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Markoff–Lagrange spectrum of one-sided shifts

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Abstract

For the Lagrange spectrum and other applications, we determine the smallest accumulation point of binary sequences that are maximal in their shift orbits. This problem is trivial for the lexicographic order, and its solution is the fixed point of a substitution for the alternating lexicographic order. For orders defined by cylinders, we show that the solutions are S-adic sequences, where S is a certain infinite set of substitutions that contains Sturmian morphisms. We also consider a similar problem for symmetric ternary shifts, which is applicable to the multiplicative version of the Markoff–Lagrange spectrum.

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1 | INTRODUCTION

Let A^∞ be the set of infinite words on a finite alphabet A, equipped with a total order $\le$ and the ultrametric d given by $d(a_1a_2\cdots, b_1b_2\cdots) = 2^{-\min\{n\geqslant 1\,:\,a_n\neq b_n\}}$ for $a_1a_2\cdots \neq b_1b_2\cdots$. We study properties of the set of *sup*-words

$$\mathcal{M}_\leqslant := \{s_\leqslant(\mathbf{a})\,:\,\mathbf{a}\in A^\infty\},\quad \text{with}\quad s_\leqslant(a_1a_2\cdots) := \sup_{n\geqslant 1} a_na_{n+1}\cdots,$$

for a large class of orders on A^∞. In particular, we are interested in the smallest accumulation point $\mathbf{m}_\le$ of $\mathcal{M}_\le$. For the lexicographic order $\leqslant_{\mathrm{lex}}$, words in $\mathcal{M}_{\leqslant_{\mathrm{lex}}}$ occur as (quasi-greedy) β-expansions of 1 for real bases $\beta>1$ (see [20]), with $\mathbf{m}_{\leqslant_{\mathrm{lex}}} = 1000\cdots$ being the limit of these expansions as $\beta\to 1$. For the alternating lexicographic order $\leqslant_{\mathrm{alt}}$, most elements of $\mathcal{M}_{\leqslant_{\mathrm{alt}}}$ are $(-\beta)$-expansions

of $\frac{-\beta}{\beta+1}$ in the sense of [16, 24], and again, $\mathbf{m}_{\leqslant_{\text{alt}}}$ is the limit of these expansions as $\beta \to 1$. An image of $\mathcal{M}_{\leqslant_{\text{alt}}}$ occurs in a multiplicative version of the (Markoff–)Lagrange spectrum w.r.t. an integer base, which is defined in terms of well approximable numbers [2, 12]; see Proposition 10. Below the image of $\mathbf{m}_{\leqslant_{\text{alt}}}$, which is the fixed point of a substitution [3, 5, 13], we find the discrete part of this spectrum. The classical Markoff and Lagrange spectra are given by two-sided versions of $\mathcal{M}_{\leqslant_{\text{alt}}}$ (and the Lagrange spectrum is defined by lim sup instead of sup). The unimodal order $\leqslant_{\text{uni}}$ yields kneading sequences of unimodal maps [19], and $\mathbf{m}_{\leqslant_{\text{uni}}}$ is the fixed point of the period-doubling (or Feigenbaum) substitution. Sup-words are also closely related to infinite Lyndon words, which are defined by $a_1a_2\cdots < a_na_{n+1}\cdots$ for all $n \geqslant 2$; see [23].

We consider orders satisfying that

$$\mathbf{a} \leqslant \mathbf{b} \leqslant \mathbf{c} \quad \text{implies} \quad d(\mathbf{a},\mathbf{b}) \leqslant d(\mathbf{a},\mathbf{c}) \quad \text{for all } \mathbf{a},\mathbf{b},\mathbf{c} \in A^\infty \tag{1}$$

(note that $d(\mathbf{a},\mathbf{b}) \leqslant d(\mathbf{a},\mathbf{c})$ is equivalent to $d(\mathbf{b},\mathbf{c}) \leqslant d(\mathbf{a},\mathbf{c})$ by the strong triangle inequality), and we call them *cylinder orders* because the elements of each cylinder of words are contiguous. Here, the *cylinder* (of length n) given by $a_1\cdots a_n \in A^n$ is

$$[a_1\cdots a_n] := \{a'_1a'_2\cdots \in A^\infty \ : \ a'_1\cdots a'_n = a_1\cdots a_n\},$$

and we write $[a_1\cdots a_n] < [b_1\cdots b_n]$ if $(a_1\cdots a_n)^\infty < (b_1\cdots b_n)^\infty$ (or, equivalently, $\mathbf{a} < \mathbf{b}$ for all $\mathbf{a} \in [a_1\cdots a_n]$, $\mathbf{b} \in [b_1\cdots b_n]$), where w^∞ denotes the infinite repetition of a word w. Note that (1) is equivalent to the condition (*) of [21], which states that $ux < vy$ for all $x, y \in A^\infty, u, v \in A^n, n \geqslant 1$, such that $u^\infty < v^\infty$. Generalized lexicographic orders as considered in [22] satisfy this condition.

In Section 2, we give basic properties of $\mathcal{M}_{\leqslant}$. Section 3 contains our main result, an algorithm for determining the smallest accumulation point $\mathbf{m}_{\leqslant}$ of $\mathcal{M}_{\leqslant}$, for any cylinder order $(A^\infty, \leqslant)$. We also give a complete description, in terms of S-adic sequences, of all words $\mathbf{m}_{\leqslant}$ obtained by cylinder orders and of the discrete part of $\mathcal{M}_{\leqslant}$, that is, all its elements below $\mathbf{m}_{\leqslant}$. Since the words $\mathbf{m}_{\leqslant}$ have linear factor complexity, real numbers having such β-expansions are in $\mathbb{Q}(\beta)$ or transcendental, for all Pisot or Salem bases $\beta \geqslant 2$. In Section 4, we determine $\mathbf{m}_{\leqslant}$ for some classical examples of cylinder orders and show that all Sturmian sequences that are sup-words can occur. We consider cylinder orders on symmetric alphabets in Section 5 and apply our results to the multiplicative Lagrange spectrum and other problems in Section 6,

2 | MARKOFF–LAGRANGE SPECTRUM

We first show that the set of periodic words in $\mathcal{M}_{\leqslant}$ is dense, and that $\mathcal{M}_{\leqslant}$ is equal to

$$\mathcal{L}_{\leqslant} := \{\ell_{\leqslant}(\mathbf{a}) \ : \ \mathbf{a} \in A^\infty\}, \quad \text{with} \quad \ell_{\leqslant}(a_1a_2\cdots) := \limsup_{n\to\infty} a_na_{n+1}\cdots.$$

Note that $\mathcal{M}_{\leqslant}$ and $\mathcal{L}_{\leqslant}$ can be seen as generalizations of the *Markoff* and *Lagrange spectrum*, respectively. In the classical case, these spectra are defined by two-sided sequences, and the Lagrange spectrum is a strict subset of the Markoff spectrum [11, 14].

Theorem 1. *Let $\leq$ be a cylinder order on A^∞. Then*

$$\mathcal{L}_{\leqslant} = \mathcal{M}_{\leqslant} = \text{closure}(\{s_{\leqslant}(\mathbf{a}) \ : \ \mathbf{a} \in A^\infty \textit{ purely periodic}\}).$$

In particular, the set $\mathcal{M}_{\leqslant}$ is closed.

In the proof of the theorem, we use the following characterization of cylinder orders.

Lemma 2. *An order $\leq$ on A^∞ is a cylinder order if and only if*

$$\mathbf{a} \leqslant \mathbf{b} \text{ implies } \mathbf{a}' \leqslant \mathbf{b}' \quad \text{for all } \mathbf{a},\mathbf{b},\mathbf{a}',\mathbf{b}' \in A^\infty \text{ with } \max(d(\mathbf{a},\mathbf{a}'),d(\mathbf{b},\mathbf{b}')) < d(\mathbf{a},\mathbf{b}). \tag{2}$$

Proof. Let $\leq$ be a cylinder order, and $\mathbf{a},\mathbf{b},\mathbf{a}',\mathbf{b}' \in A^\infty$ such that $\mathbf{a} \leqslant \mathbf{b}$, $d(\mathbf{a},\mathbf{a}') < d(\mathbf{a},\mathbf{b})$, $d(\mathbf{b},\mathbf{b}') < d(\mathbf{a},\mathbf{b})$. Then both $\mathbf{a} \leqslant \mathbf{b} \leqslant \mathbf{a}'$ and $\mathbf{b}' \leqslant \mathbf{a}' \leqslant \mathbf{b}$ are impossible by (1), using that $d(\mathbf{a}',\mathbf{b}') = d(\mathbf{a},\mathbf{b})$ by the strong triangle inequality. This implies that $\mathbf{b} > \mathbf{a}'$ and thus $\mathbf{b}' > \mathbf{a}'$, that is, (2) holds. Let now $\leq$ be an order satisfying (2). Then, $\mathbf{a} \leqslant \mathbf{b}$ and $d(\mathbf{a},\mathbf{c}) < d(\mathbf{a},\mathbf{b})$ imply that $\mathbf{c} < \mathbf{b}$, thus $\mathbf{a} \leqslant \mathbf{b} \leqslant \mathbf{c}$ with $d(\mathbf{a},\mathbf{c}) < d(\mathbf{a},\mathbf{b})$ is impossible, that is, (1) holds. □

Proof of Theorem 1. We show first that $\mathcal{M}_\leq$ is the closure of $s_\leqslant(\mathbf{a}')$ with purely periodic $\mathbf{a}' \in A^\infty$. Since $s_\leqslant(s_\leqslant(\mathbf{a})) = s_\leqslant(\mathbf{a})$, it suffices to consider $\mathbf{a} \in A^\infty$ with $s_\leqslant(\mathbf{a}) = \mathbf{a}$. We have $(a_1 \cdots a_n)^\infty \in \mathcal{M}_\leq$ whenever

$$2^{-n} d(\mathbf{a}, a_{n+1}a_{n+2}\cdots) < 2^{-i} d(\mathbf{a}, a_{i+1}a_{i+2}\cdots) \quad \text{for all } i \in [1,n). \tag{3}$$

Indeed, we have, for all $i \in [1,n)$, that $a_{i+1}a_{i+2}\cdots \leqslant \mathbf{a}$ (because $s_\leqslant(\mathbf{a}) = \mathbf{a}$) and

$$\begin{aligned} d(\mathbf{a}, a_{i+1}a_{i+2}\cdots) &> 2^{i-n} d(\mathbf{a}, a_{n+1}a_{n+2}\cdots) = 2^{i-n} d((a_1\cdots a_n)^\infty, a_{n+1}a_{n+2}\cdots) \\ &= 2^i d((a_1\cdots a_n)^\infty, \mathbf{a}) = d(a_{i+1}\cdots a_n(a_1\cdots a_n)^\infty, a_{i+1}a_{i+2}\cdots), \end{aligned}$$

hence (2) gives that $a_{i+1}\cdots a_n(a_1\cdots a_n)^\infty \leqslant (a_1\cdots a_n)^\infty$, thus $(a_1\cdots a_n)^\infty \in \mathcal{M}_\leq$. Since $\lim_{n\to\infty} 2^{-n} d(\mathbf{a}, a_{n+1}a_{n+2}\cdots) = 0$, we have either $d(\mathbf{a}, a_{n+1}a_{n+2}\cdots) = 0$ for some $n \geqslant 1$, that is, $\mathbf{a}$ is purely periodic, or there are infinitely many n such that (3) holds and thus $(a_1\cdots a_n)^\infty \in \mathcal{M}_\leq$ infinitely often. Therefore, $\mathcal{M}_\leq$ is contained in the closure of $\{s_\leqslant(\mathbf{a}) : \mathbf{a} \in A^\infty \text{ purely periodic}\}$. For the opposite inclusion, we have to show that $\mathcal{M}_\leq$ is closed. Consider $\mathbf{a} = \lim_{k\to\infty} \mathbf{a}^{(k)}$ with $\mathbf{a}^{(k)} \in \mathcal{M}_\leq$. If $a_{n+1}a_{n+2}\cdots > \mathbf{a}$ for some $n \geqslant 1$, then we would also have $a^{(k)}_{n+1}a^{(k)}_{n+2}\cdots > \mathbf{a}^{(k)}$ for all large enough k, contradicting that $\mathbf{a}^{(k)} \in \mathcal{M}_\leq$. This implies that $s_\leqslant(\mathbf{a}) = \mathbf{a}$, that is, $\mathbf{a} \in \mathcal{M}_\leq$.

Since $s_\leqslant(\ell_\leqslant(\mathbf{a})) = \ell_\leqslant(\mathbf{a})$ for all $\mathbf{a} \in A^\infty$, we have $\mathcal{L}_\leqslant \subseteq \mathcal{M}_\leq$. For the opposite inclusion, let $\mathbf{a} = \lim_{k\to\infty} \mathbf{a}^{(k)}$ for some purely periodic words $\mathbf{a}^{(k)} \in \mathcal{M}_\leq$, let (p_k) be an increasing sequence satisfying $\mathbf{a}^{(k)} = (a^{(k)}_1 \cdots a^{(k)}_{p_k})^\infty$, and let $\mathbf{b} = a^{(1)}_1 \cdots a^{(1)}_{p_1} a^{(2)}_1 \cdots a^{(2)}_{p_2} \cdots$. Then, $\ell_\leqslant(\mathbf{b}) \geqslant \mathbf{a}$. If $a^{(k)}_{i+1}\cdots a^{(k)}_{p_k} a^{(k+1)}_1 \cdots a^{(k+1)}_{p_{k+1}} \cdots > \mathbf{a}$ for some $k \geqslant 1$, $0 \leqslant i < p_k$, then

$$\delta_{i,k} := d(a^{(k)}_{i+1}\cdots a^{(k)}_{p_k} a^{(k+1)}_1 \cdots a^{(k+1)}_{p_{k+1}} \cdots, \mathbf{a}) \leqslant \max(d(\mathbf{a}^{(k)},\mathbf{a}), d(\mathbf{a}^{(k)},\mathbf{a}^{(k+1)}), 2^{-p_k}). \tag{4}$$

Indeed, for $1 \leqslant j \leqslant p_k - i$, we cannot have $[a^{(k)}_{i+1}\cdots a^{(k)}_{i+j}] > [a_1\cdots a_j] = [a^{(k)}_1 \cdots a^{(k)}_j]$ because this contradicts $\mathbf{a}^{(k)} \in \mathcal{M}_\leq$, thus $\delta_{i,k} = 2^{-j}$ implies $d(\mathbf{a}^{(k)},\mathbf{a}) \geqslant 2^{-j}$; similarly, for $p_k - i < j \leqslant 2p_k - i$,

$$[a^{(k)}_{i+1}\cdots a^{(k)}_{p_k} a^{(k)}_1 \cdots a^{(k)}_{i+j-p_k}] = [a^{(k)}_{i+1}\cdots a^{(k)}_{p_k} a^{(k+1)}_1 \cdots a^{(k+1)}_{i+j-p_k}] > [a_1\cdots a_j] = [a^{(k)}_1 \cdots a^{(k)}_j]$$

is impossible, thus $\delta_{i,k} = 2^{-j}$ implies $d(\mathbf{a}^{(k)},\mathbf{a}^{(k+1)}) \geqslant 2^{p_k - i - j}$ or $d(\mathbf{a}^{(k)},\mathbf{a}) \geqslant 2^{-j}$; hence, we have $\delta_{i,k} < 2^{i-2p_k}$ or $\delta_{i,k} \leqslant 2^{i-p_k} d(\mathbf{a}^{(k)},\mathbf{a}^{(k+1)})$ or $\delta_{i,k} \leqslant d(\mathbf{a}^{(k)},\mathbf{a})$, which implies (4). Since the right-hand side of (4) tends to 0 as $k \to \infty$, we have $\ell_\leqslant(\mathbf{b}) \leqslant \mathbf{a}$, thus $\mathbf{a} = \ell_\leqslant(\mathbf{b}) \in \mathcal{L}_\leq$. □

3 | SMALLEST ACCUMULATION POINT OF $\mathcal{M}_{\preceq}$

For determining the smallest accumulation point $\mathbf{m}_{\leqslant}$ of $\mathcal{M}_{\preceq}$ for a cylinder order $(A^\infty, \leqslant)$, we can restrict to two-letter alphabets, w.l.o.g., $A = \{0, 1\}$. Indeed, if $\{0, 1\} \subseteq A$, $[0] < [1]$, then $\mathcal{M}_{\preceq}$ has the accumulation point 10^∞ (because $(10^n)^\infty \in \mathcal{M}_{\preceq}$ for all $n \geqslant 1$), and we clearly have $s_{\leqslant}(\mathbf{a}) > 10^\infty$ if $\mathbf{a} \in A^\infty$ contains a letter $a_n \in A$ with $[a_n] > [1]$.

We use substitutions (also called word morphisms) and limit words (or S-adic sequences). Let A^* be the monoid of finite words over the alphabet A, with concatenation as operation. A *substitution* $\sigma : A^* \to A^*$ satisfies $\sigma(vw) = \sigma(v)\sigma(w)$ for all $v, w \in A^*$ and is extended naturally to A^∞; it suffices to give $\sigma(a)$ for $a \in A$ to define σ. For a sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\geqslant 1}$ of substitutions on the alphabet A and an infinite word $\mathbf{a} \in A^\infty$, the *limit word* is

$$\boldsymbol{\sigma}(\mathbf{a}) := \lim_{n\to\infty} \sigma_{[1,n]}(\mathbf{a}),$$

if this limit exists; we use the notation $\sigma_{[1,n]} := \sigma_1 \circ \sigma_2 \circ \cdots \circ \sigma_n$ for $n \geqslant 0$, with $\sigma_{[1,0]}$ being the identity map. For a set of substitutions S, we denote the monoid generated by the composition of substitutions in S by S^*. We use the set of substitutions

$$S = \{\tau_{j,k} \ : \ 0 \leqslant j < k\}, \quad \text{with} \quad \begin{matrix} \tau_{j,k} : 0 \mapsto 10^j, \\ 1 \mapsto 10^k. \end{matrix}$$

Our main result is the following characterization of the smallest accumulation point $\mathbf{m}_{\preceq}$ and the discrete part of $\mathcal{M}_{\leqslant}$ for cylinder orders $\preceq$.

Theorem 3. *Let $\mathbf{m} \in \{0, 1\}^\infty$. Then, $\mathbf{m} = \mathbf{m}_{\preceq}$ for some cylinder order $\preceq$ on $\{0, 1\}^\infty$ with $0^\infty < 1^\infty$ if and only if $\mathbf{m} = \sigma(10^\infty)$ for some $\sigma \in S^*$ or $\mathbf{m} = \boldsymbol{\sigma}(1^\infty)$ for some $\boldsymbol{\sigma} \in S^\infty$.*

If $\mathbf{m}_{\leqslant} = \boldsymbol{\sigma}(1^\infty)$, $\boldsymbol{\sigma} = (\sigma_n)_{n\geqslant 1} \in S^\infty$, then

$$\{\mathbf{a} \in \mathcal{M}_{\leqslant} \ : \ \mathbf{a} < \mathbf{m}_{\leqslant}\} = \{(\sigma_{[1,n]}(0))^\infty \ : \ n \geqslant 0\}. \tag{5}$$

If $\mathbf{m}_{\leqslant} = \sigma_{[1,h]}(10^\infty)$, $\sigma_1, \dots, \sigma_h \in S$, $h \geqslant 0$, then

$$\begin{aligned} \{\mathbf{a} \in \mathcal{M}_{\leqslant} \ : \ \mathbf{a} < \mathbf{m}_{\leqslant}\} = {} & \{(\sigma_{[1,n]}(0))^\infty \ : \ 0 \leqslant n \leqslant h\} \\ & \cup \{\sigma_{[1,h]}((10^j)^\infty) \ : \ j \geqslant 0,\ \sigma_{[1,h]}((10^j)^\infty) < \mathbf{m}_{\leqslant}\}, \end{aligned} \tag{6}$$

and there is at most one $j \geqslant 0$ such that $\sigma_{[1,h]}((10^j)^\infty) < \mathbf{m}_{\preceq}$.

The following proposition constitutes the core of the proof of Theorem 3 and provides an algorithm for calculating $\mathbf{m}_{\preceq}$.

Proposition 4. *Let $\preceq$ be a cylinder order on $\{0, 1\}^\infty$ with $[0] < [1]$.*

If $[10^j1] < [10^j0]$ holds for at most one $j \geqslant 0$, then $\mathbf{m}_{\leqslant} = 10^\infty$, and $\mathbf{a} < \mathbf{m}_{\preceq}$ implies that $\mathbf{a} = 0^\infty$ or $\mathbf{a} = (10^j)^\infty$ (in case $[10^j1] < [10^j0]$).

Otherwise, we have $\mathbf{m}_{\leqslant} = \tau_{j,k}(\mathbf{m}_{\preceq})$ for the cylinder order $\preceq$ on $\{0, 1\}^\infty$ defined by $\mathbf{a} \preceq \mathbf{b}$ if $\tau_{j,k}(\mathbf{a}) \leqslant \tau_{j,k}(\mathbf{b})$, where j, k are minimal such that $0 \leqslant j < k$, $[10^j1] < [10^j0]$, $[10^k1] < [10^k0]$; we have $[0] \prec [1]$ and $\{\mathbf{a} \in \mathcal{M}_{\leqslant} \ : \ \mathbf{a} < \mathbf{m}_{\leqslant}\} = \{0^\infty\} \cup \{\tau_{j,k}(\mathbf{a}) \ : \ \mathbf{a} \in \mathcal{M}_{\preceq}, \ \mathbf{a} \prec \mathbf{m}_{\preceq}\}$.

Proof. We have $\mathbf{m}_{\leqslant} \in [1]$ because $\mathcal{M}_{\leqslant} \cap [0] = \{0^\infty\}$, and $\mathbf{m}_{\leqslant} \leqslant 10^\infty$ because $(10^n)^\infty \in \mathcal{M}_{\leq}$ for all $n \geqslant 0$. If $[10^j 0] < [10^j 1]$ for all $j \geqslant 0$, that is, $\min[1] = 10^\infty$, then $\mathbf{m}_{\leqslant} = 10^\infty$. Otherwise, let j be minimal such that $[10^j 1] < [10^j 0]$, that is, $\min[1] \in [10^j]$. Then $\mathcal{M}_{\leqslant} \cap [10^j 1] = \{(10^j)^\infty\}$ implies that $\mathbf{m}_{\leqslant} \in [10^j 0]$. If $[10^k 0] < [10^k 1]$ for all $k > j$, that is, $\min[10^j 0] = 10^\infty$, then $\mathbf{m}_{\leqslant} = 10^\infty$. Otherwise, let $k > j$ be minimal such that $[10^k 1] < [10^k 0]$, that is, $\min[10^j 0] \in [10^k 1]$. We have $\mathbf{m}_{\leqslant} \leqslant \tau_{j,k}(10^\infty)$ because $\tau_{j,k}((10^n)^\infty) \in \mathcal{M}_{\leq}$ for all $n \geqslant 0$, thus $\mathbf{m}_{\leqslant} \in [10^k 1]$. Each word in $\mathcal{M}_{\leqslant} \cap [10^k 1]$ is a concatenation of blocks 10^j and 10^k, thus $\mathcal{M}_{\leqslant} \cap [10^k 1] \subseteq \tau_{j,k}([1])$. This proves that $\mathbf{m}_{\leqslant} = \tau_{j,k}(\mathbf{m}_{\leq})$, where $\mathbf{a} \preceq \mathbf{b}$ if $\tau_{j,k}(\mathbf{a}) \leqslant \tau_{j,k}(\mathbf{b})$; note that $\mathbf{m}_{\leq} \in [1]$ because $[0] \prec [1]$. Since $\tau_{j,k}([w0]) \subseteq [\tau_{j,k}(w0)1]$ and $\tau_{j,k}([w1]) \subseteq [\tau_{j,k}(w0)0]$ for all $w \in \{0,1\}^*$, $\preceq$ is a cylinder order. If $\mathbf{a} \in \mathcal{M}_{\leq}$ with $\mathbf{a} < \mathbf{m}_{\leq}$, then $\mathbf{a} = 0^\infty$ or $\mathbf{a} = (10^j)^\infty = \tau_{j,k}(0^\infty)$ or $\mathbf{a} = \tau_{j,k}(\mathbf{a}')$ with $\mathbf{a}' \in \mathcal{M}_{\leq} \cap [1]$, $\mathbf{a}' \prec \mathbf{m}_{\leq}$. □

The following lemma is used in the construction of a cylinder order $\leq$ such that $\mathbf{m}_{\leqslant} = \mathbf{m}$ for a given word $\mathbf{m}$. Here, ε denotes the empty word.

Lemma 5. *Let $\boldsymbol{\sigma} = (\sigma_n)_{n\geqslant 1} \in S^\infty$, $w_n = \sigma_{[1,n]}(0)\cdots\sigma_{[1,1]}(0)$ for $n \geqslant 0$, with $w_0 = \varepsilon$.*
For all even $n \geqslant 0$, we have $\sigma_{[1,n]}([0]) \subseteq [w_n 0]$, $\sigma_{[1,n]}([1]) \subseteq [w_n 1]$.
For all odd $n \geqslant 1$, we have $\sigma_{[1,n]}([0]) \subseteq [w_n 1]$, $\sigma_{[1,n]}([1]) \subseteq [w_n 0]$.

Proof. Since $\sigma_{[1,0]}$ is the identity, the statement is trivial for $n = 0$. Suppose that it is true for all $\boldsymbol{\sigma} \in S^\infty$ for some even $n \geqslant 0$. Then,

$$\sigma_{[1,n+1]}([0]) = \sigma_1 \circ \sigma_{[2,n+1]}([0]) \subseteq \sigma_1([\sigma_{[2,n+1]}(0)\cdots\sigma_{[2,2]}(0)0]) \subseteq [\sigma_{[1,n+1]}(0)\cdots\sigma_{[1,2]}(0)\sigma_1(0)1],$$

$$\sigma_{[1,n+1]}([1]) = \sigma_1 \circ \sigma_{[2,n+1]}([1]) \subseteq \sigma_1([\sigma_{[2,n+1]}(0)\cdots\sigma_{[2,2]}(0)1]) \subseteq [\sigma_{[1,n+1]}(0)\cdots\sigma_{[1,2]}(0)\sigma_1(0)0],$$

thus the statement is true for all $\boldsymbol{\sigma} \in S^\infty$ for $n+1$. The case of odd n is similar. □

Proof of Theorem 3. Let first $\mathbf{m}_{\leqslant}$ be a cylinder order with $[0] < [1]$. By iterating Proposition 4, we obtain a finite sequence $\sigma_1, \dots, \sigma_h \in S$ such that $\mathbf{m}_{\leqslant} = \sigma_{[1,h]}(10^\infty)$ or an infinite sequence $\boldsymbol{\sigma} = (\sigma_n)_{n\geqslant 1} \in S^\infty$ such that $\mathbf{m}_{\leqslant} \in \sigma_{[1,n]}([1])$ for all $n \geqslant 1$. Since $\sigma_{[1,n+1]}(1)$ starts with $\sigma_{[1,n]}(1)$ and is longer than $\sigma_{[1,n]}(1)$ for all $n \geqslant 1$, we have $\bigcap_{n\geqslant 1} \sigma_{[1,n]}([1]) = \{\boldsymbol{\sigma}(1^\infty)\}$. Equations (5) and (6), respectively, follow from Proposition 4.

Let now $\boldsymbol{\sigma} = (\sigma_n)_{n\geqslant 1} = (\tau_{j_n,k_n})_{n\geqslant 1} \in S^\infty$. By Proposition 4 and Lemma 5, we have $\boldsymbol{\sigma}(1^\infty) = \mathbf{m}_{\leq}$ for all cylinder orders $\leq$ satisfying

$$\begin{aligned} &[\sigma_{[1,n]}(10^i)w_n 0] < [\sigma_{[1,n]}(10^i)w_n 1] && \text{for all even } n \geqslant 0,\ j_{n+1} \neq i < k_{n+1},\\ &&& \text{and for all odd } n \geqslant 1,\ i \in \{j_{n+1}, k_{n+1}\},\\ &[\sigma_{[1,n]}(10^i)w_n 1] < [\sigma_{[1,n]}(10^i)w_n 0] && \text{for all odd } n \geqslant 1,\ j_{n+1} \neq i < k_{n+1},\\ &&& \text{and for all even } n \geqslant 0,\ i \in \{j_{n+1}, k_{n+1}\}. \end{aligned} \tag{7}$$

Such cylinder orders exist since $\sigma_{[1,n+1]}(1)w_{n+1}$ is longer than $\sigma_{[1,n]}(10^{k_{n+1}})w_n = \sigma_{[1,n+1]}(1)w_n$ for all $n \geqslant 0$. To obtain $\mathbf{m}_{\leqslant} = \sigma_{[1,h]}(10^\infty)$, $h \geqslant 0$, we use cylinder orders $\leq$ such that (7) holds only for $n < h$ and such that, for all $i \geqslant 0$, $[\sigma_{[1,h]}(10^i)w_h 0] < [\sigma_{[1,h]}(10^i)w_h 1]$ if h is even, $[\sigma_{[1,h]}(10^i)w_h 1] < [\sigma_{[1,h]}(10^i)w_h 0]$ if h is odd. □

The sequence $\mathbf{m}_{\leqslant}$ is thus either eventually periodic or an S-adic sequence. A word $\mathbf{a} = a_1 a_2 \cdots$ is eventually periodic if and only if the factor complexity

$$p_{a_1 a_2 \cdots}(n) := \#\{a_{k+1} a_{k+2} \cdots a_{k+n} \ : \ k \geqslant 0\}$$

is bounded; see, for example, [18, Theorem 1.3.13]. The smallest complexity for an aperiodic sequence is $p_{\mathbf{a}}(n) = n+1$, which is attained precisely by Sturmian sequences; see, for example, [18, Theorem 2.1.5]. By [10, Proposition 2.1], all aperiodic words with $\limsup \frac{p_{\mathbf{a}}(n)}{n} < \frac{4}{3}$ are essentially equal to $\boldsymbol{\sigma}(1^\infty)$ with $\boldsymbol{\sigma} = (\tau_{j_n,k_n})_{n\geqslant 1} \in S^\infty$ (and $k_n \leqslant 2j_n+1$ or $(j_n, k_n) = (0, 2)$). Without conditions on j_n, k_n, we get the following upper bound for $p_{\boldsymbol{\sigma}(1^\infty)}(n)$, which is optimal since $p_{\tau_{0,k}^\infty(1^\infty)}(n) = 3n-2$ for all $k \geqslant 2$, $2 \leqslant n \leqslant k$.

Proposition 6. *Let $\leq$ be a cylinder order. Then $p_{\mathbf{m}_{\leqslant}}(n) \leqslant 3n-2$ for all $n \geqslant 2$.*

Proof. The proof is similar to that of [6, Theorem 17]; see also [10, Proposition 4.1]. Recall that the set of factors of a word $\mathbf{a} = a_1 a_2 \cdots \in \{0,1\}^\infty$ is $\{a_{k+1} a_{k+2} \cdots a_{k+n} \ : \ k, n \geqslant 0\}$, a factor v of $\mathbf{a}$ is *strong bispecial* if all four words $0v0, 0v1, 1v0, 1v1$ are factors of $\mathbf{a}$, *weak bispecial* if $0v1, 1v0$ are factors and $0v0, 1v1$ are not factors of $\mathbf{a}$. Then, v is a strong/weak bispecial factor of $\boldsymbol{\sigma}(1^\infty)$, $\boldsymbol{\sigma} = (\sigma_n)_{n\geqslant 1} = (\tau_{j_n,k_n})_{n\geqslant 1} \in S^\infty$, if and only if $v = 0^{j_1}/0^{k_1-1}$, $k_1 \geqslant j_1+2$, or $v = 0^{j_1}\sigma_1(v'0)$, where v' is a strong/weak bispecial factor of $\lim_{n\to\infty} \boldsymbol{\sigma}_{[2,n]}(1^\infty)$. By iterating, we obtain that all strong/weak bispecial factors of $\boldsymbol{\sigma}(1^\infty)$ are of the form

$$w_{\ell,h} = \sigma_{[1,0]}(0^{j_1}) \cdots \sigma_{[1,h-1]}(0^{j_h})\sigma_{[1,h]}(0^{\ell})\sigma_{[1,h]}(0) \cdots \sigma_{[1,1]}(0),$$

with $h \geqslant 0$ such that $k_{h+1} \geqslant j_{h+1}+2$, where $\ell = j_{h+1}$ for a strong bispecial factor, $\ell = k_{h+1}-1$ for a weak one; here, $w_{\ell,0} = 0^{\ell}$. For any recurrent word $\mathbf{a} \in \{0,1\}^\infty$ (where recurrence means that each prefix of $\mathbf{a}$ occurs infinitely often in $\mathbf{a}$), the difference of $p_{\mathbf{a}}(n+2) - p_{\mathbf{a}}(n+1)$ and $p_{\mathbf{a}}(n+1) - p_{\mathbf{a}}(n)$ equals the difference of the number of strong and weak bispecial factors of $\mathbf{a}$ of length n; see [7, Proposition 3.2]. By telescoping and since $p_{\mathbf{a}}(1) - p_{\mathbf{a}}(0) = 1$, $p_{\mathbf{a}}(n+2) - p_{\mathbf{a}}(n+1) - 1$ is equal to the difference of the number of strong and weak bispecial factors of $\mathbf{a}$ up to length n. Since $|w_{k_{h+1}-1,h}| < |\sigma_{[1,h+1]}(1)| \leqslant |\sigma_{[1,h+2]}(0)| < |w_{j_{h+3},h+2}|$ for all $h \geqslant 0$, this difference for $\mathbf{a} = \boldsymbol{\sigma}(1^\infty)$ is at most 2 for all $n \geqslant 0$; note that $\boldsymbol{\sigma}(1^\infty)$ is recurrent since $\sigma_{[n,n+1]}(1)$ starts with $10^{k_n}1$ for all $n \geqslant 1$. Since $p_{\mathbf{a}}(2) \leqslant 4$, we have thus $p_{\boldsymbol{\sigma}(1^\infty)} \leqslant 3n-2$ for all $n \geqslant 2$. Since $p_{\sigma_{[1,h]}(10^\infty)}(n) \leqslant p_{\boldsymbol{\sigma}(1^\infty)}(n)$ if k_{h+1} is sufficiently large, we also have $p_{\sigma_{[1,h]}(10^\infty)}(n) \leqslant 3n-2$ for all $n \geqslant 2$, which proves the proposition by Theorem 3. □

Since the factor complexity is bounded by a linear function, we can apply the results of [1]. For $\beta > 1$, let

$$\pi_\beta(a_1 a_2 \cdots) := \sum_{n=1}^{\infty} \frac{a_n}{\beta^n}.$$

Recall that Pisot and Salem numbers are algebraic integers $\beta > 1$ with all Galois conjugates (except β itself) having absolute value $\leqslant 1$; β is a Salem number if a conjugate lies on the unit circle, a Pisot number otherwise. In particular, all integers $\beta \geqslant 2$ are Pisot numbers.

Proposition 7. *Let $\beta \geqslant 2$ be a Pisot or Salem number, and let $\leq$ be a cylinder order on $\{0,1\}^\infty$. Then, $\pi_\beta(\mathbf{m}_{\leqslant})$ is in $\mathbb{Q}(\beta)$ or transcendental.*

Proof. This is a direct consequence of Proposition 6 and [1, Theorem 1A]. □

4 | EXAMPLES

4.1 | Lexicographic order

The classical order on the set of infinite words A^∞ with an ordered alphabet $(A, \leqslant)$ is the lexicographic order, defined by $[wa] <_{\text{lex}} [wb]$ for all $w \in A^*$, $a, b \in A$ with $a < b$. For $A = \{0,1\}$, we have $\mathbf{m}_{\leqslant_{\text{lex}}} = 10^\infty = \min \mathcal{M}_{\leqslant_{\text{lex}}} \setminus \{0^\infty\}$.

By [20, §2], a sequence $\mathbf{a} \in \{0,1\}^\infty$ is the greedy β-expansion of 1 for some $\beta \in (1,2)$ if and only if $\mathbf{a} \in \mathcal{M}_{\leqslant_{\text{lex}}} \setminus \{10^\infty\}$ and $\mathbf{a}$ is not purely periodic; it is the quasi-greedy β-expansion of 1 for some $\beta \in (1,2]$ if and only if $\mathbf{a} \in \mathcal{M}_{\leqslant_{\text{lex}}}$ and $\mathbf{a}$ does not end with 0^∞. Here, the greedy β-expansion of 1 is the lexicographically largest sequence $\mathbf{a} \in \{0,1\}^\infty$ with $\pi_\beta(\mathbf{a}) = 1$, the quasi-greedy β-expansion of 1 is the largest such sequence that does not end with 0^∞. By [15, Theorem 1], we also have that $(0^\infty, \mathbf{a})$ is the pair of kneading sequences of a Lorenz map if and only if $\mathbf{a} \in \mathcal{M}_{\leqslant_{\text{lex}}}$ does not end with 0^∞.

4.2 | Alternating lexicographic order

The alternating lexicographic order is defined by $[wa] <_{\text{alt}} [wb]$ if $a < b$ and $|w|$ is even, or $a > b$ and $|w|$ is odd, where $|w|$ denotes the length of a word $w \in A^*$. For $A = \{0,1\}$ with $0 < 1$, we have $[11] <_{\text{alt}} [10]$, $[101] >_{\text{alt}} [100]$, $[1001] <_{\text{alt}} [1000]$, thus $\mathbf{m}_{\leqslant_{\text{alt}}} = \tau_{0,2}(\mathbf{m}_{\preceq})$ by Proposition 4, with $\mathbf{a} \preceq \mathbf{b}$ if $\tau_{0,2}(\mathbf{a}) \leqslant_{\text{alt}} \tau_{0,2}(\mathbf{b})$. Since $\preceq$ is equal to $\leqslant_{\text{alt}}$, we obtain that $\mathbf{m}_{\leqslant_{\text{alt}}}$ is the fixed point of $\tau_{0,2}$, that is,

$$\mathbf{m}_{\leqslant_{\text{alt}}} = \tau_{0,2}(\mathbf{m}_{\leqslant_{\text{alt}}}) = 100111001001001110011\cdots;$$

see also [3, 5, 13]. By Theorem 3, we have

$$\{\mathbf{a} \in \mathcal{M}_{\leqslant_{\text{alt}}} \,:\, \mathbf{a} <_{\text{alt}} \mathbf{m}_{\leqslant_{\text{alt}}}\} = \{(\tau_{0,2}^n(0))^\infty \,:\, n \geqslant 0\} = \{0^\infty, 1^\infty, (100)^\infty, (10011)^\infty, \dots\}. \tag{8}$$

According to [16], the $(-\beta)$-expansion of $x \in \left[\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$, $\beta > 1$, is the sequence $a_1 a_2 \cdots$ with $a_n = \left\lfloor \frac{\beta}{\beta+1} - \beta T_{-\beta}^{n-1}(x) \right\rfloor$, given by the $(-\beta)$-transformation $T_{-\beta}(y) := -\beta y - \left\lfloor \frac{\beta}{\beta+1} - \beta y \right\rfloor$, and the set of $(-\beta)$-expansions is characterized by that of $\frac{-\beta}{\beta+1}$. By [24, Theorem 2], a sequence $\mathbf{a}$ is the $(-\beta)$-expansion of $\frac{-\beta}{\beta+1}$ for some $\beta \in (1,2)$ if and only if $\mathbf{a} \in \mathcal{M}_{\leqslant_{\text{alt}}} \setminus \{(10)^\infty\}$, $\mathbf{a} >_{\text{alt}} \mathbf{m}_{\leqslant_{\text{alt}}}$, and $\mathbf{a} \notin \{w1, w00\}^\infty \setminus \{(w1)^\infty\}$ for all $w \in \{0,1\}^*$ such that $(w1)^\infty >_{\text{alt}} \mathbf{m}_{\leqslant_{\text{alt}}}$. Note that continued fractions are also ordered by the alternating lexicographic order on the sequences of partial quotients, and $\mathbf{m}_{\leqslant_{\text{alt}}}$ occurs, for example, in [8, Theorem 2] and [17, Remark 11.1].

4.3 | Unimodal maps

Let $A = \{0, 1\}$ and define the *unimodal order* by $[w0] <_{\text{uni}} [w1]$ if $|w|_1$ is even, $[w0] >_{\text{uni}} [w1]$ if $|w|_1$ is odd, where $|w|_1$ denotes the number of occurrences of 1 in $w \in \{0, 1\}^*$. Then we have $[11] <_{\text{uni}} [10]$, $[101] <_{\text{uni}} [100]$, and

$$\mathbf{m}_{\leqslant_{\text{uni}}} = \tau_{0,1}(\mathbf{m}_{\leqslant_{\text{alt}}}) = 10111010101110111011101010111010 \cdots .$$

This is the fixed point of the period-doubling (or Feigenbaum) substitution $0 \mapsto 11$, $1 \mapsto 10$. The set $\mathcal{M}_{\leqslant_{\text{uni}}}$ is the set of kneading sequences of unimodal maps [9, 19].

We define the *flipped unimodal order* by $[w0] <_{\text{flip}} [w1]$ if $|w|_0$ is even, $[w0] >_{\text{flip}} [w1]$ if $|w|_0$ is odd, where $|w|_0$ denotes the number of occurrences of 0 in $w \in \{0, 1\}^*$. Then we have $[11] >_{\text{flip}} [10]$, $[101] <_{\text{flip}} [100]$, $[1001] >_{\text{flip}} [1000]$, $[10001] >_{\text{flip}} [10000]$, and

$$\mathbf{m}_{\leqslant_{\text{flip}}} = \tau_{1,3}(\mathbf{m}_{\leqslant_{\text{alt}}}) = 10001010100010001000101010001010100010101000 1000 \cdots .$$

Note that $0\mathbf{m}_{\leqslant_{\text{flip}}} = F(\mathbf{m}_{\leqslant_{\text{uni}}})$, where $F(a_1 a_2 \cdots) := (1-a_1)(1-a_2) \cdots$, and we have

$$\mathcal{M}_{\leqslant_{\text{uni}}} = 1F(\mathcal{M}_{\leqslant_{\text{flip}}}) \cup \{0^\infty\}.$$

4.4 | Sturmian sequences

The set of substitutions $\{\theta_k : k \geqslant 1\}$ defined by $\theta_k(0) = 0^{k-1}1$, $\theta_k(1) = 0^{k-1}10$, generates the standard Sturmian words; see [18, Corollary 2.2.22]. Since $\tau_{k-1,k}$ is rotationally conjugate to θ_k, more precisely $\theta_k(w)0^{k-1} = 0^{k-1}\tau_{k-1,k}(w)$ for all $w \in \{0, 1\}^*$, the set of substitutions $\{\tau_{k-1,k} : k \geqslant 1\}$ generates the same shifts as $\{\theta_k : k \geqslant 1\}$. Therefore, the limit words of sequences in $\{\tau_{k-1,k} : k \geqslant 1\}^\infty$ provide elements of all Sturmian shifts. For example, the limit word of the sequence $(\tau_{0,1})^\infty$ is the Fibonacci word. Note that Sturmian sequences that are sup-words are also considered in [4].

5 | SYMMETRIC ALPHABETS

For a real number $q > 1$, the set

$$\tilde{\mathcal{L}}_q := \Big\{ \limsup_{n\to\infty} \|xq^n\| \ : \ x \in \mathbb{R} \Big\}, \tag{9}$$

where $\|.\|$ denotes the distance to the nearest integer, is a multiplicative version of the Lagrange spectrum, and was studied in [2, 12]. If q is an integer, then representing $x = \sum_{k=-\infty}^{\infty} a_k q^{-k}$ with $a_k \in \mathbb{Z}$, $a_k \neq 0$ for finitely many $k \leqslant 0$, $|\sum_{k=n+1}^{\infty} a_k q^{n-k}| \leqslant 1/2$, gives that $\|xq^n\| = |\sum_{k=n+1}^{\infty} a_k q^{n-k}|$; see also Proposition 10 below. This leads us to consider

$$\mathcal{M}^{\text{abs}}_{\leqslant} = \{s^{\text{abs}}_{\leqslant}(\mathbf{a}) \ : \ \mathbf{a} \in \{0, \pm 1\}^\infty\} \quad \text{with} \quad s^{\text{abs}}_{\leqslant}(a_1 a_2 \cdots) = \sup_{n\geqslant 1} \text{abs}(a_n a_{n+1} \cdots),$$

where

$$\operatorname{abs}(\mathbf{a}) = \begin{cases} \mathbf{a} & \text{if } \mathbf{a} \geqslant_{\text{lex}} 0^\infty, \\ -\mathbf{a} & \text{if } \mathbf{a} \leqslant_{\text{lex}} 0^\infty, \end{cases} \quad \text{with } -(a_1 a_2 \cdots) = (-a_1)(-a_2)\cdots.$$

We denote the smallest accumulation point of $\mathcal{M}_{\leq}^{\text{abs}}$ by $\mathbf{m}_{\leq}^{\text{abs}}$.

The same proof as for Theorem 1 shows for all cylinder orders $\leq$ on $\{0,\pm1\}^\infty$ that

$$\mathcal{L}_{\leqslant}^{\text{abs}} = \mathcal{M}_{\leqslant}^{\text{abs}} = \text{closure}(\{s_{\leqslant}^{\text{abs}}(\mathbf{a}) \ : \ \mathbf{a} \in \{0,\pm1\}^\infty \text{ purely periodic}\}),$$

where $\mathcal{L}_{\leqslant}^{\text{abs}} := \{\limsup_{n\geqslant 1} \operatorname{abs}(a_n a_{n+1}\cdots) \ : \ a_1 a_2 \cdots \in \{0,\pm1\}^\infty\}$.

In the following, we assume that a cylinder order on $\{0,\pm1\}^\infty$ is *consistent* (with the natural order on $\{0,\pm1\}$), which means that, for each $w \in \{0,\pm1\}^*$, we have $[w(-1)] < [w0] < [w1]$ or $[w(-1)] > [w0] > [w1]$. In order to describe $\mathbf{m}_{\leq}^{\text{abs}}$, we define maps $\varrho_0, \varrho_1, \varrho_2$ from $\{0,1\}^*$ to $\{0,\pm1\}^*$ by $\varrho_0(\varepsilon) = \varrho_1(\varepsilon) = \varrho_2(\varepsilon) = \varepsilon$ for the empty word ε, and

$$\varrho_0(w0) = \begin{cases} \varrho_0(w)1 & \text{if } |w|_0 \text{ is even}, \\ \varrho_0(w)(-1) & \text{if } |w|_0 \text{ is odd}, \end{cases} \qquad \varrho_0(w1) = \begin{cases} \varrho_0(w)10 & \text{if } |w|_0 \text{ is even}, \\ \varrho_0(w)(-1)0 & \text{if } |w|_0 \text{ is odd}, \end{cases}$$

$$\varrho_1(w0) = \begin{cases} \varrho_1(w)1 & \text{if } |w|_1 \text{ is even}, \\ \varrho_1(w)(-1) & \text{if } |w|_1 \text{ is odd}, \end{cases} \qquad \varrho_1(w1) = \begin{cases} \varrho_1(w)10 & \text{if } |w|_1 \text{ is even}, \\ \varrho_1(w)(-1)0 & \text{if } |w|_1 \text{ is odd}, \end{cases}$$

$$\varrho_2(w0) = \begin{cases} \varrho_2(w)1 & \text{if } |w| \text{ is even}, \\ \varrho_2(w)(-1) & \text{if } |w| \text{ is odd}, \end{cases} \qquad \varrho_2(w1) = \begin{cases} \varrho_2(w)10 & \text{if } |w| \text{ is even}, \\ \varrho_2(w)(-1)0 & \text{if } |w| \text{ is odd}, \end{cases}$$

for all $w \in \{0,1\}^*$, where $|w|$ is the length of a word w and $|w|_i$ the number of occurrences of the letter i in w. As for substitutions, the maps ϱ_i are extended naturally to $\{0,1\}^\infty$.

Theorem 8. *Let* $\mathbf{m} \in \{0,\pm1\}^\infty$. *Then* $\mathbf{m} = \mathbf{m}_{\leq}^{\text{abs}}$ *for some consistent cylinder order* $\leq$ *on* $\{0,\pm1\}^\infty$ *with* $0^\infty < 1^\infty$ *if and only if* $\mathbf{m} = \sigma(\mathbf{m}_{\preceq})$ *for some* $\sigma \in \{\varrho_0, \varrho_1, \varrho_2, \tau_{0,1}\}$ *and some cylinder order* $\preceq$ *on* $\{0,1\}^\infty$ *with* $0^\infty \prec 1^\infty$.

If $\mathbf{m}_{\leqslant}^{\text{abs}} = \sigma(\mathbf{m}_{\preceq})$, *then we can assume that* $\mathbf{a} \preceq \mathbf{b}$ *if and only if* $\sigma(\mathbf{a}) \leqslant \sigma(\mathbf{b})$, *and we have*

$$\{\mathbf{a} \in \mathcal{M}_{\leqslant}^{\text{abs}} \ : \ \mathbf{a} < \mathbf{m}_{\leqslant}^{\text{abs}}\} = \{0^\infty\} \cup \{\sigma(\mathbf{a}) \ : \ \mathbf{a} \in \mathcal{M}_{\preceq}, \ \mathbf{a} \prec \mathbf{m}_{\preceq}\}. \tag{10}$$

Proof. Let first $\leq$ be a consistent cylinder order on $\{0,\pm1\}^\infty$. Then, the order $\preceq$ defined by $\mathbf{a} \preceq \mathbf{b}$ if $\sigma(\mathbf{a}) \leqslant \sigma(\mathbf{b})$ is a cylinder order for all $\sigma \in \{\varrho_0, \varrho_1, \varrho_2, \tau_{0,1}\}$. Indeed, for any $w \in \{0,1\}^*$, we have $\sigma([w0]) \subset [\sigma(w)xy]$, $\sigma([w1]) \subset [\sigma(w)x0]$, where $x, y \in \{\pm1\}$.

Assume first that $[1\bar{1}] < [10]$; here and in the following, we use the notation $\bar{1} = -1$. Then, $\mathcal{M}_{\leqslant}^{\text{abs}} \cap [1\bar{1}] = \{(1\bar{1})^\infty\}$, thus $\mathbf{m}_{\leqslant}^{\text{abs}} \in [10]$. If $[10\bar{1}] < [100]$, then each 1 in a word in $\mathcal{M}_{\leqslant}^{\text{abs}} \cap [10\bar{1}]$ is followed by $\bar{1}$ or $0\bar{1}$, and each $\bar{1}$ is followed by 1 or 01, that is,

$$\mathcal{M}_{\leqslant}^{\text{abs}} \cap [10\bar{1}] \subseteq 10\,(\{\bar{1}, \bar{1}0\}\{1, 10\})^\infty = \varrho_2([1]).$$

Therefore, $\mathbf{m}_{\leqslant}^{\text{abs}} = \varrho_2(\mathbf{m}_{\preceq})$ for the cylinder order $\preceq$ defined by $\mathbf{a} \preceq \mathbf{b}$ if $\varrho_2(\mathbf{a}) \leqslant \varrho_2(\mathbf{b})$. If $[101] < [100]$, then each 1 in a word in $\mathcal{M}_{\leqslant}^{\text{abs}} \cap [101]$ is followed by $\bar{1}$ or 01, and each $\bar{1}$ is followed by 1 or

$0\bar{1}$, that is,

$$\mathcal{M}^{\mathrm{abs}}_{\leqslant} \cap [101] \subseteq 10((10)^*1(\bar{1}0)^*\bar{1})^\infty \cup 10((10)^*1(\bar{1}0)^*\bar{1})^*(10)^\infty \cup 10((10)^*1(\bar{1}0)^*\bar{1})^*(10)^*1(\bar{1}0)^\infty$$
$$= \varrho_0(1(1^*01^*0)^\infty) \cup \varrho_0(1(1^*01^*0)^*1^\infty) \cup \varrho_0(1(1^*01^*0)^*1^*01^\infty) = \varrho_0([1]).$$

Therefore, we have $\mathbf{m}^{\mathrm{abs}}_{\leqslant} = \varrho_0(\mathbf{m}_{\preceq})$ for the cylinder order $\preceq$ defined by $\mathbf{a} \preceq \mathbf{b}$ if $\varrho_0(\mathbf{a}) \leqslant \varrho_0(\mathbf{b})$. Assume now $[11] < [10]$. Then $\mathcal{M}^{\mathrm{abs}}_{\leqslant} \cap [11] = \{1^\infty\}$, thus $\mathbf{m}_{\leqslant} \in [10]$. If $[10\bar{1}] < [100]$, then

$$\mathcal{M}^{\mathrm{abs}}_{\leqslant} \cap [10\bar{1}] \subseteq 10(\bar{1}^*\bar{1}01^*10)^\infty \cup 10(\bar{1}^*\bar{1}01^*10)^*\bar{1}^\infty \cup 10(\bar{1}^*\bar{1}01^*10)^*\bar{1}^*\bar{1}01^\infty = \varrho_1([1]),$$

thus $\mathbf{m}^{\mathrm{abs}}_{\leqslant} = \varrho_1(\mathbf{m}_{\preceq})$, with $\preceq$ defined by $\mathbf{a} \preceq \mathbf{b}$ if $\varrho_1(\mathbf{a}) \leqslant \varrho_1(\mathbf{b})$. If $[101] < [100]$, then

$$\mathcal{M}^{\mathrm{abs}}_{\leqslant} \cap [101] \subseteq 10\{1, 10\}^\infty = \tau_{0,1}([1]),$$

thus $\mathbf{m}^{\mathrm{abs}}_{\leqslant} = \tau_{0,1}(\mathbf{m}_{\preceq})$ for the cylinder order $\preceq$ defined by $\mathbf{a} \preceq \mathbf{b}$ if $\tau_{0,1}(\mathbf{a}) \leqslant \tau_{0,1}(\mathbf{b})$. Since $\varrho_0(0^\infty) = \varrho_2(0^\infty) = (1\bar{1})^\infty$ and $\varrho_1(0^\infty) = \tau_{0,1}(0^\infty) = 1^\infty$, Equation (10) holds.

Let now $\preceq$ be a cylinder order on $\{0,1\}^\infty$ with $0^\infty \prec 1^\infty$ and $\sigma \in \{\varrho_0, \varrho_1, \varrho_2, \tau_{0,1}\}$. Then, there exists a consistent cylinder order $\preceq$ on $\{0, \pm1\}^\infty$ satisfying $\sigma(\mathbf{a}) \leqslant \sigma(\mathbf{b})$ if $\mathbf{a} \preceq \mathbf{b}$ and $0^\infty < 1^\infty$. Indeed, for $w \in \{0, \pm1\}^*$ and distinct $a, b \in \{0, \pm1\}$, we set $[wa] < [wb]$ if $\mathbf{a}' \prec \mathbf{b}'$ for some $\mathbf{a}', \mathbf{b}' \in \{0,1\}^\infty$ with $\sigma(\mathbf{a}') \in [wa], \sigma(\mathbf{b}') \in [wb]$; by Lemma 2, this does not depend on the choice of $\mathbf{a}', \mathbf{b}'$. Moreover, since $\sigma(\mathbf{a}) \in [w\bar{1}]$ and $\sigma(\mathbf{b}) \in [w1]$ is impossible, we have no obstruction to a consistent cylinder order. Since $0^\infty \prec 1^\infty$, we have $[1\bar{1}] < [10]$ in case $\sigma \in \{\varrho_0, \varrho_2\}$, $[11] < [10]$ in case $\sigma \in \{\varrho_1, \tau_{0,1}\}$. We can set $[10\bar{1}] < [100]$ in case $\sigma \in \{\varrho_1, \varrho_2\}$ because $([100] \cup [101]) \cap \sigma(\{0,1\})^\infty = \emptyset$; similarly, we can set $[101] < [100]$ in case $\sigma \in \{\varrho_0, \tau_{0,1}\}$. Then, we have $\mathbf{m}^{\mathrm{abs}}_{\leqslant} = \sigma(\mathbf{m}_{\leqslant})$. □

Proposition 9. *Let $\beta \geqslant 3$ be a Pisot or Salem number, and let $\preceq$ be a consistent cylinder order on $\{0, \pm1\}^\infty$. Then $\pi_\beta(\mathbf{m}^{\mathrm{abs}}_{\leqslant})$ is in $\mathbb{Q}(\beta)$ or transcendental.*

Proof. Let $G(a_1a_2\cdots) = |a_1|\,|a_2|\cdots$. Then $G \circ \varrho_i = \tau_{0,1}$ for all $i \in \{0,1,2\}$, thus $p_{G(\mathbf{m}^{\mathrm{abs}}_{\leqslant})}(n) \leqslant 3n-2$ by Proposition 6, Theorems 3 and 8. Moreover, the map G is 2-to-1 from the set of factors of $\mathbf{m}^{\mathrm{abs}}_{\leqslant}$ to the set of factors of $G(\mathbf{m}^{\mathrm{abs}}_{\leqslant})$, thus $p_{\mathbf{m}^{\mathrm{abs}}_{\leqslant}}(n) \leqslant 6n-4$. By [1, Theorem 1A] and by adding 1 to each digit of $\mathbf{m}^{\mathrm{abs}}_{\leqslant}$, we obtain that $\pi_\beta(\mathbf{m}^{\mathrm{abs}}_{\leqslant}) + \frac{1}{\beta-1}$ is in $\mathbb{Q}(\beta)$ or transcendental; thus, also $\pi_\beta(\mathbf{m}^{\mathrm{abs}}_{\leqslant})$ is in $\mathbb{Q}(\beta)$ or transcendental. □

6 | EXAMPLES OF ORDERS ON SYMMETRIC SHIFT SPACES

6.1 | Lexicographic order

For the lexicographic order on $\{0, \pm1\}^\infty$ (with $-1 < 0 < 1$), we have $[1\bar{1}] < [10]$, $[10\bar{1}] < [100]$, and we obtain that

$$\mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}} = \varrho_2(\mathbf{m}_{\leqslant_{\mathrm{alt}}}) = 10\bar{1}1\bar{1}010\bar{1}01\bar{1}10\bar{1}1\bar{1}01\bar{1}10\bar{1}010\bar{1}1\bar{1}010\cdots.$$

The following proposition relates the Lagrange spectrum $\tilde{\mathcal{L}}_q$, defined in (9), and its smallest accumulation point $\tilde{\mathbf{m}}_q$ to $\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}$ and $\mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}$; it slightly improves results of [2, 12].

Proposition 10. *We have*

$$\tilde{\mathcal{L}}_2 = \pi_2(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}) \cap \left[0, \tfrac{1}{2}\right] \neq \pi_2(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}), \quad \tilde{\mathcal{L}}_3 = \pi_3(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}), \tag{11}$$

$$\pi_q(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}) = \tilde{\mathcal{L}}_q \cap \left[0, \tfrac{1}{q-1}\right] \neq \tilde{\mathcal{L}}_q \quad \textit{for all integers } q \geqslant 4. \tag{12}$$

For all integers $q \geqslant 2$, we have $\tilde{\mathbf{m}}_q = \pi_q(\mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}) = \pi_q(\varrho_2(\mathbf{m}_{\leqslant_{\mathrm{alt}}}))$ and

$$\tilde{\mathcal{L}}_q \cap [0, \tilde{\mathbf{m}}_q) = \pi_q(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}) \cap [0, \tilde{\mathbf{m}}_q) = \{0\} \cup \{\pi_q(\varrho_2(\tau^n_{0,2}(0^\infty))) \ : \ n \geqslant 0\}.$$

Proof. As mentioned at the beginning of Section 5, for integer $q \geqslant 2$, $\|xq^n\|$ can be determined by a symmetric q-expansion of x. We can assume w.l.o.g. $|x| \leqslant \frac{1}{2}$. Let

$$\mathcal{A}_q := \left\{a_1 a_2 \cdots \in A_q^\infty \ : \ |\pi_q(a_k a_{k+1} \cdots)| \leqslant \tfrac{1}{2} \text{ for all } k \geqslant 1\right\}, \text{ with } A_q := \{0, \pm 1, \dots, \pm\lfloor q/2 \rfloor\}.$$

For each $x \in [-\frac{1}{2}, \frac{1}{2}]$, we obtain a sequence $\mathbf{a} = a_1 a_2 \cdots \in \mathcal{A}_q$ satisfying $x = \pi_q(\mathbf{a})$ by taking $a_k = \lfloor q\tilde{T}_q^{k-1}(x) + \frac{1}{2} \rfloor$ where $\tilde{T}_q(y) := qy - \lfloor qy + \frac{1}{2} \rfloor$. Then,

$$\|xq^n\| = |\pi_q(a_{n+1}a_{n+2}\cdots)| = \pi_q(\mathrm{abs}(a_{n+1}a_{n+2}\cdots)).$$

Note that $\mathbf{a} \leqslant_{\mathrm{lex}} \mathbf{b}$ implies $\pi_q(\mathbf{a}) \leqslant \pi_q(\mathbf{b})$ for all $\mathbf{a}, \mathbf{b} \in \mathcal{A}_q$. Since $\mathcal{L}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}} = \mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}$ (and a similar relation holds for larger alphabets), we obtain that

$$\tilde{\mathcal{L}}_q = \{\pi_q(s^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}(\mathbf{a})) \ : \ \mathbf{a} \in \mathcal{A}_q\}.$$

For $q \in \{2, 3\}$, we have $A_q = \{0, \pm 1\}$, thus $\tilde{\mathcal{L}}_q \subseteq \pi_q(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}})$. For $q \geqslant 3$, we have $\{0, \pm 1\}^\infty \subseteq \mathcal{A}_q$, thus $\pi_q(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}) \subseteq \tilde{\mathcal{L}}_q$. Since $\pi_2(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}) \cap [0, \frac{1}{2}] \subseteq \tilde{\mathcal{L}}_2$ and $1 = \pi_2(1^\infty) \in \pi_2(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}) \setminus \tilde{\mathcal{L}}_2$, this proves (11). For $q \geqslant 4$, we have $\pi_q(s^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}(\mathbf{a})) \geqslant \pi_q((2\overline{2})^\infty) = \frac{2}{q+1} > \frac{1}{q-1}$ for all $\mathbf{a} \in A_q^\infty \setminus \{0, \pm 1\}^\infty$, thus $\tilde{\mathcal{L}}_q \cap [0, \frac{1}{q-1}] \subseteq \pi_q(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}})$. Together with $\frac{2}{q+1} \in \tilde{\mathcal{L}}_q \setminus \pi_q(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}})$, we obtain (12). Since π_q is order-preserving on $\mathcal{A}_q$, we obtain that $\tilde{\mathbf{m}}_q = \pi_q(\mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}})$ and that $\tilde{\mathcal{L}}_q$ and $\pi_q(\mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}})$ agree on $[0, \tilde{\mathbf{m}}_q)$. Since $\{\mathbf{a} \in \mathcal{M}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}} \ : \ \mathbf{a} < \mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathrm{lex}}}\}$ is equal to $\{0^\infty\} \cup \{\varrho_2(\tau^n_{0,2}(0^\infty)) \ : \ n \geqslant 0\}$ by Theorem 8 and (8), this completes the proof of the proposition. □

6.2 | Alternating lexicographic order

For the alternating lexicographic order on $\{0, \pm 1\}^\infty$ (with $-1 < 0 < 1$), we have $[11] <_{\mathrm{alt}} [10]$ and $[10\bar{1}] <_{\mathrm{alt}} [100]$,

$$\mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathrm{alt}}} = \varrho_1(\mathbf{m}_{\leqslant_{\mathrm{alt}}}) = 10\bar{1}\bar{1}\bar{1}010\bar{1}01110\bar{1}\bar{1}\bar{1}01110\bar{1}010\bar{1}\bar{1}\bar{1}010\cdots.$$

6.3 | Bimodal order

Similarly to the unimodal order, we define the *bimodal order* on $\{0,\pm1\}^\infty$ by $[wa] <_{\mathrm{bi}} [wb]$ if $a < b$ (with $-1 < 0 < 1$) and $|w|_1 + |w|_{-1}$ is even, or $a > b$ and $|w|_1 + |w|_{-1}$ is odd. Then $\mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathrm{bi}}} = \tau_{0,1}(\mathbf{m}_{\leqslant_{\mathrm{alt}}}) = \mathbf{m}_{\leqslant_{\mathrm{uni}}}$. We get the same result for the order defined by $[wa] < [wb]$ if $a < b$ and $|w|_1$ is even, or $a > b$ and $|w|_1$ is odd.

We also define the *flipped bimodal order* on $\{0,\pm1\}^\infty$ by $[wa] <_{\mathrm{biflip}} [wb]$ if $a < b$ and $|w|_0$ is even, or $a > b$ and $|w|_0$ is odd. Then

$$\mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathrm{biflip}}} = \varrho_0(\mathbf{m}_{\leqslant_{\mathrm{alt}}}) = 101\bar{1}1010101\bar{1}101\bar{1}101\bar{1}10101010\bar{1}1010\cdots..$$

6.4 | Other orders

For $\mathbf{e} \in \{\pm1\}^\infty$, we define a cylinder order $\leqslant_{\mathbf{e}}$ on $\{0,\pm1\}^\infty$ by

$$\mathbf{a} \leqslant_{\mathbf{e}} \mathbf{b} \quad \text{if } \mathbf{e}\cdot\mathbf{a} \leqslant_{\mathrm{lex}} \mathbf{e}\cdot\mathbf{b},$$

where $(e_1e_2\cdots)\cdot(a_1a_2\cdots) = (e_1a_1)(e_2a_2)\cdots$. We know from Proposition 7 that $\pi_\beta(\mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathbf{e}}})$ is in $\mathbb{Q}(\beta)$ or transcendental for all Pisot or Salem numbers β. However, here the value of $\pi_\beta(\mathbf{e}\cdot\mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathbf{e}}})$, which is the smallest accumulation point of $\{\limsup_{n\to\infty} |\sum_{k=n+1}^{\infty} \frac{e_k a_k}{\beta^{k-n}}| \ : \ a_1a_2\cdots \in \{0,\pm1\}^\infty\}$ when $e_1 = 1$ and $\beta \geqslant 3$, is more relevant. If $\mathbf{e}$ is periodic with period k, then $p_{\mathbf{e}\cdot\mathbf{a}}(n) \leqslant k\, p_{\mathbf{a}}(n)$, and hence, $\pi_\beta(\mathbf{e}\cdot\mathbf{m}^{\mathrm{abs}}_{\leqslant_{\mathbf{e}}})$ is also in $\mathbb{Q}(\beta)$ or transcendental for all Pisot or Salem numbers β. We do not know whether the same result holds when $\mathbf{e}$ is aperiodic.

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