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Markoff–Lagrange spectrum of one-sided shifts

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Abstract

For the Lagrange spectrum and other applications, we determine the smallest accumulation point of binary sequences that are maximal in their shift orbits. This problem is trivial for the lexicographic order, and its solution is the fixed point of a substitution for the alternating lexicographic order. For orders defined by cylinders, we show that the solutions are S -adic sequences, where S is a certain infinite set of substitutions that contains Sturmian morphisms. We also consider a similar problem for symmetric ternary shifts, which is applicable to the multiplicative version of the Markoff–Lagrange spectrum.

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1 | INTRODUCTION

Let A^∞ be the set of infinite words on a finite alphabet A , equipped with a total order \leq and the ultrametric d given by $d(a_1 a_2 \dots, b_1 b_2 \dots) = 2^{-\min\{n \geq 1 : a_n \neq b_n\}}$ for $a_1 a_2 \dots \neq b_1 b_2 \dots$. We study properties of the set of *sup*-words

$$\mathcal{M}_{\leq} := \{s_{\leq}(\mathbf{a}) : \mathbf{a} \in A^\infty\}, \quad \text{with} \quad s_{\leq}(a_1 a_2 \dots) := \sup_{n \geq 1} a_n a_{n+1} \dots,$$

for a large class of orders on A^∞ . In particular, we are interested in the smallest accumulation point \mathbf{m}_{\leq} of \mathcal{M}_{\leq} . For the lexicographic order \leq_{lex} , words in $\mathcal{M}_{\leq_{\text{lex}}}$ occur as (quasi-greedy) β -expansions of 1 for real bases $\beta > 1$ (see [20]), with $\mathbf{m}_{\leq_{\text{lex}}} = 1000 \dots$ being the limit of these expansions as $\beta \rightarrow 1$. For the alternating lexicographic order \leq_{alt} , most elements of $\mathcal{M}_{\leq_{\text{alt}}}$ are $(-\beta)$ -expansions

of $\frac{-\beta}{\beta+1}$ in the sense of [16, 24], and again, $\mathbf{m}_{\leq\text{alt}}$ is the limit of these expansions as $\beta \rightarrow 1$. An image of $\mathcal{M}_{\leq\text{alt}}$ occurs in a multiplicative version of the (Markoff-)Lagrange spectrum w.r.t. an integer base, which is defined in terms of well approximable numbers [2, 12]; see Proposition 10. Below the image of $\mathbf{m}_{\leq\text{alt}}$, which is the fixed point of a substitution [3, 5, 13], we find the discrete part of this spectrum. The classical Markoff and Lagrange spectra are given by two-sided versions of $\mathcal{M}_{\leq\text{alt}}$ (and the Lagrange spectrum is defined by \limsup instead of \sup). The unimodal order \leq_{uni} yields kneading sequences of unimodal maps [19], and $\mathbf{m}_{\leq\text{uni}}$ is the fixed point of the period-doubling (or Feigenbaum) substitution. Sup-words are also closely related to infinite Lyndon words, which are defined by $a_1 a_2 \dots < a_n a_{n+1} \dots$ for all $n \geq 2$; see [23].

We consider orders satisfying that

$$\mathbf{a} \leq \mathbf{b} \leq \mathbf{c} \text{ implies } d(\mathbf{a}, \mathbf{b}) \leq d(\mathbf{a}, \mathbf{c}) \text{ for all } \mathbf{a}, \mathbf{b}, \mathbf{c} \in A^\infty \tag{1}$$

(note that $d(\mathbf{a}, \mathbf{b}) \leq d(\mathbf{a}, \mathbf{c})$ is equivalent to $d(\mathbf{b}, \mathbf{c}) \leq d(\mathbf{a}, \mathbf{c})$ by the strong triangle inequality), and we call them *cylinder orders* because the elements of each cylinder of words are contiguous. Here, the *cylinder* (of length n) given by $a_1 \dots a_n \in A^n$ is

$$[a_1 \dots a_n] := \{a'_1 a'_2 \dots \in A^\infty : a'_1 \dots a'_n = a_1 \dots a_n\},$$

and we write $[a_1 \dots a_n] < [b_1 \dots b_n]$ if $(a_1 \dots a_n)^\infty < (b_1 \dots b_n)^\infty$ (or, equivalently, $\mathbf{a} < \mathbf{b}$ for all $\mathbf{a} \in [a_1 \dots a_n]$, $\mathbf{b} \in [b_1 \dots b_n]$), where w^∞ denotes the infinite repetition of a word w . Note that (1) is equivalent to the condition (*) of [21], which states that $ux < vy$ for all $x, y \in A^\infty$, $u, v \in A^n$, $n \geq 1$, such that $u^\infty < v^\infty$. Generalized lexicographic orders as considered in [22] satisfy this condition.

In Section 2, we give basic properties of \mathcal{M}_{\leq} . Section 3 contains our main result, an algorithm for determining the smallest accumulation point \mathbf{m}_{\leq} of \mathcal{M}_{\leq} , for any cylinder order (A^∞, \leq) . We also give a complete description, in terms of S -adic sequences, of all words \mathbf{m}_{\leq} obtained by cylinder orders and of the discrete part of \mathcal{M}_{\leq} , that is, all its elements below \mathbf{m}_{\leq} . Since the words \mathbf{m}_{\leq} have linear factor complexity, real numbers having such β -expansions are in $\mathbb{Q}(\beta)$ or transcendental, for all Pisot or Salem bases $\beta \geq 2$. In Section 4, we determine \mathbf{m}_{\leq} for some classical examples of cylinder orders and show that all Sturmian sequences that are sup-words can occur. We consider cylinder orders on symmetric alphabets in Section 5 and apply our results to the multiplicative Lagrange spectrum and other problems in Section 6,

2 | MARKOFF-LAGRANGE SPECTRUM

We first show that the set of periodic words in \mathcal{M}_{\leq} is dense, and that \mathcal{M}_{\leq} is equal to

$$\mathcal{L}_{\leq} := \{\ell_{\leq}(\mathbf{a}) : \mathbf{a} \in A^\infty\}, \text{ with } \ell_{\leq}(a_1 a_2 \dots) := \limsup_{n \rightarrow \infty} a_n a_{n+1} \dots$$

Note that \mathcal{M}_{\leq} and \mathcal{L}_{\leq} can be seen as generalizations of the *Markoff* and *Lagrange spectrum*, respectively. In the classical case, these spectra are defined by two-sided sequences, and the Lagrange spectrum is a strict subset of the Markoff spectrum [11, 14].

Theorem 1. *Let \leq be a cylinder order on A^∞ . Then*

$$\mathcal{L}_{\leq} = \mathcal{M}_{\leq} = \text{closure}(\{s_{\leq}(\mathbf{a}) : \mathbf{a} \in A^\infty \text{ purely periodic}\}).$$

In particular, the set \mathcal{M}_{\leq} is closed.

In the proof of the theorem, we use the following characterization of cylinder orders.

Lemma 2. *An order \leq on A^∞ is a cylinder order if and only if*

$$\mathbf{a} \leq \mathbf{b} \text{ implies } \mathbf{a}' \leq \mathbf{b}' \text{ for all } \mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}' \in A^\infty \text{ with } \max(d(\mathbf{a}, \mathbf{a}'), d(\mathbf{b}, \mathbf{b}')) < d(\mathbf{a}, \mathbf{b}). \quad (2)$$

Proof. Let \leq be a cylinder order, and $\mathbf{a}, \mathbf{b}, \mathbf{a}', \mathbf{b}' \in A^\infty$ such that $\mathbf{a} \leq \mathbf{b}$, $d(\mathbf{a}, \mathbf{a}') < d(\mathbf{a}, \mathbf{b})$, $d(\mathbf{b}, \mathbf{b}') < d(\mathbf{a}, \mathbf{b})$. Then both $\mathbf{a} \leq \mathbf{b} \leq \mathbf{a}'$ and $\mathbf{b}' \leq \mathbf{a}' \leq \mathbf{b}$ are impossible by (1), using that $d(\mathbf{a}', \mathbf{b}') = d(\mathbf{a}, \mathbf{b})$ by the strong triangle inequality. This implies that $\mathbf{b} > \mathbf{a}'$ and thus $\mathbf{b}' > \mathbf{a}'$, that is, (2) holds. Let now \leq be an order satisfying (2). Then, $\mathbf{a} \leq \mathbf{b}$ and $d(\mathbf{a}, \mathbf{c}) < d(\mathbf{a}, \mathbf{b})$ imply that $\mathbf{c} < \mathbf{b}$, thus $\mathbf{a} \leq \mathbf{b} \leq \mathbf{c}$ with $d(\mathbf{a}, \mathbf{c}) < d(\mathbf{a}, \mathbf{b})$ is impossible, that is, (1) holds. \square

Proof of Theorem 1. We show first that \mathcal{M}_\leq is the closure of $s_{\leq}(\mathbf{a}')$ with purely periodic $\mathbf{a}' \in A^\infty$. Since $s_{\leq}(s_{\leq}(\mathbf{a})) = s_{\leq}(\mathbf{a})$, it suffices to consider $\mathbf{a} \in A^\infty$ with $s_{\leq}(\mathbf{a}) = \mathbf{a}$. We have $(a_1 \dots a_n)^\infty \in \mathcal{M}_\leq$ whenever

$$2^{-n}d(\mathbf{a}, a_{n+1}a_{n+2} \dots) < 2^{-i}d(\mathbf{a}, a_{i+1}a_{i+2} \dots) \text{ for all } i \in [1, n]. \quad (3)$$

Indeed, we have, for all $i \in [1, n]$, that $a_{i+1}a_{i+2} \dots \leq \mathbf{a}$ (because $s_{\leq}(\mathbf{a}) = \mathbf{a}$) and

$$\begin{aligned} d(\mathbf{a}, a_{i+1}a_{i+2} \dots) &> 2^{i-n}d(\mathbf{a}, a_{n+1}a_{n+2} \dots) = 2^{i-n}d((a_1 \dots a_n)^\infty, a_{n+1}a_{n+2} \dots) \\ &= 2^i d((a_1 \dots a_n)^\infty, \mathbf{a}) = d(a_{i+1} \dots a_n(a_1 \dots a_n)^\infty, a_{i+1}a_{i+2} \dots), \end{aligned}$$

hence (2) gives that $a_{i+1} \dots a_n(a_1 \dots a_n)^\infty \leq (a_1 \dots a_n)^\infty$, thus $(a_1 \dots a_n)^\infty \in \mathcal{M}_\leq$. Since $\lim_{n \rightarrow \infty} 2^{-n}d(\mathbf{a}, a_{n+1}a_{n+2} \dots) = 0$, we have either $d(\mathbf{a}, a_{n+1}a_{n+2} \dots) = 0$ for some $n \geq 1$, that is, \mathbf{a} is purely periodic, or there are infinitely many n such that (3) holds and thus $(a_1 \dots a_n)^\infty \in \mathcal{M}_\leq$ infinitely often. Therefore, \mathcal{M}_\leq is contained in the closure of $\{s_{\leq}(\mathbf{a}) : \mathbf{a} \in A^\infty \text{ purely periodic}\}$. For the opposite inclusion, we have to show that \mathcal{M}_\leq is closed. Consider $\mathbf{a} = \lim_{k \rightarrow \infty} \mathbf{a}^{(k)}$ with $\mathbf{a}^{(k)} \in \mathcal{M}_\leq$. If $a_{n+1}a_{n+2} \dots > \mathbf{a}$ for some $n \geq 1$, then we would also have $a_{n+1}^{(k)}a_{n+2}^{(k)} \dots > \mathbf{a}^{(k)}$ for all large enough k , contradicting that $\mathbf{a}^{(k)} \in \mathcal{M}_\leq$. This implies that $s_{\leq}(\mathbf{a}) = \mathbf{a}$, that is, $\mathbf{a} \in \mathcal{M}_\leq$.

Since $s_{\leq}(\ell_{\leq}(\mathbf{a})) = \ell_{\leq}(\mathbf{a})$ for all $\mathbf{a} \in A^\infty$, we have $\mathcal{L}_\leq \subseteq \mathcal{M}_\leq$. For the opposite inclusion, let $\mathbf{a} = \lim_{k \rightarrow \infty} \mathbf{a}^{(k)}$ for some purely periodic words $\mathbf{a}^{(k)} \in \mathcal{M}_\leq$, let (p_k) be an increasing sequence satisfying $\mathbf{a}^{(k)} = (a_1^{(k)} \dots a_{p_k}^{(k)})^\infty$, and let $\mathbf{b} = a_1^{(1)} \dots a_{p_1}^{(1)} a_1^{(2)} \dots a_{p_2}^{(2)} \dots$. Then, $\ell_{\leq}(\mathbf{b}) \geq \mathbf{a}$. If $a_{i+1}^{(k)} \dots a_{p_k}^{(k)} a_1^{(k+1)} \dots a_{p_{k+1}}^{(k+1)} \dots > \mathbf{a}$ for some $k \geq 1, 0 \leq i < p_k$, then

$$\delta_{i,k} := d(a_{i+1}^{(k)} \dots a_{p_k}^{(k)} a_1^{(k+1)} \dots a_{p_{k+1}}^{(k+1)} \dots, \mathbf{a}) \leq \max(d(\mathbf{a}^{(k)}, \mathbf{a}), d(\mathbf{a}^{(k)}, \mathbf{a}^{(k+1)}), 2^{-p_k}). \quad (4)$$

Indeed, for $1 \leq j \leq p_k - i$, we cannot have $[a_{i+1}^{(k)} \dots a_{i+j}^{(k)}] > [a_1 \dots a_j] = [a_1^{(k)} \dots a_j^{(k)}]$ because this contradicts $\mathbf{a}^{(k)} \in \mathcal{M}_\leq$, thus $\delta_{i,k} = 2^{-j}$ implies $d(\mathbf{a}^{(k)}, \mathbf{a}) \geq 2^{-j}$; similarly, for $p_k - i < j \leq 2p_k - i$,

$$[a_{i+1}^{(k)} \dots a_{p_k}^{(k)} a_1^{(k)} \dots a_{i+j-p_k}^{(k)}] = [a_{i+1}^{(k)} \dots a_{p_k}^{(k)} a_1^{(k+1)} \dots a_{i+j-p_k}^{(k+1)}] > [a_1 \dots a_j] = [a_1^{(k)} \dots a_j^{(k)}]$$

is impossible, thus $\delta_{i,k} = 2^{-j}$ implies $d(\mathbf{a}^{(k)}, \mathbf{a}^{(k+1)}) \geq 2^{p_k - i - j}$ or $d(\mathbf{a}^{(k)}, \mathbf{a}) \geq 2^{-j}$; hence, we have $\delta_{i,k} < 2^{i-2p_k}$ or $\delta_{i,k} \leq 2^{i-p_k}d(\mathbf{a}^{(k)}, \mathbf{a}^{(k+1)})$ or $\delta_{i,k} \leq d(\mathbf{a}^{(k)}, \mathbf{a})$, which implies (4). Since the right-hand side of (4) tends to 0 as $k \rightarrow \infty$, we have $\ell_{\leq}(\mathbf{b}) \leq \mathbf{a}$, thus $\mathbf{a} = \ell_{\leq}(\mathbf{b}) \in \mathcal{L}_\leq$. \square

3 | SMALLEST ACCUMULATION POINT OF \mathcal{M}_{\leq}

For determining the smallest accumulation point \mathbf{m}_{\leq} of \mathcal{M}_{\leq} for a cylinder order (A^∞, \leq) , we can restrict to two-letter alphabets, w.l.o.g., $A = \{0, 1\}$. Indeed, if $\{0, 1\} \subseteq A$, $[0] < [1]$, then \mathcal{M}_{\leq} has the accumulation point 10^∞ (because $(10^n)^\infty \in \mathcal{M}_{\leq}$ for all $n \geq 1$), and we clearly have $s_{\leq}(\mathbf{a}) > 10^\infty$ if $\mathbf{a} \in A^\infty$ contains a letter $a_n \in A$ with $[a_n] > [1]$.

We use substitutions (also called word morphisms) and limit words (or S -adic sequences). Let A^* be the monoid of finite words over the alphabet A , with concatenation as operation. A *substitution* $\sigma : A^* \rightarrow A^*$ satisfies $\sigma(vw) = \sigma(v)\sigma(w)$ for all $v, w \in A^*$ and is extended naturally to A^∞ ; it suffices to give $\sigma(a)$ for $a \in A$ to define σ . For a sequence $\sigma = (\sigma_n)_{n \geq 1}$ of substitutions on the alphabet A and an infinite word $\mathbf{a} \in A^\infty$, the *limit word* is

$$\sigma(\mathbf{a}) := \lim_{n \rightarrow \infty} \sigma_{[1,n]}(\mathbf{a}),$$

if this limit exists; we use the notation $\sigma_{[1,n]} := \sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_n$ for $n \geq 0$, with $\sigma_{[1,0]}$ being the identity map. For a set of substitutions S , we denote the monoid generated by the composition of substitutions in S by S^* . We use the set of substitutions

$$S = \{\tau_{j,k} : 0 \leq j < k\}, \quad \text{with} \quad \begin{matrix} \tau_{j,k} : 0 \mapsto 10^j, \\ 1 \mapsto 10^k. \end{matrix}$$

Our main result is the following characterization of the smallest accumulation point \mathbf{m}_{\leq} and the discrete part of \mathcal{M}_{\leq} for cylinder orders \leq .

Theorem 3. *Let $\mathbf{m} \in \{0, 1\}^\infty$. Then, $\mathbf{m} = \mathbf{m}_{\leq}$ for some cylinder order \leq on $\{0, 1\}^\infty$ with $0^\infty < 1^\infty$ if and only if $\mathbf{m} = \sigma(10^\infty)$ for some $\sigma \in S^*$ or $\mathbf{m} = \sigma(1^\infty)$ for some $\sigma \in S^\infty$.*

If $\mathbf{m}_{\leq} = \sigma(1^\infty)$, $\sigma = (\sigma_n)_{n \geq 1} \in S^\infty$, then

$$\{\mathbf{a} \in \mathcal{M}_{\leq} : \mathbf{a} < \mathbf{m}_{\leq}\} = \{(\sigma_{[1,n]}(0))^\infty : n \geq 0\}. \tag{5}$$

If $\mathbf{m}_{\leq} = \sigma_{[1,h]}(10^\infty)$, $\sigma_1, \dots, \sigma_h \in S$, $h \geq 0$, then

$$\begin{aligned} \{\mathbf{a} \in \mathcal{M}_{\leq} : \mathbf{a} < \mathbf{m}_{\leq}\} &= \{(\sigma_{[1,n]}(0))^\infty : 0 \leq n \leq h\} \\ &\cup \{\sigma_{[1,h]}((10^j)^\infty) : j \geq 0, \sigma_{[1,h]}((10^j)^\infty) < \mathbf{m}_{\leq}\}, \end{aligned} \tag{6}$$

and there is at most one $j \geq 0$ such that $\sigma_{[1,h]}((10^j)^\infty) < \mathbf{m}_{\leq}$.

The following proposition constitutes the core of the proof of Theorem 3 and provides an algorithm for calculating \mathbf{m}_{\leq} .

Proposition 4. *Let \leq be a cylinder order on $\{0, 1\}^\infty$ with $[0] < [1]$.*

If $[10^j 1] < [10^j 0]$ holds for at most one $j \geq 0$, then $\mathbf{m}_{\leq} = 10^\infty$, and $\mathbf{a} < \mathbf{m}_{\leq}$ implies that $\mathbf{a} = 0^\infty$ or $\mathbf{a} = (10^j)^\infty$ (in case $[10^j 1] < [10^j 0]$).

Otherwise, we have $\mathbf{m}_{\leq} = \tau_{j,k}(\mathbf{m}_{\leq})$ for the cylinder order \leq on $\{0, 1\}^\infty$ defined by $\mathbf{a} \leq \mathbf{b}$ if $\tau_{j,k}(\mathbf{a}) \leq \tau_{j,k}(\mathbf{b})$, where j, k are minimal such that $0 \leq j < k$, $[10^j 1] < [10^j 0]$, $[10^k 1] < [10^k 0]$; we have $[0] < [1]$ and $\{\mathbf{a} \in \mathcal{M}_{\leq} : \mathbf{a} < \mathbf{m}_{\leq}\} = \{0^\infty\} \cup \{\tau_{j,k}(\mathbf{a}) : \mathbf{a} \in \mathcal{M}_{\leq}, \mathbf{a} < \mathbf{m}_{\leq}\}$.

Proof. We have $\mathbf{m}_{\leq} \in [1]$ because $\mathcal{M}_{\leq} \cap [0] = \{0^\infty\}$, and $\mathbf{m}_{\leq} \leq 10^\infty$ because $(10^n)^\infty \in \mathcal{M}_{\leq}$ for all $n \geq 0$. If $[10^j 0] < [10^j 1]$ for all $j \geq 0$, that is, $\min[1] = 10^\infty$, then $\mathbf{m}_{\leq} = 10^\infty$. Otherwise, let j be minimal such that $[10^j 1] < [10^j 0]$, that is, $\min[1] \in [10^j]$. Then $\mathcal{M}_{\leq} \cap [10^j 1] = \{(10^j)^\infty\}$ implies that $\mathbf{m}_{\leq} \in [10^j 0]$. If $[10^k 0] < [10^k 1]$ for all $k > j$, that is, $\min[10^j 0] = 10^\infty$, then $\mathbf{m}_{\leq} = 10^\infty$. Otherwise, let $k > j$ be minimal such that $[10^k 1] < [10^k 0]$, that is, $\min[10^j 0] \in [10^k 1]$. We have $\mathbf{m}_{\leq} \leq \tau_{j,k}(10^\infty)$ because $\tau_{j,k}((10^n)^\infty) \in \mathcal{M}_{\leq}$ for all $n \geq 0$, thus $\mathbf{m}_{\leq} \in [10^k 1]$. Each word in $\mathcal{M}_{\leq} \cap [10^k 1]$ is a concatenation of blocks 10^j and 10^k , thus $\mathcal{M}_{\leq} \cap [10^k 1] \subseteq \tau_{j,k}([1])$. This proves that $\mathbf{m}_{\leq} = \tau_{j,k}(\mathbf{m}_{\leq})$, where $\mathbf{a} \leq \mathbf{b}$ if $\tau_{j,k}(\mathbf{a}) \leq \tau_{j,k}(\mathbf{b})$; note that $\mathbf{m}_{\leq} \in [1]$ because $[0] < [1]$. Since $\tau_{j,k}([w0]) \subseteq [\tau_{j,k}(w0)1]$ and $\tau_{j,k}([w1]) \subseteq [\tau_{j,k}(w0)0]$ for all $w \in \{0, 1\}^*$, \leq is a cylinder order. If $\mathbf{a} \in \mathcal{M}_{\leq}$ with $\mathbf{a} < \mathbf{m}_{\leq}$, then $\mathbf{a} = 0^\infty$ or $\mathbf{a} = (10^j)^\infty = \tau_{j,k}(0^\infty)$ or $\mathbf{a} = \tau_{j,k}(\mathbf{a}')$ with $\mathbf{a}' \in \mathcal{M}_{\leq} \cap [1]$, $\mathbf{a}' < \mathbf{m}_{\leq}$. □

The following lemma is used in the construction of a cylinder order \leq such that $\mathbf{m}_{\leq} = \mathbf{m}$ for a given word \mathbf{m} . Here, ε denotes the empty word.

Lemma 5. Let $\sigma = (\sigma_n)_{n \geq 1} \in S^\infty$, $w_n = \sigma_{[1,n]}(0) \cdots \sigma_{[1,1]}(0)$ for $n \geq 0$, with $w_0 = \varepsilon$.

For all even $n \geq 0$, we have $\sigma_{[1,n]}([0]) \subseteq [w_n 0]$, $\sigma_{[1,n]}([1]) \subseteq [w_n 1]$.

For all odd $n \geq 1$, we have $\sigma_{[1,n]}([0]) \subseteq [w_n 1]$, $\sigma_{[1,n]}([1]) \subseteq [w_n 0]$.

Proof. Since $\sigma_{[1,0]}$ is the identity, the statement is trivial for $n = 0$. Suppose that it is true for all $\sigma \in S^\infty$ for some even $n \geq 0$. Then,

$$\sigma_{[1,n+1]}([0]) = \sigma_1 \circ \sigma_{[2,n+1]}([0]) \subseteq \sigma_1([\sigma_{[2,n+1]}(0) \cdots \sigma_{[2,2]}(0)0]) \subseteq [\sigma_{[1,n+1]}(0) \cdots \sigma_{[1,2]}(0)\sigma_1(0)1],$$

$$\sigma_{[1,n+1]}([1]) = \sigma_1 \circ \sigma_{[2,n+1]}([1]) \subseteq \sigma_1([\sigma_{[2,n+1]}(0) \cdots \sigma_{[2,2]}(0)1]) \subseteq [\sigma_{[1,n+1]}(0) \cdots \sigma_{[1,2]}(0)\sigma_1(0)0],$$

thus the statement is true for all $\sigma \in S^\infty$ for $n+1$. The case of odd n is similar. □

Proof of Theorem 3. Let first \mathbf{m}_{\leq} be a cylinder order with $[0] < [1]$. By iterating Proposition 4, we obtain a finite sequence $\sigma_1, \dots, \sigma_h \in S$ such that $\mathbf{m}_{\leq} = \sigma_{[1,h]}(10^\infty)$ or an infinite sequence $\sigma = (\sigma_n)_{n \geq 1} \in S^\infty$ such that $\mathbf{m}_{\leq} \in \sigma_{[1,n]}([1])$ for all $n \geq 1$. Since $\sigma_{[1,n+1]}(1)$ starts with $\sigma_{[1,n]}(1)$ and is longer than $\sigma_{[1,n]}(1)$ for all $n \geq 1$, we have $\bigcap_{n \geq 1} \sigma_{[1,n]}([1]) = \{\sigma(1^\infty)\}$. Equations (5) and (6), respectively, follow from Proposition 4.

Let now $\sigma = (\sigma_n)_{n \geq 1} = (\tau_{j_n, k_n})_{n \geq 1} \in S^\infty$. By Proposition 4 and Lemma 5, we have $\sigma(1^\infty) = \mathbf{m}_{\leq}$ for all cylinder orders \leq satisfying

$$\begin{aligned} [\sigma_{[1,n]}(10^i)w_n 0] &< [\sigma_{[1,n]}(10^i)w_n 1] && \text{for all even } n \geq 0, j_{n+1} \neq i < k_{n+1}, \\ &&& \text{and for all odd } n \geq 1, i \in \{j_{n+1}, k_{n+1}\}, \\ [\sigma_{[1,n]}(10^i)w_n 1] &< [\sigma_{[1,n]}(10^i)w_n 0] && \text{for all odd } n \geq 1, j_{n+1} \neq i < k_{n+1}, \\ &&& \text{and for all even } n \geq 0, i \in \{j_{n+1}, k_{n+1}\}. \end{aligned} \tag{7}$$

Such cylinder orders exist since $\sigma_{[1,n+1]}(1)w_{n+1}$ is longer than $\sigma_{[1,n]}(10^{k_{n+1}})w_n = \sigma_{[1,n+1]}(1)w_n$ for all $n \geq 0$. To obtain $\mathbf{m}_{\leq} = \sigma_{[1,h]}(10^\infty)$, $h \geq 0$, we use cylinder orders \leq such that (7) holds only for $n < h$ and such that, for all $i \geq 0$, $[\sigma_{[1,h]}(10^i)w_h 0] < [\sigma_{[1,h]}(10^i)w_h 1]$ if h is even, $[\sigma_{[1,h]}(10^i)w_h 1] < [\sigma_{[1,h]}(10^i)w_h 0]$ if h is odd. □

The sequence \mathbf{m}_{\leq} is thus either eventually periodic or an S -adic sequence. A word $\mathbf{a} = a_1 a_2 \dots$ is eventually periodic if and only if the factor complexity

$$p_{a_1 a_2 \dots}(n) := \#\{a_{k+1} a_{k+2} \dots a_{k+n} : k \geq 0\}$$

is bounded; see, for example, [18, Theorem 1.3.13]. The smallest complexity for an aperiodic sequence is $p_{\mathbf{a}}(n) = n+1$, which is attained precisely by Sturmian sequences; see, for example, [18, Theorem 2.1.5]. By [10, Proposition 2.1], all aperiodic words with $\limsup \frac{p_{\mathbf{a}}(n)}{n} < \frac{4}{3}$ are essentially equal to $\sigma(1^\infty)$ with $\sigma = (\tau_{j_n, k_n})_{n \geq 1} \in S^\infty$ (and $k_n \leq 2j_n + 1$ or $(j_n, k_n) = (0, 2)$). Without conditions on j_n, k_n , we get the following upper bound for $p_{\sigma(1^\infty)}(n)$, which is optimal since $p_{\tau_{0,k}^\infty(1^\infty)}(n) = 3n - 2$ for all $k \geq 2, 2 \leq n \leq k$.

Proposition 6. *Let \leq be a cylinder order. Then $p_{\mathbf{m}_{\leq}}(n) \leq 3n - 2$ for all $n \geq 2$.*

Proof. The proof is similar to that of [6, Theorem 17]; see also [10, Proposition 4.1]. Recall that the set of factors of a word $\mathbf{a} = a_1 a_2 \dots \in \{0, 1\}^\infty$ is $\{a_{k+1} a_{k+2} \dots a_{k+n} : k, n \geq 0\}$, a factor v of \mathbf{a} is *strong bispecial* if all four words $0v0, 0v1, 1v0, 1v1$ are factors of \mathbf{a} , *weak bispecial* if $0v1, 1v0$ are factors and $0v0, 1v1$ are not factors of \mathbf{a} . Then, v is a strong/weak bispecial factor of $\sigma(1^\infty)$, $\sigma = (\sigma_n)_{n \geq 1} = (\tau_{j_n, k_n})_{n \geq 1} \in S^\infty$, if and only if $v = 0^{j_1} / 0^{k_1 - 1}, k_1 \geq j_1 + 2$, or $v = 0^{j_1} \sigma_1(v'0)$, where v' is a strong/weak bispecial factor of $\lim_{n \rightarrow \infty} \sigma_{[2, n]}(1^\infty)$. By iterating, we obtain that all strong/weak bispecial factors of $\sigma(1^\infty)$ are of the form

$$w_{\ell, h} = \sigma_{[1, 0]}(0^{j_1}) \dots \sigma_{[1, h-1]}(0^{j_h}) \sigma_{[1, h]}(0^\ell) \sigma_{[1, h]}(0) \dots \sigma_{[1, 1]}(0),$$

with $h \geq 0$ such that $k_{h+1} \geq j_{h+1} + 2$, where $\ell = j_{h+1}$ for a strong bispecial factor, $\ell = k_{h+1} - 1$ for a weak one; here, $w_{\ell, 0} = 0^\ell$. For any recurrent word $\mathbf{a} \in \{0, 1\}^\infty$ (where recurrence means that each prefix of \mathbf{a} occurs infinitely often in \mathbf{a}), the difference of $p_{\mathbf{a}}(n+2) - p_{\mathbf{a}}(n+1)$ and $p_{\mathbf{a}}(n+1) - p_{\mathbf{a}}(n)$ equals the difference of the number of strong and weak bispecial factors of \mathbf{a} of length n ; see [7, Proposition 3.2]. By telescoping and since $p_{\mathbf{a}}(1) - p_{\mathbf{a}}(0) = 1, p_{\mathbf{a}}(n+2) - p_{\mathbf{a}}(n+1) - 1$ is equal to the difference of the number of strong and weak bispecial factors of \mathbf{a} up to length n . Since $|w_{k_{h+1}-1, h}| < |\sigma_{[1, h+1]}(1)| \leq |\sigma_{[1, h+2]}(0)| < |w_{j_{h+3}, h+2}|$ for all $h \geq 0$, this difference for $\mathbf{a} = \sigma(1^\infty)$ is at most 2 for all $n \geq 0$; note that $\sigma(1^\infty)$ is recurrent since $\sigma_{[n, n+1]}(1)$ starts with $10^{k_n} 1$ for all $n \geq 1$. Since $p_{\mathbf{a}}(2) \leq 4$, we have thus $p_{\sigma(1^\infty)} \leq 3n - 2$ for all $n \geq 2$. Since $p_{\sigma_{[1, h]}(10^\infty)}(n) \leq p_{\sigma(1^\infty)}(n)$ if k_{h+1} is sufficiently large, we also have $p_{\sigma_{[1, h]}(10^\infty)}(n) \leq 3n - 2$ for all $n \geq 2$, which proves the proposition by Theorem 3. \square

Since the factor complexity is bounded by a linear function, we can apply the results of [1]. For $\beta > 1$, let

$$\pi_\beta(a_1 a_2 \dots) := \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}.$$

Recall that Pisot and Salem numbers are algebraic integers $\beta > 1$ with all Galois conjugates (except β itself) having absolute value ≤ 1 ; β is a Salem number if a conjugate lies on the unit circle, a Pisot number otherwise. In particular, all integers $\beta \geq 2$ are Pisot numbers.

Proposition 7. *Let $\beta \geq 2$ be a Pisot or Salem number, and let \leq be a cylinder order on $\{0, 1\}^\infty$. Then, $\pi_\beta(\mathbf{m}_{\leq})$ is in $\mathbb{Q}(\beta)$ or transcendental.*

Proof. This is a direct consequence of Proposition 6 and [1, Theorem 1A]. □

4 | EXAMPLES

4.1 | Lexicographic order

The classical order on the set of infinite words A^∞ with an ordered alphabet (A, \leq) is the lexicographic order, defined by $[wa] <_{\text{lex}} [wb]$ for all $w \in A^*$, $a, b \in A$ with $a < b$. For $A = \{0, 1\}$, we have $\mathbf{m}_{\leq_{\text{lex}}} = 10^\infty = \min \mathcal{M}_{\leq_{\text{lex}}} \setminus \{0^\infty\}$.

By [20, §2], a sequence $\mathbf{a} \in \{0, 1\}^\infty$ is the greedy β -expansion of 1 for some $\beta \in (1, 2)$ if and only if $\mathbf{a} \in \mathcal{M}_{\leq_{\text{lex}}} \setminus \{10^\infty\}$ and \mathbf{a} is not purely periodic; it is the quasi-greedy β -expansion of 1 for some $\beta \in (1, 2]$ if and only if $\mathbf{a} \in \mathcal{M}_{\leq_{\text{lex}}}$ and \mathbf{a} does not end with 0^∞ . Here, the greedy β -expansion of 1 is the lexicographically largest sequence $\mathbf{a} \in \{0, 1\}^\infty$ with $\pi_\beta(\mathbf{a}) = 1$, the quasi-greedy β -expansion of 1 is the largest such sequence that does not end with 0^∞ . By [15, Theorem 1], we also have that $(0^\infty, \mathbf{a})$ is the pair of kneading sequences of a Lorenz map if and only if $\mathbf{a} \in \mathcal{M}_{\leq_{\text{lex}}}$ does not end with 0^∞ .

4.2 | Alternating lexicographic order

The alternating lexicographic order is defined by $[wa] <_{\text{alt}} [wb]$ if $a < b$ and $|w|$ is even, or $a > b$ and $|w|$ is odd, where $|w|$ denotes the length of a word $w \in A^*$. For $A = \{0, 1\}$ with $0 < 1$, we have $[11] <_{\text{alt}} [10]$, $[101] >_{\text{alt}} [100]$, $[1001] <_{\text{alt}} [1000]$, thus $\mathbf{m}_{\leq_{\text{alt}}} = \tau_{0,2}(\mathbf{m}_{\leq})$ by Proposition 4, with $\mathbf{a} \leq \mathbf{b}$ if $\tau_{0,2}(\mathbf{a}) \leq_{\text{alt}} \tau_{0,2}(\mathbf{b})$. Since \leq is equal to \leq_{alt} , we obtain that $\mathbf{m}_{\leq_{\text{alt}}}$ is the fixed point of $\tau_{0,2}$, that is,

$$\mathbf{m}_{\leq_{\text{alt}}} = \tau_{0,2}(\mathbf{m}_{\leq_{\text{alt}}}) = 100111001001001110011 \dots;$$

see also [3, 5, 13]. By Theorem 3, we have

$$\{\mathbf{a} \in \mathcal{M}_{\leq_{\text{alt}}} : \mathbf{a} <_{\text{alt}} \mathbf{m}_{\leq_{\text{alt}}}\} = \{(\tau_{0,2}^n(0))^\infty : n \geq 0\} = \{0^\infty, 1^\infty, (100)^\infty, (10011)^\infty, \dots\}. \quad (8)$$

According to [16], the $(-\beta)$ -expansion of $x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, $\beta > 1$, is the sequence $a_1 a_2 \dots$ with $a_n = \lfloor \frac{\beta}{\beta+1} - \beta T_{-\beta}^{n-1}(x) \rfloor$, given by the $(-\beta)$ -transformation $T_{-\beta}(y) := -\beta y - \lfloor \frac{\beta}{\beta+1} - \beta y \rfloor$, and the set of $(-\beta)$ -expansions is characterized by that of $\frac{-\beta}{\beta+1}$. By [24, Theorem 2], a sequence \mathbf{a} is the $(-\beta)$ -expansion of $\frac{-\beta}{\beta+1}$ for some $\beta \in (1, 2)$ if and only if $\mathbf{a} \in \mathcal{M}_{\leq_{\text{alt}}} \setminus \{(10)^\infty\}$, $\mathbf{a} >_{\text{alt}} \mathbf{m}_{\leq_{\text{alt}}}$, and $\mathbf{a} \notin \{w1, w00\}^\infty \setminus \{(w1)^\infty\}$ for all $w \in \{0, 1\}^*$ such that $(w1)^\infty >_{\text{alt}} \mathbf{m}_{\leq_{\text{alt}}}$. Note that continued fractions are also ordered by the alternating lexicographic order on the sequences of partial quotients, and $\mathbf{m}_{\leq_{\text{alt}}}$ occurs, for example, in [8, Theorem 2] and [17, Remark 11.1].

4.3 | Unimodal maps

Let $A = \{0, 1\}$ and define the *unimodal order* by $[w0] <_{\text{uni}} [w1]$ if $|w|_1$ is even, $[w0] >_{\text{uni}} [w1]$ if $|w|_1$ is odd, where $|w|_1$ denotes the number of occurrences of 1 in $w \in \{0, 1\}^*$. Then we have $[11] <_{\text{uni}} [10]$, $[101] <_{\text{uni}} [100]$, and

$$\mathbf{m}_{\leq_{\text{uni}}} = \tau_{0,1}(\mathbf{m}_{\leq_{\text{alt}}}) = 10111010101110111011101010111010 \dots$$

This is the fixed point of the period-doubling (or Feigenbaum) substitution $0 \mapsto 11, 1 \mapsto 10$. The set $\mathcal{M}_{\leq_{\text{uni}}}$ is the set of kneading sequences of unimodal maps [9, 19].

We define the *flipped unimodal order* by $[w0] <_{\text{flip}} [w1]$ if $|w|_0$ is even, $[w0] >_{\text{flip}} [w1]$ if $|w|_0$ is odd, where $|w|_0$ denotes the number of occurrences of 0 in $w \in \{0, 1\}^*$. Then we have $[11] >_{\text{flip}} [10]$, $[101] <_{\text{flip}} [100]$, $[1001] >_{\text{flip}} [1000]$, $[10001] >_{\text{flip}} [10000]$, and

$$\mathbf{m}_{\leq_{\text{flip}}} = \tau_{1,3}(\mathbf{m}_{\leq_{\text{alt}}}) = 100010101000100010001010100010101000101010001000 \dots$$

Note that $0\mathbf{m}_{\leq_{\text{flip}}} = F(\mathbf{m}_{\leq_{\text{uni}}})$, where $F(a_1 a_2 \dots) := (1 - a_1)(1 - a_2) \dots$, and we have

$$\mathcal{M}_{\leq_{\text{uni}}} = 1F(\mathcal{M}_{\leq_{\text{flip}}}) \cup \{0^\infty\}.$$

4.4 | Sturmian sequences

The set of substitutions $\{\theta_k : k \geq 1\}$ defined by $\theta_k(0) = 0^{k-1}1, \theta_k(1) = 0^{k-1}10$, generates the standard Sturmian words; see [18, Corollary 2.2.22]. Since $\tau_{k-1,k}$ is rotationally conjugate to θ_k , more precisely $\theta_k(w)0^{k-1} = 0^{k-1}\tau_{k-1,k}(w)$ for all $w \in \{0, 1\}^*$, the set of substitutions $\{\tau_{k-1,k} : k \geq 1\}$ generates the same shifts as $\{\theta_k : k \geq 1\}$. Therefore, the limit words of sequences in $\{\tau_{k-1,k} : k \geq 1\}^\infty$ provide elements of all Sturmian shifts. For example, the limit word of the sequence $(\tau_{0,1})^\infty$ is the Fibonacci word. Note that Sturmian sequences that are sup-words are also considered in [4].

5 | SYMMETRIC ALPHABETS

For a real number $q > 1$, the set

$$\tilde{\mathcal{L}}_q := \left\{ \limsup_{n \rightarrow \infty} \|xq^n\| : x \in \mathbb{R} \right\}, \tag{9}$$

where $\|\cdot\|$ denotes the distance to the nearest integer, is a multiplicative version of the Lagrange spectrum, and was studied in [2, 12]. If q is an integer, then representing $x = \sum_{k=-\infty}^\infty a_k q^{-k}$ with $a_k \in \mathbb{Z}, a_k \neq 0$ for finitely many $k \leq 0, |\sum_{k=n+1}^\infty a_k q^{n-k}| \leq 1/2$, gives that $\|xq^n\| = |\sum_{k=n+1}^\infty a_k q^{n-k}|$; see also Proposition 10 below. This leads us to consider

$$\mathcal{M}_{\leq}^{\text{abs}} = \{s_{\leq}^{\text{abs}}(\mathbf{a}) : \mathbf{a} \in \{0, \pm 1\}^\infty\} \quad \text{with} \quad s_{\leq}^{\text{abs}}(a_1 a_2 \dots) = \sup_{n \geq 1} \text{abs}(a_n a_{n+1} \dots),$$

where

$$\text{abs}(\mathbf{a}) = \begin{cases} \mathbf{a} & \text{if } \mathbf{a} \geq_{\text{lex}} 0^\infty, \\ -\mathbf{a} & \text{if } \mathbf{a} \leq_{\text{lex}} 0^\infty, \end{cases} \quad \text{with } -(a_1 a_2 \dots) = (-a_1)(-a_2) \dots.$$

We denote the smallest accumulation point of $\mathcal{M}_{\leq}^{\text{abs}}$ by $\mathbf{m}_{\leq}^{\text{abs}}$.

The same proof as for Theorem 1 shows for all cylinder orders \leq on $\{0, \pm 1\}^\infty$ that

$$\mathcal{L}_{\leq}^{\text{abs}} = \mathcal{M}_{\leq}^{\text{abs}} = \text{closure}(\{s_{\leq}^{\text{abs}}(\mathbf{a}) : \mathbf{a} \in \{0, \pm 1\}^\infty \text{ purely periodic}\}),$$

where $\mathcal{L}_{\leq}^{\text{abs}} := \{\limsup_{n \geq 1} \text{abs}(a_n a_{n+1} \dots) : a_1 a_2 \dots \in \{0, \pm 1\}^\infty\}$.

In the following, we assume that a cylinder order on $\{0, \pm 1\}^\infty$ is *consistent* (with the natural order on $\{0, \pm 1\}$), which means that, for each $w \in \{0, \pm 1\}^*$, we have $[w(-1)] < [w0] < [w1]$ or $[w(-1)] > [w0] > [w1]$. In order to describe $\mathbf{m}_{\leq}^{\text{abs}}$, we define maps $\varrho_0, \varrho_1, \varrho_2$ from $\{0, 1\}^*$ to $\{0, \pm 1\}^*$ by $\varrho_0(\varepsilon) = \varrho_1(\varepsilon) = \varrho_2(\varepsilon) = \varepsilon$ for the empty word ε , and

$$\begin{aligned} \varrho_0(w0) &= \begin{cases} \varrho_0(w)1 & \text{if } |w|_0 \text{ is even,} \\ \varrho_0(w)(-1) & \text{if } |w|_0 \text{ is odd,} \end{cases} & \varrho_0(w1) &= \begin{cases} \varrho_0(w)10 & \text{if } |w|_0 \text{ is even,} \\ \varrho_0(w)(-1)0 & \text{if } |w|_0 \text{ is odd,} \end{cases} \\ \varrho_1(w0) &= \begin{cases} \varrho_1(w)1 & \text{if } |w|_1 \text{ is even,} \\ \varrho_1(w)(-1) & \text{if } |w|_1 \text{ is odd,} \end{cases} & \varrho_1(w1) &= \begin{cases} \varrho_1(w)10 & \text{if } |w|_1 \text{ is even,} \\ \varrho_1(w)(-1)0 & \text{if } |w|_1 \text{ is odd,} \end{cases} \\ \varrho_2(w0) &= \begin{cases} \varrho_2(w)1 & \text{if } |w| \text{ is even,} \\ \varrho_2(w)(-1) & \text{if } |w| \text{ is odd,} \end{cases} & \varrho_2(w1) &= \begin{cases} \varrho_2(w)10 & \text{if } |w| \text{ is even,} \\ \varrho_2(w)(-1)0 & \text{if } |w| \text{ is odd,} \end{cases} \end{aligned}$$

for all $w \in \{0, 1\}^*$, where $|w|$ is the length of a word w and $|w|_i$ the number of occurrences of the letter i in w . As for substitutions, the maps ϱ_i are extended naturally to $\{0, 1\}^\infty$.

Theorem 8. *Let $\mathbf{m} \in \{0, \pm 1\}^\infty$. Then $\mathbf{m} = \mathbf{m}_{\leq}^{\text{abs}}$ for some consistent cylinder order \leq on $\{0, \pm 1\}^\infty$ with $0^\infty < 1^\infty$ if and only if $\mathbf{m} = \sigma(\mathbf{m}_{\leq})$ for some $\sigma \in \{\varrho_0, \varrho_1, \varrho_2, \tau_{0,1}\}$ and some cylinder order \leq on $\{0, 1\}^\infty$ with $0^\infty < 1^\infty$.*

If $\mathbf{m}_{\leq}^{\text{abs}} = \sigma(\mathbf{m}_{\leq})$, then we can assume that $\mathbf{a} \leq \mathbf{b}$ if and only if $\sigma(\mathbf{a}) \leq \sigma(\mathbf{b})$, and we have

$$\{\mathbf{a} \in \mathcal{M}_{\leq}^{\text{abs}} : \mathbf{a} < \mathbf{m}_{\leq}^{\text{abs}}\} = \{0^\infty\} \cup \{\sigma(\mathbf{a}) : \mathbf{a} \in \mathcal{M}_{\leq}, \mathbf{a} < \mathbf{m}_{\leq}\}. \tag{10}$$

Proof. Let first \leq be a consistent cylinder order on $\{0, \pm 1\}^\infty$. Then, the order \leq defined by $\mathbf{a} \leq \mathbf{b}$ if $\sigma(\mathbf{a}) \leq \sigma(\mathbf{b})$ is a cylinder order for all $\sigma \in \{\varrho_0, \varrho_1, \varrho_2, \tau_{0,1}\}$. Indeed, for any $w \in \{0, 1\}^*$, we have $\sigma([w0]) \subset [\sigma(w)xy]$, $\sigma([w1]) \subset [\sigma(w)x0]$, where $x, y \in \{\pm 1\}$.

Assume first that $[1\bar{1}] < [10]$; here and in the following, we use the notation $\bar{1} = -1$. Then, $\mathcal{M}_{\leq}^{\text{abs}} \cap [1\bar{1}] = \{(1\bar{1})^\infty\}$, thus $\mathbf{m}_{\leq}^{\text{abs}} \in [10]$. If $[10\bar{1}] < [100]$, then each 1 in a word in $\mathcal{M}_{\leq}^{\text{abs}} \cap [10\bar{1}]$ is followed by $\bar{1}$ or $0\bar{1}$, and each $\bar{1}$ is followed by 1 or $0\bar{1}$, that is,

$$\mathcal{M}_{\leq}^{\text{abs}} \cap [10\bar{1}] \subseteq 10(\{\bar{1}, \bar{1}0\}1, 10)^\infty = \varrho_2([1]).$$

Therefore, $\mathbf{m}_{\leq}^{\text{abs}} = \varrho_2(\mathbf{m}_{\leq})$ for the cylinder order \leq defined by $\mathbf{a} \leq \mathbf{b}$ if $\varrho_2(\mathbf{a}) \leq \varrho_2(\mathbf{b})$. If $[101] < [100]$, then each 1 in a word in $\mathcal{M}_{\leq}^{\text{abs}} \cap [101]$ is followed by $\bar{1}$ or $0\bar{1}$, and each $\bar{1}$ is followed by 1 or

$0\bar{1}$, that is,

$$\begin{aligned} \mathcal{M}_{\leq}^{\text{abs}} \cap [101] &\subseteq 10((10)^*1(\bar{1}0)^*\bar{1})^\infty \cup 10((10)^*1(\bar{1}0)^*\bar{1})^*(10)^\infty \cup 10((10)^*1(\bar{1}0)^*\bar{1})^*(10)^*1(\bar{1}0)^\infty \\ &= \varrho_0(1(1^*01^*0)^\infty) \cup \varrho_0(1(1^*01^*0)^*1^\infty) \cup \varrho_0(1(1^*01^*0)^*1^*01^\infty) = \varrho_0([1]). \end{aligned}$$

Therefore, we have $\mathbf{m}_{\leq}^{\text{abs}} = \varrho_0(\mathbf{m}_{\leq})$ for the cylinder order \leq defined by $\mathbf{a} \leq \mathbf{b}$ if $\varrho_0(\mathbf{a}) \leq \varrho_0(\mathbf{b})$. Assume now $[11] < [10]$. Then $\mathcal{M}_{\leq}^{\text{abs}} \cap [11] = \{1^\infty\}$, thus $\mathbf{m}_{\leq} \in [10]$. If $[10\bar{1}] < [100]$, then

$$\mathcal{M}_{\leq}^{\text{abs}} \cap [10\bar{1}] \subseteq 10(\bar{1}^*\bar{1}01^*10)^\infty \cup 10(\bar{1}^*\bar{1}01^*10)^*\bar{1}^\infty \cup 10(\bar{1}^*\bar{1}01^*10)^*\bar{1}^*\bar{1}01^\infty = \varrho_1([1]),$$

thus $\mathbf{m}_{\leq}^{\text{abs}} = \varrho_1(\mathbf{m}_{\leq})$, with \leq defined by $\mathbf{a} \leq \mathbf{b}$ if $\varrho_1(\mathbf{a}) \leq \varrho_1(\mathbf{b})$. If $[101] < [100]$, then

$$\mathcal{M}_{\leq}^{\text{abs}} \cap [101] \subseteq 10\{1, 10\}^\infty = \tau_{0,1}([1]),$$

thus $\mathbf{m}_{\leq}^{\text{abs}} = \tau_{0,1}(\mathbf{m}_{\leq})$ for the cylinder order \leq defined by $\mathbf{a} \leq \mathbf{b}$ if $\tau_{0,1}(\mathbf{a}) \leq \tau_{0,1}(\mathbf{b})$. Since $\varrho_0(0^\infty) = \varrho_2(0^\infty) = (1\bar{1})^\infty$ and $\varrho_1(0^\infty) = \tau_{0,1}(0^\infty) = 1^\infty$, Equation (10) holds.

Let now \leq be a cylinder order on $\{0, 1\}^\infty$ with $0^\infty < 1^\infty$ and $\sigma \in \{\varrho_0, \varrho_1, \varrho_2, \tau_{0,1}\}$. Then, there exists a consistent cylinder order \leq on $\{0, \pm 1\}^\infty$ satisfying $\sigma(\mathbf{a}) \leq \sigma(\mathbf{b})$ if $\mathbf{a} \leq \mathbf{b}$ and $0^\infty < 1^\infty$. Indeed, for $w \in \{0, \pm 1\}^*$ and distinct $a, b \in \{0, \pm 1\}$, we set $[wa] < [wb]$ if $\mathbf{a}' < \mathbf{b}'$ for some $\mathbf{a}', \mathbf{b}' \in \{0, 1\}^\infty$ with $\sigma(\mathbf{a}') \in [wa]$, $\sigma(\mathbf{b}') \in [wb]$; by Lemma 2, this does not depend on the choice of \mathbf{a}', \mathbf{b}' . Moreover, since $\sigma(\mathbf{a}) \in [w\bar{1}]$ and $\sigma(\mathbf{b}) \in [w1]$ is impossible, we have no obstruction to a consistent cylinder order. Since $0^\infty < 1^\infty$, we have $[1\bar{1}] < [10]$ in case $\sigma \in \{\varrho_0, \varrho_2\}$, $[11] < [10]$ in case $\sigma \in \{\varrho_1, \tau_{0,1}\}$. We can set $[10\bar{1}] < [100]$ in case $\sigma \in \{\varrho_1, \varrho_2\}$ because $([100] \cup [101]) \cap \sigma(\{0, 1\}^\infty) = \emptyset$; similarly, we can set $[101] < [100]$ in case $\sigma \in \{\varrho_0, \tau_{0,1}\}$. Then, we have $\mathbf{m}_{\leq}^{\text{abs}} = \sigma(\mathbf{m}_{\leq})$. \square

Proposition 9. *Let $\beta \geq 3$ be a Pisot or Salem number, and let \leq be a consistent cylinder order on $\{0, \pm 1\}^\infty$. Then $\pi_\beta(\mathbf{m}_{\leq}^{\text{abs}})$ is in $\mathbb{Q}(\beta)$ or transcendental.*

Proof. Let $G(a_1 a_2 \dots) = |a_1| |a_2| \dots$. Then $G \circ \varrho_i = \tau_{0,1}$ for all $i \in \{0, 1, 2\}$, thus $p_{G(\mathbf{m}_{\leq}^{\text{abs}})}(n) \leq 3n - 2$ by Proposition 6, Theorems 3 and 8. Moreover, the map G is 2-to-1 from the set of factors of $\mathbf{m}_{\leq}^{\text{abs}}$ to the set of factors of $G(\mathbf{m}_{\leq}^{\text{abs}})$, thus $p_{\mathbf{m}_{\leq}^{\text{abs}}}(n) \leq 6n - 4$. By [1, Theorem 1A] and by adding 1 to each digit of $\mathbf{m}_{\leq}^{\text{abs}}$, we obtain that $\pi_\beta(\mathbf{m}_{\leq}^{\text{abs}}) + \frac{1}{\beta - 1}$ is in $\mathbb{Q}(\beta)$ or transcendental; thus, also $\pi_\beta(\mathbf{m}_{\leq}^{\text{abs}})$ is in $\mathbb{Q}(\beta)$ or transcendental. \square

6 | EXAMPLES OF ORDERS ON SYMMETRIC SHIFT SPACES

6.1 | Lexicographic order

For the lexicographic order on $\{0, \pm 1\}^\infty$ (with $-1 < 0 < 1$), we have $[1\bar{1}] < [10]$, $[10\bar{1}] < [100]$, and we obtain that

$$\mathbf{m}_{\leq_{\text{lex}}}^{\text{abs}} = \varrho_2(\mathbf{m}_{\leq_{\text{alt}}}) = 10\bar{1}\bar{1}\bar{1}010\bar{1}0\bar{1}\bar{1}10\bar{1}\bar{1}\bar{1}01\bar{1}\bar{1}10\bar{1}010\bar{1}\bar{1}\bar{1}010 \dots$$

The following proposition relates the Lagrange spectrum $\tilde{\mathcal{L}}_q$, defined in (9), and its smallest accumulation point $\tilde{\mathbf{m}}_q$ to $\mathcal{M}_{\leq \text{lex}}^{\text{abs}}$ and $\mathbf{m}_{\leq \text{lex}}^{\text{abs}}$; it slightly improves results of [2, 12].

Proposition 10. *We have*

$$\tilde{\mathcal{L}}_2 = \pi_2(\mathcal{M}_{\leq \text{lex}}^{\text{abs}}) \cap \left[0, \frac{1}{2}\right] \neq \pi_2(\mathcal{M}_{\leq \text{lex}}^{\text{abs}}), \quad \tilde{\mathcal{L}}_3 = \pi_3(\mathcal{M}_{\leq \text{lex}}^{\text{abs}}), \tag{11}$$

$$\pi_q(\mathcal{M}_{\leq \text{lex}}^{\text{abs}}) = \tilde{\mathcal{L}}_q \cap \left[0, \frac{1}{q-1}\right] \neq \tilde{\mathcal{L}}_q \quad \text{for all integers } q \geq 4. \tag{12}$$

For all integers $q \geq 2$, we have $\tilde{\mathbf{m}}_q = \pi_q(\mathbf{m}_{\leq \text{lex}}^{\text{abs}}) = \pi_q(\varrho_2(\mathbf{m}_{\leq \text{alt}}))$ and

$$\tilde{\mathcal{L}}_q \cap [0, \tilde{\mathbf{m}}_q) = \pi_q(\mathcal{M}_{\leq \text{lex}}^{\text{abs}}) \cap [0, \tilde{\mathbf{m}}_q) = \{0\} \cup \{\pi_q(\varrho_2(\tau_{0,2}^n(0^\infty))) : n \geq 0\}.$$

Proof. As mentioned at the beginning of Section 5, for integer $q \geq 2$, $\|xq^n\|$ can be determined by a symmetric q -expansion of x . We can assume w.l.o.g. $|x| \leq \frac{1}{2}$. Let

$$\mathcal{A}_q := \{a_1 a_2 \dots \in A_q^\infty : |\pi_q(a_k a_{k+1} \dots)| \leq \frac{1}{2} \text{ for all } k \geq 1\}, \text{ with } A_q := \{0, \pm 1, \dots, \pm \lfloor q/2 \rfloor\}.$$

For each $x \in [-\frac{1}{2}, \frac{1}{2}]$, we obtain a sequence $\mathbf{a} = a_1 a_2 \dots \in \mathcal{A}_q$ satisfying $x = \pi_q(\mathbf{a})$ by taking $a_k = \lfloor q\tilde{T}_q^{k-1}(x) + \frac{1}{2} \rfloor$ where $\tilde{T}_q(y) := qy - \lfloor qy + \frac{1}{2} \rfloor$. Then,

$$\|xq^n\| = |\pi_q(a_{n+1} a_{n+2} \dots)| = \pi_q(\text{abs}(a_{n+1} a_{n+2} \dots)).$$

Note that $\mathbf{a} \leq_{\text{lex}} \mathbf{b}$ implies $\pi_q(\mathbf{a}) \leq \pi_q(\mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathcal{A}_q$. Since $\mathcal{L}_{\leq \text{lex}}^{\text{abs}} = \mathcal{M}_{\leq \text{lex}}^{\text{abs}}$ (and a similar relation holds for larger alphabets), we obtain that

$$\tilde{\mathcal{L}}_q = \{\pi_q(s_{\leq \text{lex}}^{\text{abs}}(\mathbf{a})) : \mathbf{a} \in \mathcal{A}_q\}.$$

For $q \in \{2, 3\}$, we have $A_q = \{0, \pm 1\}$, thus $\tilde{\mathcal{L}}_q \subseteq \pi_q(\mathcal{M}_{\leq \text{lex}}^{\text{abs}})$. For $q \geq 3$, we have $\{0, \pm 1\}^\infty \subseteq \mathcal{A}_q$, thus $\pi_q(\mathcal{M}_{\leq \text{lex}}^{\text{abs}}) \subseteq \tilde{\mathcal{L}}_q$. Since $\pi_2(\mathcal{M}_{\leq \text{lex}}^{\text{abs}}) \cap [0, \frac{1}{2}] \subseteq \tilde{\mathcal{L}}_2$ and $1 = \pi_2(1^\infty) \in \pi_2(\mathcal{M}_{\leq \text{lex}}^{\text{abs}}) \setminus \tilde{\mathcal{L}}_2$, this proves (11). For $q \geq 4$, we have $\pi_q(s_{\leq \text{lex}}^{\text{abs}}(\mathbf{a})) \geq \pi_q((2\bar{2})^\infty) = \frac{2}{q+1} > \frac{1}{q-1}$ for all $\mathbf{a} \in A_q^\infty \setminus \{0, \pm 1\}^\infty$, thus $\tilde{\mathcal{L}}_q \cap [0, \frac{1}{q-1}] \subseteq \pi_q(\mathcal{M}_{\leq \text{lex}}^{\text{abs}})$. Together with $\frac{2}{q+1} \in \tilde{\mathcal{L}}_q \setminus \pi_q(\mathcal{M}_{\leq \text{lex}}^{\text{abs}})$, we obtain (12). Since π_q is order-preserving on \mathcal{A}_q , we obtain that $\tilde{\mathbf{m}}_q = \pi_q(\mathbf{m}_{\leq \text{lex}}^{\text{abs}})$ and that $\tilde{\mathcal{L}}_q$ and $\pi_q(\mathcal{M}_{\leq \text{lex}}^{\text{abs}})$ agree on $[0, \tilde{\mathbf{m}}_q)$. Since $\{\mathbf{a} \in \mathcal{M}_{\leq \text{lex}}^{\text{abs}} : \mathbf{a} < \mathbf{m}_{\leq \text{lex}}^{\text{abs}}\}$ is equal to $\{0^\infty\} \cup \{\varrho_2(\tau_{0,2}^n(0^\infty)) : n \geq 0\}$ by Theorem 8 and (8), this completes the proof of the proposition. \square

6.2 | Alternating lexicographic order

For the alternating lexicographic order on $\{0, \pm 1\}^\infty$ (with $-1 < 0 < 1$), we have $[11] <_{\text{alt}} [10]$ and $[10\bar{1}] <_{\text{alt}} [100]$,

$$\mathbf{m}_{\leq \text{alt}}^{\text{abs}} = \varrho_1(\mathbf{m}_{\leq \text{alt}}) = 10\bar{1}\bar{1}\bar{1}010\bar{1}01110\bar{1}\bar{1}\bar{1}01110\bar{1}0101\bar{1}\bar{1}\bar{1}010 \dots.$$

6.3 | Bimodal order

Similarly to the unimodal order, we define the *bimodal order* on $\{0, \pm 1\}^\infty$ by $[wa] <_{bi} [wb]$ if $a < b$ (with $-1 < 0 < 1$) and $|w|_1 + |w|_{-1}$ is even, or $a > b$ and $|w|_1 + |w|_{-1}$ is odd. Then $\mathbf{m}_{\leq_{bi}}^{abs} = \tau_{0,1}(\mathbf{m}_{\leq_{alt}}) = \mathbf{m}_{\leq_{uni}}$. We get the same result for the order defined by $[wa] < [wb]$ if $a < b$ and $|w|_1$ is even, or $a > b$ and $|w|_1$ is odd.

We also define the *flipped bimodal order* on $\{0, \pm 1\}^\infty$ by $[wa] <_{biflip} [wb]$ if $a < b$ and $|w|_0$ is even, or $a > b$ and $|w|_0$ is odd. Then

$$\mathbf{m}_{\leq_{biflip}}^{abs} = \varrho_0(\mathbf{m}_{\leq_{alt}}) = 101\bar{1}1010101\bar{1}101\bar{1}101\bar{1}1010101\bar{1}1010 \dots$$

6.4 | Other orders

For $\mathbf{e} \in \{\pm 1\}^\infty$, we define a cylinder order \leq_e on $\{0, \pm 1\}^\infty$ by

$$\mathbf{a} \leq_e \mathbf{b} \text{ if } \mathbf{e} \cdot \mathbf{a} \leq_{lex} \mathbf{e} \cdot \mathbf{b},$$

where $(e_1 e_2 \dots) \cdot (a_1 a_2 \dots) = (e_1 a_1)(e_2 a_2) \dots$. We know from Proposition 7 that $\pi_\beta(\mathbf{m}_{\leq_e}^{abs})$ is in $\mathbb{Q}(\beta)$ or transcendental for all Pisot or Salem numbers β . However, here the value of $\pi_\beta(\mathbf{e} \cdot \mathbf{m}_{\leq_e}^{abs})$, which is the smallest accumulation point of $\{\limsup_{n \rightarrow \infty} |\sum_{k=n+1}^\infty \frac{e_k a_k}{\beta^{k-n}}| : a_1 a_2 \dots \in \{0, \pm 1\}^\infty\}$ when $e_1 = 1$ and $\beta \geq 3$, is more relevant. If \mathbf{e} is periodic with period k , then $p_{\mathbf{e} \cdot \mathbf{a}}(n) \leq k p_{\mathbf{a}}(n)$, and hence, $\pi_\beta(\mathbf{e} \cdot \mathbf{m}_{\leq_e}^{abs})$ is also in $\mathbb{Q}(\beta)$ or transcendental for all Pisot or Salem numbers β . We do not know whether the same result holds when \mathbf{e} is aperiodic.

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