, then W is stable by u.*

Proof. Let $\lambda_1, \dots, \lambda_k$ be the distinct eigenvalues of u, and let P be a polynomial such that for any i, $P(e^{\lambda_i}) = \lambda_i$. Then $P(\exp(u)) = u$, since this equality can be checked on a basis of eigenvectors of u. The result follows directly. □

Proposition 3.26. *We have*

$$N_{G^{\mathbb{C}}}(H^{\mathbb{C}}) = (N_G(H))^{\mathbb{C}}.$$

Proof. If g is in $N_G(H)$, then, $gHg^{-1} \subset H \subset H^{\mathbb{C}}$. Consider the holomorphic application

$$C_g\colon H^{\mathbb{C}} \to G^{\mathbb{C}}/H^{\mathbb{C}}, \qquad h \mapsto ghg^{-1}.$$

Then, C_g vanishes on H, and thanks to corollary 3.7, C_g vanishes on $H^{\mathbb{C}}$. The same argument applies for g^{-1} so g normalizes $H^{\mathbb{C}}$ in $G^{\mathbb{C}}$. Hence $N_G(H) \subset N_{G^{\mathbb{C}}}(H^{\mathbb{C}})$ and by theorem 3.3 we get

$$(N_G(H))^{\mathbb{C}} \subset N_{G^{\mathbb{C}}}(H^{\mathbb{C}}).$$

Let us prove the reverse inclusion. Let g be in $N_{G^{\mathbb{C}}}(H^{\mathbb{C}})$. By theorem 3.3, we can write $g = ke^{iX}$ with k in G and X in $\mathfrak{g}$. We need to prove that k (resp. X) belongs to $N(H)$ (resp. $\mathrm{Lie}(N(H))$. We have $\iota(g) = ke^{-iX}$. To prove that $X \in \mathrm{Lie}(N(H))$, we use the description of $\mathrm{Lie}(N(H))$ given in lemma 3.23.

On one hand, thanks to lemma 3.2 (2), we have $e^{2iX} = \iota(g)^{-1}g$. Besides, the normalizer of $H^{\mathbb{C}}$ is stable under the Cartan involution ι, and hence e^{2iX} normalizes $H^{\mathbb{C}}$. This implies that $\mathrm{Ad}(e^{2iX})(\mathfrak{h}^{\mathbb{C}}) = \mathfrak{h}^{\mathbb{C}}$, where $\mathfrak{h}$ is the Lie algebra of H. Now $\mathrm{Ad}(e^{2iX}) = \exp(2i\mathrm{ad}(X))$. Since X is in $\mathrm{u}(n)$, $2i\mathrm{ad}(X)$ is diagonalizable with real eigenvalues. Thanks to lemma 3.25, $\mathrm{ad}(X)(\mathfrak{h}^{\mathbb{C}}) \subset \mathfrak{h}^{\mathbb{C}}$. Consequently, $X \in N(\mathfrak{h}^{\mathbb{C}})$. Since X is in $\mathfrak{u}(n)$, X belongs to $N(\mathfrak{h})$.

On the other hand, let γ in Γ. Then $e^{2iX}\gamma e^{-2iX} \in H^{\mathbb{C}}$ since $e^{2iX} \in N(H^{\mathbb{C}})$. Now we calculate $\iota(e^{2iX}\gamma e^{-2iX})^{-1}(e^{2iX}\gamma e^{-2iX})$ which on one side gives e^{2iZ} with $Z \in \mathfrak{h}$ and on the other side, we have

$$\begin{aligned}\iota(e^{2iX}\gamma e^{-2iX})^{-1}(e^{2iX}\gamma e^{-2iX}) &= (e^{-2iX}\gamma e^{2iX})^{-1}(e^{2iX}\gamma e^{-2iX}) \\ &= e^{-2iX}\,\mathrm{Ad}(\gamma^{-1})(e^{4iX})e^{-2iX} \\ &= e^{-2iX}e^{4i\,\mathrm{Ad}(\gamma^{-1})(X)}e^{-2iX}.\end{aligned}$$

Therefore, combining the two equalities together, we get

$$e^{2iZ} = e^{-2iX}e^{4i\,\mathrm{Ad}(\gamma^{-1})(X)}e^{-2iX}$$

that is

$$e^{4i\,\mathrm{Ad}(\gamma^{-1})(X)} = e^{2iX}e^{2iZ}e^{2iX}.$$

Applying lemma 3.24 with $A = 2\,\mathrm{Ad}(\gamma^{-1})(X) \in \mathfrak{u}(n)$ and $B = 2X \in N(\mathfrak{h})$, we have finally $\mathrm{Ad}(\gamma^{-1})(X) - X \in \mathfrak{h}$, so $X - \mathrm{Ad}(\gamma)(X) \in \mathfrak{h}$. This proves that X belongs to $\mathrm{Lie}(N(H))$.

Now, both e^{iX} and ke^{iX} normalize $H^{\mathbb{C}}$ and so does $k \in N(H^{\mathbb{C}})$. We have $kH^{\mathbb{C}}k^{-1} = H^{\mathbb{C}}$ so

$$H = H^{\mathbb{C}} \cap \mathrm{U}(n) = kH^{\mathbb{C}}k^{-1} \cap \mathrm{U}(n) = k(H^{\mathbb{C}} \cap \mathrm{U}(n))k^{-1} = kHk^{-1}$$

so k belongs to $N_G(H)$. □

4. Proof of the main theorem

The main idea to prove theorem 1.1 is issued from the following erroneous proof. Let $\{J_1, \dots, J_N\}$ be a generating set for the invariant algebra $\mathbb{R}[V]^G$. Then this set is a separating set for (V, G) (see the discussion after remark 2.14). For $1 \leq k \leq N$, let j_k denote the restriction of J_k to V^H. The inclusion $V^H \hookrightarrow V$ induces the morphism

$$\mathbb{R}[V]^G \to \mathbb{R}[V^H]^{N(H)}, \qquad \{J_1, \dots, J_N\} \mapsto \{j_1, \dots, j_N\}.$$

Then, the set $\{j_1, \dots, j_N\}$ is separating for $(V^H, N(H))$. If theorem 2.15 was true for a real representation, we could conclude that F is a generating set for the invariant field $\mathbb{R}(V^H)^{N(H)}$ and this would achieve the proof. Unfortunately this is false. However, this argument can me modified to become a real proof.

Proof of theorem 1.1. Let $\{J_1, \dots, J_N\}$ be a generating set for the invariant algebra $\mathbb{R}[V]^G$. Then $\{J_1, \dots, J_N\}$ separates the orbits of (V, G). Let j_k be the restriction of J_k on V^H, then, the j_k separate the orbits of $V^H \cap \Sigma_{[H]}$.

Let $W = (V^{\mathbb{C}})^{H^{\mathbb{C}}}$. Thanks to proposition 3.16, there exists a nonempty Zariski open set U_1 of W such that for any w in U_1, $G_w^{\mathbb{C}} = H^{\mathbb{C}}$. Let $K = N_G(H)$. The group K acts on W, and thanks to proposition 3.12, there exists a nonempty Zariski open subset U_2 of W such that all $K^{\mathbb{C}}$-orbits of elements of U_2 are closed in W. Let $U = U_1 \cap U_2$, then, U is also a nonempty Zariski open set in W.

We claim that $\{j_1, \dots, j_N\}$ separates $K^{\mathbb{C}}$-orbits of elements in U. To prove this, let w_1 and w_2 be two elements in U that are not in the same $K^{\mathbb{C}}$-orbit. Assume that w_2 and w_1 are in the same $G^{\mathbb{C}}$-orbit. If $gw_1 = w_2$ for some g in $G^{\mathbb{C}}$, then $H^{\mathbb{C}} = gH^{\mathbb{C}}g^{-1}$ and g belongs to $N_{G^{\mathbb{C}}}(H^{\mathbb{C}})$. Thanks to proposition 3.26, g is in $K^{\mathbb{C}}$, which gives a contradiction. Hence $G^{\mathbb{C}}w_1 \cap G^{\mathbb{C}}w_2 = \emptyset$.

The $K^{\mathbb{C}}$-orbits of w_1 and w_2 are closed because w_1 and w_2 belong to U_2. Besides, thanks to proposition 3.26, $K^{\mathbb{C}} = N_{G^{\mathbb{C}}}(H^{\mathbb{C}})$. It follows from [29, Corollary 1] that the $G^{\mathbb{C}}$-orbits of w_1 and w_2 are also closed. Since these orbitsts are constructible, they are Zariski closed (and also $G^{\mathbb{C}}$-stable). Thanks to corollary 3.22, they can be separated by the J_k. This proves our claim.

By Popov & Vinberg's theorem 2.15, the j_k generate the field $(\mathbb{C}(W))^{K^{\mathbb{C}}}$. It follows that the map

$$(\mathbb{C}(V^{\mathbb{C}}))^{G^{\mathbb{C}}} \dashrightarrow (\mathbb{C}(W))^{K^{\mathbb{C}}}$$

is surjective. It remains to go back to real fields of invariants. Thanks to corollary 3.8, we have a diagram

$$\begin{array}{ccc} (\mathbb{C}(V^{\mathbb{C}}))^{G^{\mathbb{C}}} & \dashrightarrow & (\mathbb{C}(W))^{K^{\mathbb{C}}} \\ \| & & \| \\ (\mathbb{C}(V^{\mathbb{C}}))^{G} & \dashrightarrow & (\mathbb{C}(W))^{K} \end{array}$$

Moreover, $\mathbb{C}(V^{\mathbb{C}}) = \mathbb{R}(V) \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C}(W) = \mathbb{C}((V^H)^{\mathbb{C}}) = \mathbb{R}(V^H) \otimes_{\mathbb{R}} \mathbb{C}$. Hence the map

$$(\mathbb{R}(V))^{G} \dashrightarrow (\mathbb{R}(V^H))^{N_G(H)}$$

is also surjective because every real rational invariant is in the image of a complex rational invariant, and therefore it is also the image of its real part.

We now prove the algebraicity of strata. Thanks to proposition 3.16, there exists a Zariski subset U in $(V^{\mathbb{C}})^{H^{\mathbb{C}}}$ which is contained in the open fixed locus. Let X denotes the $G^{\mathbb{C}}$ orbit of U in $V^{\mathbb{C}}$, then by Chevalley's theorem X is constructible in $V^{\mathbb{C}}$, so its topological closure is the same as its Zariski closure.

Let us first prove that $X \cap V$ lies in Σ_H. If v is in $X \cap V$, then there exists ℓ in $G^{\mathbb{C}}$ such that $\ell \cdot v$ belongs to U. This implies $G^{\mathbb{C}}_{\ell.v} = H^{\mathbb{C}}$, and therefore $\ell\, G^{\mathbb{C}}_v\, \ell^{-1} = H^{\mathbb{C}}$. Since $G^{\mathbb{C}}_v$ and $H^{\mathbb{C}}$ are stable under the Cartan involution ι, we also have $\iota(\ell)\, G^{\mathbb{C}}_v\, \iota(\ell)^{-1} = H^{\mathbb{C}}$. It follows that $\iota(\ell)^{-1}.\ell$ normalizes $G^{\mathbb{C}}_v$. If $\ell = ge^{iZ}$, then $\iota(\ell)^{-1}.\ell = e^{2iZ}$ and thanks to proposition 3.26, Z is in the Lie algebra of $N_G(G_v)$ so e^{iZ} normalizes $G^{\mathbb{C}}_v$. It follows that

$$H^{\mathbb{C}} = ge^{iZ} G^{\mathbb{C}}_v e^{-iZ} g^{-1} = g\, G^{\mathbb{C}}_v\, g^{-1}.$$

Taking the fixed points under the Cartan involution on both sides, $H = g\, G_v\, g^{-1}$, which implies that v lies in the G-orbit of $\overset{\circ}{V}{}^H$.

Let us now prove that $\overline{X \cap V} = \overline{\Sigma}_{[H]}$. The inclusion follows by taking the closure of the inclusion $X \cap V \subset \Sigma_H$. For the other inclusion, recall that U is a Zariski open subset of W and that X is the $G^{\mathbb{C}}$-orbit or U. We have thus $U \cap V^H \subset X \cap V$ and thanks to corollary A.8, $U \cap V^H$ is a nonempty Zariski subset of V^H. Hence, $V^H \subset \overline{X \cap V}$. Since the right hand side is G-stable, we get $\overline{\Sigma}_{[H]} \subset \overline{X \cap V}$.

We can finish the proof as follows: X is Zariski open in $\overline{X}$, so $X \cap V$ is Zariski open in the algebraic set $\overline{X} \cap V$. Therefore, $\overline{X \cap V}$ is also an algebraic set, which is the union of a finite number of irreducible components of $\overline{X} \cap V$. □

Remark 4.1.

- In the first part of the proof, we made crucial use of Luna's criterion for closedness of orbits [29, Corollary 1]. For an easy proof of this theorem using Kempf-Ness theory, see [26].
- The natural candidate for $\overline{\Sigma}_{[H]}$ should be $\overline{X} \cap V$. However, we don't know if $\overline{X} \cap V$ is irreducible in full generality. Hence the best result we could get is that $\overline{\Sigma}_{[H]}$ is one of its irreducible components.

Appendix A. Lie groups and real analytic structures

In this appendix, we recall a few results we need about Lie groups, and add some folklore results on real analytic functions and real analytic manifolds.

We now focus on Lie groups, and recall the following classic formula.

Lemma A.1 (Baker-Campbell-Hausdorff formula)**.** [12, §3.4] *Let G be Lie group and $\mathfrak{g}$ its Lie algebra. Consider $X, Y \in \mathfrak{g}$. Then the solution $Z \in \mathfrak{g}$ of $e^X e^Y = e^Z$ is a formal series in iterated commutators of X and Y*

$$Z = X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}[X,[X,Y]] - \frac{1}{12}[Y,[X,Y]] + \cdots,$$

where "$\cdots$" indicates terms involving higher commutators of X and Y. If X and Y are sufficiently small elements of the Lie algebra $\mathfrak{g}$, the series is convergent.

This formula implies that the multiplication of G in exponential coordinates near the neutral element is real analytic. From this, one deduces immediately the following result.

Corollary A.2. *Let G be a Lie group. Then G has a natural real analytic structure such that:*

- *The multiplication map is globally real-analytic*
- *For any g in G the map $L_g \circ \exp$ in a neighborhood of 0 defines a local real analytic chart near g.*

Besides, if H is a Lie subgroup of G, then H is a real analytic submanifold of G. More generally, any continuous morphism between compact Lie groups is real analytic.

Lastly, let us state the following lemma, which is the differentiable version of [38, Proposition 3.7].

Lemma A.3. *Let L be a Lie group with a finite number of connected components, K be a closed subgroup of L, and assume that there exists ℓ in L such that $\ell K \ell^{-1} \subset K$. Then, ℓ normalizes K.*

Proof. The automorphism $\mathrm{Ad}(\ell)$ stabilizes $\mathfrak{k}$, so it is an automorphism of $\mathfrak{k}$. Therefore $\mathrm{Ad}(\ell^{-1})$ stabilizes $\mathfrak{k}$ so $\ell^{-1} K^e \ell \subset K^e$.

Let K/K^e be the set of connected components of K. Then K/K^e is a finite group and $\mathrm{Ad}(\ell)$ induces an action on K/K^e. Let k be a representative of an element of K/K^e which is in the kernel of $\mathrm{Ad}(\ell)$. Then $\ell k \ell^{-1}$ belongs to K^e so $k = \ell^{-1}(\ell\gamma\ell^{-1})\ell \in K^e$, which means that the class of k is 0 in K/K^e. Hence $\mathrm{Ad}(\ell)$ is injective, and since K/K^e is finite, $\mathrm{Ad}(\ell)$ is a bijection. This implies that for any k in K there exists an element of the form $\ell k' \ell^{-1}$ in the connected component of k. This means that there exists k'' in K^e such that $\ell k' \ell^{-1} = k'' k$ so we get $\ell^{-1} k \ell = \ell^{-1} \left((k'')^{-1} \ell k' \ell^{-1}\right) \ell = \left(\ell^{-1}(k'')^{-1}\ell\right) k' \in K$. □

We now recall a folklore results on propagation of local real analytic identities.

Lemma A.4. *Let n be a positive integer, let V be an open and connected subset of $\mathbb{R}^n$, and $f : V \to \mathbb{R}$ a real analytic function. If there exists $v \in V$ such that $f = 0$ on a neighborhood of v then $f = 0$ on V.*

Proof. Let S be the set of points x in U such that all partial derivatives of f vanish at x. This set is closed, but since f is real analytic it is also open. Since $v \in S$, S is nonempty so $S = V$. □

Corollary A.5. *Let n be a positive integer, and assume to be given the following data:*

- *V is an open and connected subset of $\mathbb{R}^n$,*
- *Z is a real analytic manifold,*
- *Y is a (closed) real-analytic submanifold of Z,*
- *$f : V \to Z$ is an analytic function.*

Assume that there exists a point v in V and a neighborhood U_v of v in V such that $f(U_v) \subset Y$. Then $f(V) \subset Y$.

Proof. Let W be the subset of points x in V such that there exists a neighborhood U_x of x in V such that $f(U_x) \subset Y$. By definition, W is open. Let x be a point in $\overline{W} \cap V$. As Y is closed, $f(x) \in Y$. Let $\varphi\colon U_y \to \mathbb{R}^m$ be a real analytic chart on a neighborhood U_y of y in Z, such that $U_y \cap Y$ is given by the equations $\varphi_1 = \cdots = \varphi_r = 0$, and let U_x be a connected neighborhood of x in V such that $f(U_x) \subset U_y$. Then, for $1 \leq i$, $\varphi_i \circ f$ is a real analytic function on U_x that vanishes on an open subset of U_x (since $x \in \overline{W}$). By lemma A.4, for $1 \leq i \leq r$, $\varphi_i \circ f$ vanishes in U_x, which means that $f(U_x) \subset Y$, so $x \in W$. It follows that $\overline{W} \cap V = W$, which means that W is closed in V. Since V is connected, $W = V$ and in particular $f(V) \subset Y$. □

We end this appendix with some classical results on totally real submanifolds. Let X be a complex manifold. A real submanifold S of X is *totally real* if for any s in S, $\mathrm{T}_s S$ is a totally real subspace of $\mathrm{T}_s X$, which means that $\mathrm{T}_s S \cap i\mathrm{T}_s S = \{0\}$.

Lemma A.6. *Assume that S is a totally real submanifold of a complex manifold that is moreover real analytic. Then, around each point s of S, there exists an holomorphic chart $\varphi\colon U_s \to \mathbb{C}^n$, where U_s is a neighborhood of s, such that $\varphi(U_s \cap S) = \varphi(U_s) \cap \mathbb{R}^k$.*

Proof. Let $f\colon \mathbb{R}^k \to S$ be a real analytic parametrization of S defined in a neighborhood of 0 such that $f(0) = s$. We can complexify f to a map $F \to \mathbb{C}^k$. Now, for each point z in $\mathbb{C}^k$ near 0, $dF_0(u + iv) = df_0(u) + idf_0(v)$. Since $T_s S$ is totally real, we see that dF_0 is injective and has rank k. It follows that F parameterizes a complex manifold of X containing S. The result follows. □

Corollary A.7. *Let X be a complex manifold and S be a totally real and real analytic submanifold of X of maximal dimension. Then the analytic Zariski closure of S is the union of the connected components of X which intersect S.*

Proof. It is enough to deal with the case where X is connected. Let Z be a complex analytic set containing S. For any point s in S, let Z' be an irreducible component of Z containing s and let us write locally $Z' = \bigcap_{i=1}^r \{f_i = 0\}$ around s. According to lemma A.6, we can write locally the inclusion of S in Z around s as the inclusion of $\mathbb{R}^N$ in $\mathbb{C}^N$ near 0. Then, $f_{i|\mathbb{R}^N} = 0$. Since for any multi-index $I = \{i_1, \ldots, i_N\}$ we have

$$\frac{\partial^I f}{\partial z_I}(0) = \frac{\partial^I f}{\partial x_I}(0) = 0,$$

we see that $f_i = 0$, so Z' contains a neighborhood of s in X. Hence $Z' = X$. □

Corollary A.8. *Let V be a real vector space of finite dimension. If U is a nonempty Zariski open set in $V^{\mathbb{C}}$, then $U \cap V$ is a nonempty Zariski open set in V.*

Proof. Let Z be the complement of U in $V^{\mathbb{C}}$. Then Z is a closed analytic subset of $V^{\mathbb{C}}$. Assume that $U \cap V = \emptyset$. Then Z contains V, so it contains its Zariski closure. Thanks to corollary A.7, $Z = V^{\mathbb{C}}$ so $U = \emptyset$ and we get a contradiction. Hence $U \cap V \neq \emptyset$ and the result follows. □

Appendix B. Proof of the principal orbit theorem in the linear case

In this appendix, we provide for the reader's convenience a proof of theorem 2.6 for the linear case, which combines the ideas of the proofs presented in [10] and [38].

We first need a topological lemma, which allows to propagate connexity.

Lemma B.1. *Let X be a connected topological space, U a dense subset of X, and assume than for any point x in X, there exists a neighborhood U_x of x such that $U \cap U_x$ is connected. Then U is connected.*

Proof. Assume that U can we written as the disjoint union of two nonempty open sets U_1 and U_2. Since U is dense, $X = \overline{U}_1 \cup \overline{U}_2$ and since X is connected, $\overline{U}_1 \cap \overline{U}_2 \neq \emptyset$. Let x in $\overline{U}_1 \cap \overline{U}_2$. Let U_x be a neighborhood of x such that $U \cap U_x$ is connected. Then $U \cap U_x$ is the disjoint union of $U_1 \cap U_x$ and $U_2 \cap U_x$. However these two open sets are non empty, and we get a contradiction. Hence U is connected. □

Remark B.2. This lemma can be better visualized in term of sheaf theory, let $j\colon U \hookrightarrow X$ denote the inclusion, and let us consider the morphism of sheaves $\Delta\colon \mathbb{Z}_X \to j_*\mathbb{Z}_U$. Then the two hypotheses can be translated as follows:

- U is dense in X if and only if Δ is a monomorphism.
- The second property is granted if and only if there exists a covering $(U_i)_{i\in I}$ of U such that Δ_{U_i} is an isomorphism.

We claim that these two properties imply that Δ_X is also an isomorphism. Indeed, the injectivity is straightforward and for the surjectivity, we argue as follows: if s is a global section of $j_*\mathbb{Z}_U$ on X, then for all i, we can write $s_{|U_i} = \Delta_{U_i}(t_i)$, and therefore $\Delta_{U_i\cap U_j}(t_i - t_j) = 0$. Since Δ is a monomorphism, the $(t_i)'s$ glue together to a global section t of $\mathbb{Z}_X$ such that $\Delta_X(t) = s$.

To conclude, we write

$$\mathrm{H}^0(U, \mathbb{Z}_U) = \mathrm{H}^0(X, j_*\mathbb{Z}_U) \simeq \mathrm{H}^0(X, \mathbb{Z}_X) \simeq \mathbb{Z},$$

and therefore U is connected.

Now we explain how to construct linear slices explicitly.

Lemma B.3. *Assume that G acts by orthogonal transformations on V. For any v in V, let E_v be the tangent space of the orbit $G\cdot v$ at v. Then, $E_v^\perp$ is a linear subspace of V, stable by G_v, and meets all G-orbits in V. Besides, a neighborhood of v in $E_v^\perp$ is a local slice of the G-action at v.*

Proof. The Lie algebra $\mathfrak{g}$ of G acts on V, besides this action is given by a morphism from $\mathfrak{g}$ to $\mathfrak{o}(V)$. The tangent space of $G\cdot v$ at v is given by

$$E_v = \{X\cdot v, X \in \mathfrak{g}\}$$

Since X acts by an element of $\mathfrak{o}(V)$, $(X\cdot v|v) = 0$ so v is orthogonal to E_v. Besides, if g is in G_v, w is in $E_v^\perp$ and X is in $\mathfrak{g}$, we have

$$(g\cdot w|X\cdot v) = (w|g^{-1}\cdot(X\cdot v)) = (w|\mathrm{Ad}(g^{-1})(X)\cdot v) = 0$$

so $g\cdot w$ belongs to $E_v^\perp$. Hence $E_v^\perp$ is stable by G_v.

Let $\mathcal{O}$ be an orbit in V, and assume that w is the point of $\mathcal{O}$ that minimizes the distance to v. Then w is a critical point for the function $g \mapsto \|g^{-1}\cdot w - v\|^2 = \|g\cdot v - w\|^2$, which means that for all X in $\mathfrak{g}$, $(X\cdot v|v-w) = 0$. Since v belongs to $E_v^\perp$, so does w. □

Proof of theorem 2.6, linear case. We argue by induction on the dimension on V. The case $V = \{0\}$ is obvious. We now separate two different situations.

Case 1: the representation $\rho\colon \mathfrak{g} \to \mathfrak{o}(V)$ is trivial. Then, the representation itself is trivial on the neutral component of G, which means that G factors to a representation of the finite group G/G^e. Let Γ be the image of the representation, it is a finite group acting faithfully on V. Principal points are exactly the complement of the finite union of the fixed loci V^γ for γ in $\Gamma\setminus\{e\}$. It is open and dense, we denote it by U. Let us now partition $\Gamma\setminus\{e\}$ as $\Gamma_1\cup\Gamma_2$ where

$$\Gamma_1 = \{\gamma\in\Gamma, \operatorname{codim} V^\gamma = 1\}, \quad \text{and} \quad \Gamma_2 = \{\gamma\in\Gamma, \operatorname{codim} V^\gamma \geq 2\}.$$

Let us write $\Gamma_1 = \{\gamma_1,\dots,\gamma_p\}$, $\Gamma_2 = \{\gamma_{p+1},\dots,\gamma_{p+q}\}$, and let us define inductively open subsets $(\Omega_i)_{1\leq i\leq p+q}$ of V as follows: $\Omega_0 = \Omega$ and $\Omega_i = \Omega_{i-1}\cap(V^{\gamma_i})^c$. Then, each Ω_i is dense in Ω_{i-1}. Besides, for any v in $\Omega_{i-1}\cap V^{\gamma_i}$, if B is a small ball centered in v, then we separate two cases.

- If $1\leq i\leq p$, then $B\cap(V^{\gamma_i})^c$ is connected.
- If $p+1\leq i\leq p+q$, $B\cap(V^{\gamma_i})^c$ is not connected because V^{γ_i} is an hyperplane, but γ_i has order 2 in Γ. Hence γ_i swaps the two connected components of $B\cap(V^{\gamma_i})^c$, so $\pi(B\cap(V^{\gamma_i})^c)$ is connected.

Hence it follows by induction using lemma B.1 that all $\pi(\Omega_k)$ are connected. In particular, since $\Omega_{p+q} = U_\Omega = U\cap\Omega$, $\pi(U_\Omega)$ is connected.

Case 2: the representation $\rho\colon \mathfrak{g}\to\mathfrak{o}(V)$ is not trivial. Let

$$W = \{v\in V \text{ such that } \forall X\in\mathfrak{g}, X\cdot v = 0\}$$

be the kernel of the representation ρ. Then, by hypothesis, W is a proper vector subspace of V.

First we claim that W has codimension at least 2. Indeed, if W has codimension 1, then the line $W^\perp$ is stable by the action of G^e. Since orthogonal transformations of a line are ± 1, the action of G^e is trivial on this line, which implies that $W = V$ and gives a contradiction. It follows from this that $\pi(\Omega \cap W^c)$ is connected.

Next we claim that points of W are not principal. Indeed, if w is in W, G_w contains the identity component G^e of G. If w was principal, then for any w' close to w, the same property would hold for $G_{w'}$. This would imply that w' is in W, so W would have nonempty interior. Hence we can replace Ω by $\Omega \cap W^\perp$ without changing the conclusions, that is assume that $\Omega \cap W = \emptyset$.

If v is in Ω, thanks to lemma B.3, we can apply the induction hypothesis with the action of G_v on $E_v^\perp$ which has smaller dimension. Let us consider a small G_v-stable open ball B centered at v in $E_v^\perp$. If the radius is small enough, B is a local slice of the G-action at v. Let $\mathcal{P}$ be the set of principal points in B. By induction, since B is open and connected in $E_v^\perp$, $\mathcal{P}$ is open and dense in B, and $\mathcal{P}/G_v$ is connected.

Let us now remark that since B is a local slice, for any v' in B, $(G_v)_{v'} = G'_v$. It follows that $\mathcal{P}$ consists of principal points of B, but also of V. This proves that v is in the closure of U_Ω, so U_Ω is dense in Ω.

Since B is a local slice, we can assume, up to shrinking B, that the action map $G \times B \to V$ is an open map. Hence, $U_v = G.\mathcal{P}$ is an open subset of principal points in V, and furthermore $U_v/G = \mathcal{P}/G_v$ is connected.

We can now conclude by applying lemma B.1 to the pair $(\pi(U_\Omega), \pi(\Omega))$. For v in Ω, we can consider the neighborhood $G \cdot B$ of v in V. Then $U_\Omega \cap G \cdot B = U_v$ so $\pi(U_\Omega) \cap \pi(G \cdot B) = \mathcal{P}/G_v$ is connected. Hence $\pi(U_\Omega)$ is connected. □

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