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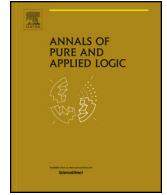


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Full Length Article

## Can you take Komjath's inaccessible away?

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## ABSTRACT

In this paper we aim to compare Kurepa trees and Aronszajn trees. Moreover, we analyze the effect of large cardinal assumptions on this comparison. Using the method of walks on ordinals, we will show it is consistent with ZFC that there is a Kurepa tree and every Kurepa tree contains an Aronszajn subtree, if there is an inaccessible cardinal. This is stronger than Komjath's theorem in [5], where he proves the same consistency from two inaccessible cardinals. Moreover, we prove it is consistent with ZFC that there is a Kurepa tree  $T$  such that if  $U \subset T$  is a Kurepa tree with the inherited order from  $T$ , then  $U$  has an Aronszajn subtree. This theorem uses no large cardinal assumption. Our last theorem immediately implies the following: If  $MA_{\omega_2}$  holds and  $\omega_2$  is not a Mahlo cardinal in  $L$  then there is a Kurepa tree with the property that every Kurepa subset has an Aronszajn subtree. Our work entails proving a new lemma about Todorčević's  $\rho$  function which might be useful in other contexts.

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## 1. Introduction

In this paper we aim to compare Kurepa trees and Aronszajn trees. Moreover, we analyze the effect of large cardinal assumptions on this comparison. We are interested in the question that to what extent do Kurepa trees contain Aronszajn subtrees. The first result regarding this question is due to Jensen. He showed that there is a Kurepa tree in the constructible universe  $L$ , which has no Aronszajn subtrees. Todorčević showed that there is a countably closed forcing which adds a Kurepa tree with no Aronszajn subtree. Both Jensen's and Todorčević's results are in the negative direction. In the positive direction, Komjath proved the following theorem.

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**Theorem 1.1.** [5] *It is consistent relative to the existence of two inaccessible cardinals that there is a Kurepa tree and every Kurepa tree has an Aronszajn subtree.*

It is natural to ask whether or not the large cardinal assumptions in Theorem 1.1 are sharp. In other words, assume every Kurepa tree has an Aronszajn subtree, then is it consistent that there are at least two inaccessible cardinals?

Let's call an  $\omega_1$ -tree *Aronszajn free* if it has no Aronszajn subtree. Without the use of large cardinals, there are various ways to show the consistency of the existence of Aronszajn free Kurepa trees. It is natural to ask, if there are no large cardinals, do Kurepa trees have to have Aronszajn free Kurepa subtrees? In other words, do we need large cardinals in order to show the existence of a Kurepa tree with no Aronszajn free Kurepa subtree?

Our work reveals a new fact about Todorcevic's  $\rho$  function. Based on this fact about  $\rho$  and a notion of capturing which was introduced in [3], we find Aronszajn subtrees in some canonical Kurepa trees without any large cardinal assumptions. It is worth mentioning that although we analyze some  $\omega_1$ -trees to prove this fact about  $\rho$ , the function  $\rho$  is defined in terms of ordinals with no reference to  $\omega_1$ -trees.

In this paper we will show the following theorem, which is stronger than Komjath's Theorem.

**Theorem 1.2.** *Assume there is an inaccessible cardinal. Then it is consistent that there is a Kurepa tree and every Kurepa tree contains an Aronszajn subtree.*

Regarding the existence of a Kurepa tree with no Aronszajn free Kurepa subtree, we show the following theorem. It is worth mentioning that the following theorem does not need any large cardinal assumption.

**Theorem 1.3.** *It is consistent that there is a Kurepa tree  $T$  such that whenever  $U \subset T$  is a Kurepa tree when it is considered with the inherited order from  $T$ , then  $U$  has an Aronszajn subtree.*

In [7] by using  $\rho$ , Todorcevic introduces a forcing which satisfies the Knaster condition and which adds a Kurepa tree. We use this forcing to prove Lemma 4.3, which reveals a new inequality about  $\rho$ . We use Lemma 4.3 to find Aronszajn subtrees and show Theorem 1.3. Since the tree  $T$  can be forced to exist in any model of  $\square_{\omega_1}$  using a ccc forcing, the following corollary trivially follows from Theorem 1.3.

**Corollary 1.4.** *Assume  $\text{MA}_{\omega_2}$  holds and  $\omega_2$  is not a Mahlo cardinal in  $L$ . Then there is a Kurepa tree with the property that every Kurepa subset has an Aronszajn subtree.*

The following question still remains unanswered.

**Question 1.5.** *Is the large cardinal assumption in Theorem 1.2 sharp? In other words, assume every Kurepa tree has an Aronszajn subtree. Then is it consistent that there is an inaccessible cardinal?*

## 2. Preliminaries

We will be using the following notation and terminology. Assume  $T$  is an  $\omega_1$ -tree. For any  $\alpha \in \omega_1$ ,  $T_\alpha$  denotes the set of all elements of  $T$  which have height  $\alpha$ .  $T_{<\alpha}$  is the set of all members of  $T$  which have height less than  $\alpha$ .  $T_{\leq\alpha}$  is defined similarly.  $\mathcal{B}(T)$  is the set of all cofinal branches of  $T$ . If  $b$  is a cofinal branch in  $T$  and  $\alpha \in \omega_1$ ,  $b(\alpha)$  refers to the element in  $b$  which is of height  $\alpha$ . If  $t \in T$  and  $\alpha \in \omega_1$  then  $t \upharpoonright \alpha$  refers to the set of all elements  $x \leq_T t$  whose height is less than  $\alpha$ . For any  $x \in T$ ,  $T_x$  is the set of all  $t \in T$  that are comparable with  $x$ . In particular the predecessors of  $x$  are in  $T_x$ .

If  $x$  is a finite set of ordinals and  $i \in |x|$  then  $x(i)$  refers to the  $i$ 'th element of  $x$ . For  $x, y$  finite sets of ordinals we say  $x < y$  if every element in  $x$  is less than every element in  $y$ . Assume  $x$  is a finite set of ordinals

and  $\langle T^\alpha : \alpha \in x \rangle$  are  $\omega_1$ -trees, then  $\bigotimes_{\alpha \in x} T^\alpha = \bigcup_{\xi \in \omega_1} \prod_{\alpha \in x} T_\xi^\alpha$ . It is easy to see that the component-wise order on this product makes it an  $\omega_1$ -tree. With this product, for every  $n \in \omega$  we can define  $T^{[n]} = \bigotimes_{i \in n} T$ . Assume  $T$  is an  $\omega_1$ -tree and  $\langle v_i : i \in n \rangle$  are pairwise distinct elements of  $T$  with the same height, then  $\bigotimes_{i \in n} T_{v_i}$  is called a derived tree of  $T$  with dimension  $n$ .

In [3] a notion of capturing is defined for linear orders. This notion can be used for  $\omega_1$ -trees as well. We will use this notion and Proposition 2.3 in order to characterize when an  $\omega_1$ -tree contains an Aronszajn subtree.

**Definition 2.1.** [3] Assume  $T$  is an  $\omega_1$ -tree,  $\kappa$  is a large enough regular cardinal,  $t \in T \cup \mathcal{B}(T)$ , and  $N \prec H_\kappa$  is countable such that  $T \in N$ . We say that  $N$  captures  $t$  if there is a chain  $c \subset T$  in  $N$  which contains all elements of  $T_{<N \cap \omega_1}$  below  $t$ , or equivalently  $t \upharpoonright (\delta_N) \subset c$ .

The following definition is a modification of Definition 3.1 in [3].

**Definition 2.2.** Assume  $T = (\omega_1, <)$  is an  $\omega_1$ -tree,  $x \in T \cup \mathcal{B}(T)$  and  $N \prec H_\theta$  is countable with  $T \in N$ . We say that  $x$  is weakly external to  $N$  if there is a stationary  $\Sigma \subset [H_{(2^{\omega_1})^+}]^\omega$  in  $N$  such that

$$\forall M \in N \cap \Sigma, M \text{ does not capture } x.$$

Note that there is a major difference between the definition above and Definition 3.1 in [3]. If we require  $\Sigma$  to be a club we obtain the definition of external elements in [3]. This is why we call  $x$  weakly external in our definition. The purpose of this definition is to find Aronszajn suborders. It turns out that the existence of weakly external elements is enough for an  $\omega_1$ -tree to have Aronszajn subtrees. This should be compared with Theorem 4.1 in [3], where the existence of external elements is required for finding Aronszajn suborders. The proof we present here uses the ideas in the proof of Theorem 4.1 in [3], but we will include it for more clarity.

**Proposition 2.3.** Let  $T = (\omega_1, <)$  be an  $\omega_1$ -tree,  $\kappa = (2^{\omega_1})^+$  and  $\Sigma \subset [H_\kappa]^\omega$  be stationary. Assume for all large enough regular cardinal  $\theta$  there are  $x \in T$  and countable  $N \prec H_\theta$  such that  $x$  is weakly external to  $N$ , witnessed by  $\Sigma$ . In other words, for all  $M \in \Sigma \cap N$ ,  $M$  does not capture  $x$ . Then  $T$  has an Aronszajn subtree.

**Proof.** Fix  $\theta$  as in the proposition. For each  $t \in T$  let  $W_t$  be the set of all countable  $N' \prec H_\theta$  such that  $\Sigma, T$  are in  $N'$  and there is  $s > t$  such that for all  $M \in \Sigma \cap N'$ ,  $M$  does not capture  $s$ . Let  $A$  be the set of all  $t \in T$  such that  $W_t$  is stationary. We will show that  $A$  is Aronszajn.

First note that  $A$  is downward closed. This is because if  $t < t'$  then  $W_{t'} \subset W_t$ . Moreover, if  $t \in T$ ,  $\delta \in \omega_1$  and  $\text{ht}(t) < \delta$  then  $W_t = \bigcup \{W_s : s > t \text{ and } \text{ht}(s) = \delta\}$ . In other words, if  $A \neq \emptyset$  then  $A$  is uncountable. So it suffices to show that  $A \neq \emptyset$  and  $A$  does not contain any uncountable branch of  $T$ .

First we will show that  $A \neq \emptyset$ . Fix a regular cardinal  $\lambda > 2^\theta$  such that  $\theta$  is definable in  $H_\lambda$ . Let  $P \prec H_\lambda$  be countable such that for some  $x \in T$   $\Sigma$  witnesses that  $x$  is weakly external to  $P$ . Let  $t \in T \cap P$  and  $t < x$ . Then  $P \cap H_\theta \in W_t \in P$ . By the fact that  $P \prec H_\lambda$ ,  $W_t$  intersects every closed unbounded subset of  $[H_\theta]^\omega$  which means that it is stationary and  $A \neq \emptyset$ .

In order to see  $A$  contains no uncountable branch of  $T$ , assume for a contradiction that  $b \subset A$  is a cofinal branch. Let  $M \prec H_\kappa$  be countable such that  $T, A, b$ , are in  $M$  and  $M \in \Sigma$ . Let  $\delta = M \cap \omega_1$  and  $t = b(\delta)$ . Let  $N \prec H_\theta$  be countable such that  $N \in W_t$  and  $M \in N$ . This is possible because  $t \in A$  and  $W_t$  is a stationary subset of  $[H_\theta]^\omega$ . Let  $s > t$  be the element in  $T$  such that for all  $Z \in \Sigma \cap N$ ,  $Z$  does not capture  $s$ . But  $M \in \Sigma \cap N$  and it captures  $s$  via  $b$ . This is a contradiction.  $\square$

If  $T$  is an  $\omega_1$ -tree with an Aronszajn subtree  $A$ ,  $N \prec H_\theta$  is countable with  $A \in N$ , and  $x \in A \setminus N$ , then  $x$  is external to  $N$ . This makes the following corollary immediate.

**Corollary 2.4.** *Assume  $T = (\omega_1, <)$  is an  $\omega_1$ -tree. Then the following are equivalent:*

- $T$  has an Aronszajn subtree.
- For all large enough regular cardinal  $\theta$  there are  $x \in T$  and countable  $N \prec H_\theta$  such that  $x$  is external to  $N$ .
- For all large enough regular cardinal  $\theta$  there are  $x \in T$  and countable  $N \prec H_\theta$  such that  $x$  is weakly external to  $N$ .

We will use the following facts from [4] which are due to Jensen and Schlichta. For more clarity we will include the sketch of their proofs.

**Fact 2.5.** [4] *Assume  $A \in V$  is a countably closed poset,  $F \subset A$  is  $V$ -generic,  $B \in V$  is a ccc poset and  $G \subset B$  is  $V[F]$ -generic. Let  $T \in V[G]$  be a normal  $\omega_1$ -tree.*

- (1) *If  $b \in V[F][G]$  is a cofinal branch in  $T$ , then  $b \in V[G]$ .*
- (2) *If  $S \in V[F][G]$  is a downward closed Souslin subtree of  $T$  then  $S \in V[G]$ .*

**Proof.** Assume for a contradiction that  $b \in V[F][G] \setminus V[G]$  is a branch in  $T$ , and let  $\dot{b}$  be the name which is forced by 1 to be outside of  $V[G]$ . For  $k \in 2$ , let  $j_k : A \times B \rightarrow A^2 \times B$  be the injections which take  $(p, q)$  to  $(1, p, q)$  and  $(p, 1, q)$ . Obviously, these injections naturally induce injections on  $(A \times B)$ -names. We will abuse the notation and use  $j_k$  for the injections on names too. Let  $j_k(\dot{b}) = \tau_k$  for  $k \in 2$ . Since  $b \notin V[G]$ ,  $1_{A^2 \times B} \Vdash \tau_0 \neq \tau_1$ .

Note that the set  $D = \{(a_0, a_1) \in A^2 : \exists \alpha \in \omega_1 (a_0, a_1, 1_B) \Vdash \tau_0(\alpha) \neq \tau_1(\alpha)\}$  is dense in  $A^2$ . This uses an argument similar to the proof of fullness lemma and the fact that countably closed posets do not add new countable subsets of the ground model. Similarly, the set  $D_\alpha = \{a \in A : \text{for some } B\text{-name } \dot{x}, (a, 1_B) \Vdash \dot{b}(\alpha) = \dot{x}\}$  is dense in  $A$ . Now construct an increasing sequence  $\alpha_n, n \in \omega$  and  $a_s, \dot{x}_s$  for  $s \in 2^{<\omega}$  such that:

- $(a_s, 1) \Vdash \dot{b}(\alpha_{|s|}) = \dot{x}_s$  where  $\dot{x}_s$  is a  $B$ -name in  $V$ ,
- $a_{s \smallfrown 0}, a_{s \smallfrown 1}$  are both below  $a_s$  and  $(a_{s \smallfrown 0}, a_{s \smallfrown 1}, 1) \Vdash \dot{x}_{s \smallfrown 0} \neq \dot{x}_{s \smallfrown 1}$ .

For each  $r \in 2^\omega \cap V$  let  $a_r$  be the lower bound for  $\langle a_s : s \subset r \rangle$ . In  $V[G]$  let  $y_r$  be the element which is forced by  $a_r$  to be the element on top of  $\langle x_s : s \subset r \rangle$ . This means that  $T$  has an uncountable level in  $V[G]$  which is a contradiction.

The proof of the statement for Souslin subtrees uses similar ideas and the following facts, which we briefly mention. First note that if  $X$  is a countable subset of  $V$  which is in  $V[F][G]$  then  $X \in V[G]$ . Also, if  $S$  is a Souslin subtree of  $T$  in  $V[G]$  then it is Souslin in  $V[F][G]$ . If  $S \in V[F][G] \setminus V[G]$  is a downward closed Souslin subtree of  $T$  then there is downward closed Souslin  $S' \subset S$  such that every cone  $S'_x$  is outside of  $V[G]$  for all  $x \in S'$ .

Now assume for a contradiction that  $S$  is a Souslin subtree of  $T$  which is in  $V[F][G] \setminus V[G]$ . Without loss of generality we can assume that every cone  $S_x$  is outside of  $V[G]$ , for every  $x \in S$ . Assume  $\dot{S}$  is the name which is forced by 1 to be outside of  $V[G]$ . Again let  $\tau_k$  be the corresponding names  $j_k(\dot{S})$  as above.

Let  $S_k$  be the Souslin tree for  $\tau_k$ , for  $k \in 2$ , in the extension by  $(F_0, F_1, G) \subset A^2 \times B$  which is  $V$ -generic. Note that  $S_0 \cap S_1 \subset T_{<\alpha}$  for some  $\alpha \in \omega_1$ . In order to see this assume  $S_0 \cap S_1$  is uncountable. Then  $S_0 \cap S_1$  is an uncountable downwards closed subtree of  $S_0 \cup S_1$ . But  $S_0 \cup S_1$  is a Souslin tree. So  $S_0 \cap S_1$  contains a cone from

$S_0 \cup S_1$ . Then for some  $x \in S_0 \cap S_1$ ,  $(S_0)_x = (S_1)_x$ . But then  $(S_0)_x = (S_1)_x \in V[F_0][G] \cap V[F_1][G] = V[G]$ , which is a contradiction. Choose an increasing sequence  $\langle \alpha_n : n \in \omega \rangle$  and a sequence  $\langle a_s, \dot{x}_s : s \in 2^{<\omega} \rangle$  such that:

- $(a_s, 1) \Vdash \dot{S} \cap T_{<\alpha_{|s|}} = \dot{x}_s$ , where  $\dot{x}_s$  is a  $B$ -name in  $V$ ,
- $a_{s \smallfrown 0}, a_{s \smallfrown 1}$  are both below  $a_s$ ,
- $(a_{s \smallfrown 0}, a_{s \smallfrown 1}, 1) \Vdash T_{\alpha_s} \cap \dot{x}_{s \smallfrown 0} \cap \dot{x}_{s \smallfrown 1} = \emptyset$ .

For each  $r \in 2^\omega \cap V$ , let  $a_r$  be a lower bound for  $\langle a_s : s \subset r \rangle$ . Also let  $\alpha = \sup\{\alpha_n : n \in \omega\}$ . Now we work in  $V[G]$ . For each  $r$  let  $x_r = \bigcup_{s \subset r} \dot{x}_s[G]$ . Note that if  $r \neq r'$  then there is no  $t \in T_\alpha$  such that the set of predecessors of  $t$  is contained in  $x_r \cap x_{r'}$ . For each  $r$ , let  $y_r \in T_\alpha$  such that  $\{t \in T : t < y_r\} \subset x_r$ . But this means that  $T_\alpha$  is uncountable which is a contradiction.  $\square$

In this paper  $\text{coll}(\omega_1, < \lambda)$  refers to the usual Levy collapse forcing with countable conditions which collapses every cardinal less than  $\lambda$  to  $\omega_1$ . Fact 2.5 immediately implies the following lemma.

**Lemma 2.6.** *Let  $\lambda \in V$  be an inaccessible cardinal,  $F \subset \text{coll}(\omega_1, < \lambda)$  be  $V$ -generic,  $\mathbb{P}$  be a ccc poset of size  $\aleph_1$  in  $V[F]$ ,  $G \subset \mathbb{P}$  be  $V[F]$ -generic and  $U \in V[F][G]$  be an  $\omega_1$ -tree. Then  $U$  has at most  $\aleph_1$  many Souslin subtrees and cofinal branches in  $V^\mathbb{P}$ .*

**Proof.** For every  $\alpha \in \lambda$ , let  $F_\alpha = F \cap \text{coll}(\omega_1, < \alpha)$ . Let  $\kappa < \lambda$  be a regular uncountable cardinal such that  $\mathbb{P} \in V[F_\kappa]$  and  $U \in V[F_\kappa][G]$ . Fact 2.5 implies the following.

- If  $b \in V[F][G]$  is a cofinal branch of  $U$  then it is in  $V[F_\kappa][G]$ .
- If  $S \in V[F][G]$  is a Souslin subtree of  $U$  then it is in  $V[F_\kappa][G]$ .

It is obvious that  $|\mathcal{B}(T) \cap V[F_\kappa][G]| = \aleph_1$ , in  $V[F][G]$ . Similarly the conclusion follows for Souslin subtrees of  $U$ .  $\square$

The following Lemma from [1] is useful in finding club embeddings between  $\omega_1$ -trees.

**Lemma 2.7** (Lemma 3.2 of [1]). *Assume  $R$  and all its derived trees are Souslin,  $A$  is an Aronszajn tree and  $R'$  is a derived tree of  $R$  whose dimension is  $n$ . Moreover assume forcing with  $R'$  adds a new branch to  $A$  and  $R'$  has the least dimension with respect to this property among the derived trees of  $R$ . Then  $R'$  club embeds into  $A$ .*

**Lemma 2.8.** *For every Aronszajn tree  $A$  there is a forcing  $P_A$  which*

- *adds an uncountable antichain to  $A$ ,*
- *preserves cardinals and*
- *adds no new cofinal branch to  $\omega_1$ -trees of the ground model.*

**Proof.** For every  $\omega_1$ -tree  $A$ , let  $P_A$  be the poset consisting of all finite antichains in  $A$ . Based on the work in [2], if  $A$  is Aronszajn and  $W$  is an uncountable collection of pairwise disjoint finite antichains of  $A$ , then there are distinct  $x, y$  in  $W$  such that  $x \cup y \in P_A$ . Moreover,  $P_A$  is ccc if and only if  $A$  is Aronszajn. Since  $P_A$  is a ccc poset of size  $\aleph_1$ , it preserves cardinals. Usual density arguments, show that  $P_A$  adds an uncountable antichain to  $A$ .

First we show that  $P_A$  does not add branches to the Aronszajn trees of the ground model. Assume  $U$  is an Aronszajn tree. Without loss of generality assume  $U, A$  are disjoint. Obviously  $A \cup U$  with  $<_A \cup <_U$  is an Aronszajn tree. Define  $\varphi$  from  $P_A \times P_U$  to  $P_{A \cup U}$  by  $\varphi(a, b) = a \cup b$ . Observe that  $\varphi$  is an isomorphism. Therefore  $P_A \times P_U$  is ccc. Hence  $U$  remains Aronszajn after forcing with  $P_A$ .

Now we show that  $P_A$  does not add new cofinal branches to  $\omega_1$ -trees of the ground model. To this end, let  $U$  be an  $\omega_1$ -tree in the ground model, to which  $P_A$  adds a new branch. Since  $P_A$  is ccc, the possible points of the new branch form a Souslin subtree of  $U$ . In particular, there is a Souslin tree in the ground model to which  $P_A$  adds a cofinal branch. But this is impossible because of what we just showed above.  $\square$

It is worth pointing out that in the presence of CH there are posets which in addition to satisfying the requirements of Lemma 2.8, do not add new reals. This is the poset introduced in Remark 5.2 of [6]. Let  $Q_S$  be the poset consisting of all  $q = (X_q, \mathcal{U}_q)$  such that:

- $X_q$  is a countable downward closed subset of  $S$  which has a last level of height  $\alpha_q$ ,
- $\mathcal{U}_q$  is a non-empty countable set and for every  $U \in \mathcal{U}_q$  there exists  $n \in \omega$  such that  $U$  is a pruned downwards closed subtree of  $S^{[n]}$ ,
- for every  $U \in \mathcal{U}_q$  there is a  $\sigma \in U$  which is a subset of the last level of  $X_q$ .

We let  $p \leq q$  if  $(X_p)_{\leq \alpha_q} = X_q$  and  $\mathcal{U}_q \subset \mathcal{U}_p$ .

Observe that for every  $q \in Q_S$  and  $s \in S$  there are  $t > s$  and  $p < q$  such that  $\alpha_p > \text{ht}(t)$  and  $t \notin X_q$ . This shows that if  $G \subset Q_S$  is generic then  $\bigcup_{p \in G} X_p$  does not contain any cone  $S_s$ . Obviously,  $\bigcup_{p \in G} X_p$  is uncountable downward closed. Therefore, the minimal elements of  $S \setminus \bigcup_{p \in G} X_p$  form an uncountable antichain in  $S$ .

Lemma 5.3 of [6] asserts that there exists a poset which projects onto  $Q_S$  and which does not add new branches to  $\omega_1$ -trees of the ground model. Therefore,  $Q_S$  does not add new branches to  $\omega_1$ -trees of the ground model. The fact that  $Q_S$  preserves cardinals follows from Remark 5.2 in [6]. CH is only used for preserving  $\omega_2$ . The same remark also explains why  $Q_S$  does not add new reals.

We will use  $\square_{\omega_1}$  in order to have the structure of walks on ordinals up to  $\omega_2$ . The following is the standard definition of  $\square_{\omega_1}$ .

**Definition 2.9.** A sequence  $\langle C_\alpha : \alpha \text{ is limit and } \omega_1 < \alpha < \omega_2 \rangle$  is said to be a  $\square_{\omega_1}$ -sequence if

- $C_\alpha$  is a closed unbounded subset of  $\alpha$ ,
- $\text{otp}(C_\alpha) < \alpha$  and
- if  $\alpha$  is a limit point of  $C_\beta$  then  $C_\beta \cap \alpha = C_\alpha$ .

The assertion that there is a  $\square_{\omega_1}$ -sequence is called  $\square_{\omega_1}$ .

The following proposition is obtained from standard argument using  $\square_{\omega_1}$ -sequences.

**Proposition 2.10.** *If  $\square_{\omega_1}$  holds then there is a sequence  $\langle C_\alpha : \alpha \in \omega_2 \rangle$  such that*

- $C_\alpha$  is a closed unbounded subset of  $\alpha$ ,
- $C_{\alpha+1} = \{\alpha\}$ ,
- $\text{otp}(C_\alpha) \leq \omega_1$  and if  $\text{cf}(\alpha) = \omega$  then  $\text{otp}(C_\alpha) < \omega_1$ ,
- if  $\alpha \in C_\beta$  and  $\beta$  is limit then  $\text{cf}(\alpha) \leq \omega$ ,
- if  $\alpha$  is a limit point of  $C_\beta$  then  $C_\beta \cap \alpha = C_\alpha$ .

We only consider  $\square_{\omega_1}$ -sequences which have the properties mentioned in the proposition above. We will also use the following standard fact.

**Fact 2.11.** *Assume  $\lambda$  is a regular cardinal which is not Mahlo in  $L$ . Let  $G \subset \text{coll}(\omega_1, < \lambda)$  be  $L$ -generic. Then  $\square_{\omega_1}$  holds in  $L[G]$ .*

Now we briefly review some definitions and facts about walks on ordinals, from sections 7.3, 7.4, and 7.5 of [7] unless otherwise is mentioned. We fix a  $\square_{\omega_1}$ -sequence  $\langle C_\alpha : \alpha \in \omega_2 \rangle$  which satisfies the properties in Proposition 2.10.

We will use the following notation in the rest of the paper. For all  $X$ ,  $\alpha_X = \sup(X \cap \omega_2)$ . For each  $\alpha \in \omega_2$  we let  $L_\alpha$  be the set of all  $\beta \in \omega_2$  such that  $\alpha \in \lim(C_\beta)$ . For each  $\alpha < \beta$  in  $\omega_2$ , let  $\Lambda(\alpha, \beta)$  be the maximal limit point of  $C_\beta \cap (\alpha + 1)$  when such a limit point exists, otherwise  $\Lambda(\alpha, \beta) = 0$ .

**Definition 2.12** (See section 7.3 in [7]). The function  $\rho : [\omega_2]^2 \rightarrow \omega_1$  is defined recursively as follows: for  $\alpha < \beta$ ,

$$\rho(\alpha, \beta) = \max\{\text{otp}(C_\beta \cap \alpha), \rho(\alpha, \min(C_\beta \setminus \alpha)), \rho(\xi, \alpha) : \xi \in C_\beta \cap [\Lambda(\alpha, \beta), \alpha)\}.$$

We define  $\rho(\alpha, \alpha) = 0$  for all  $\alpha \in \omega_2$ . When the order between  $\alpha, \beta$  is not known we use  $\rho\{\alpha, \beta\}$  instead of  $\rho(\alpha, \beta)$ . More precisely,  $\rho\{\alpha, \beta\} = \rho(\alpha, \beta)$  if  $\alpha \leq \beta$  and  $\rho\{\alpha, \beta\} = \rho(\beta, \alpha)$  if  $\beta \leq \alpha$ .

**Lemma 2.13** (Lemma 7.3.6 of [7]). *Assume  $\xi \in \alpha$  and  $\alpha$  is a limit point of  $C_\beta$ . Then  $\rho(\xi, \alpha) = \rho(\xi, \beta)$ .*

**Lemma 2.14** (Lemma 7.3.11 of [7]). *If  $\alpha < \beta$ ,  $\alpha$  is a limit ordinal such that there is a cofinal sequence of  $\xi \in \alpha$ , with  $\rho(\xi, \beta) \leq \nu$  then  $\rho(\alpha, \beta) \leq \nu$ .*

**Lemma 2.15** (Lemma 7.3.8 of [7]). *For all  $\nu \in \omega_1$  and  $\alpha \in \omega_2$ , the set  $\{\xi \in \alpha : \rho(\xi, \alpha) \leq \nu\}$  is countable.*

**Lemma 2.16** (Lemma 7.3.7 of [7]). *Assume  $\alpha \leq \beta \leq \gamma$ . Then*

- $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$ ,
- $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$ .

The following lemma can be obtained in the same way as Lemma 3.1.3 of [7].

**Lemma 2.17.** [7] *Assume  $\alpha < \beta < \gamma$ . We have  $\rho(\alpha, \gamma) = \rho(\alpha, \beta)$ , if  $\rho(\beta, \gamma) < \max\{\rho(\alpha, \beta), \rho(\alpha, \gamma)\}$ .*

**Proof.** We only prove that if  $\rho(\alpha, \gamma) > \rho(\beta, \gamma)$  then  $\rho(\alpha, \gamma) = \rho(\alpha, \beta)$ . The other half of the statement can be proved by similar argument.  $\rho(\alpha, \beta) \leq \max\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\} = \rho(\alpha, \gamma)$ . On the other hand,  $\rho(\alpha, \gamma) \leq \max\{\rho(\alpha, \beta), \rho(\beta, \gamma)\}$  and  $\rho(\alpha, \gamma) > \rho(\beta, \gamma)$ . So  $\rho(\alpha, \gamma) \leq \rho(\alpha, \beta)$ . And this finishes the proof.  $\square$

**Lemma 2.18** (Lemma 7.3.10 of [7]). *Assume  $\beta \in \lim(\omega_2)$ , and  $\gamma > \beta$ . Then there is  $\beta' \in \beta$  such that for all  $\alpha \in (\beta', \beta)$ ,  $\rho(\alpha, \gamma) \geq \rho(\alpha, \beta)$ .*

**Lemma 2.19** (Lemma 7.4.7 of [7]). *Assume  $A$  is an uncountable family of finite subsets of  $\omega_2$  and  $\nu \in \omega_1$ . Then there is an uncountable  $B \subset A$  such that  $B$  forms a  $\Delta$ -system with root  $r$  and for all  $a, b$  in  $B$ :*

- $\rho\{\alpha, \beta\} > \nu$  for all  $\alpha \in a \setminus b$  and  $\beta \in b \setminus a$ ,
- $\rho\{\alpha, \beta\} \geq \min\{\rho(\alpha, \gamma), \rho(\beta, \gamma)\}$  for all  $\alpha \in a \setminus b$ ,  $\beta \in b \setminus a$  and  $\gamma \in a \cap b$ .



The following forcing is from the proof of Theorem 7.5.9 in [7].

**Definition 2.20.** [7] Assume  $A \subset \omega_2$ .  $Q_A$  is the poset consisting of all finite functions  $p$  such that the following holds.

- (1)  $\text{dom}(p) \subset A$ .
- (2) For all  $\alpha \in \text{dom}(p)$ ,  $p(\alpha) \in [\omega_1]^{<\omega}$  such that for all  $\nu \in \omega_1$ ,  $p(\alpha) \cap [\nu, \nu + \omega)$  has at most one element.
- (3) For all  $\alpha, \beta$  in  $\text{dom}(p)$ ,  $p(\alpha) \cap p(\beta)$  is an initial segment of both  $p(\alpha)$  and  $p(\beta)$ .
- (4) For all  $\alpha < \beta$  in  $\text{dom}(p)$ ,  $\max(p(\alpha) \cap p(\beta)) < \rho(\alpha, \beta)$  or  $p(\alpha) \cap p(\beta) = \emptyset$ .

We let  $q \leq p$  if  $\text{dom}(p) \subset \text{dom}(q)$  and  $\forall \alpha \in \text{dom}(p)$ ,  $p(\alpha) \subset q(\alpha)$ . We use  $Q$  in order to refer to  $Q_{\omega_2}$ . The poset  $Q_c$  consists of all conditions  $p$  in  $Q$  with the additional condition that for all  $\alpha \in \text{dom}(p)$ ,  $\text{cf}(\alpha) \leq \omega$ .

**Definition 2.21.** Assume  $G$  is generic for  $Q$ . Then for each  $\xi \in \omega_2$ ,  $b_\xi = \bigcup \{p(\xi) : p \in G\}$ .

Recall that a poset  $P$  satisfies the *Knaster condition* if every uncountable subset  $A$  of  $P$  contains an uncountable subset  $B$  such that the elements of  $B$  are pairwise compatible. Note that Knaster condition is stronger than ccc. Moreover, if  $P$  satisfies the Knaster condition then it does not add new cofinal branches to  $\omega_1$ -trees and its iteration with any ccc poset is ccc.

**Proposition 2.22** (*Theorem 7.5.9. in [7]*). *The poset  $Q$  satisfies the Knaster condition.*

We finish this section by some simple observations regarding the poset  $Q$ . Let  $G \subset Q$  be generic. For  $t \in s \in \omega_1$ , we let  $t < s$  if there is  $\alpha \in \omega_2$  and  $p \in G$  such that  $\alpha \in \text{dom}(p)$  and  $t, s$  are in  $p(\alpha)$ . By Condition 3 of Definition 2.20,  $<$  is transitive and  $T = (\omega_1, <)$  forms a tree. Also note that for all  $\alpha \in \omega_2$  and  $\nu \in \{0\} \cup \text{lim}(\omega_1)$  the set of all conditions  $q \in Q$  such that  $\alpha \in \text{dom}(q)$  and  $q(\alpha) \cap [\nu, \nu + \omega) \neq \emptyset$  is a dense subset of  $Q$ . So for each  $\alpha \in \omega_2$  and  $\nu \in \{0\} \cup \text{lim}(\omega_1)$ ,  $|b_\alpha \cap [\nu, \nu + \omega)| = 1$ . This means that for each  $\alpha \in \omega_2$ ,  $b_\alpha$  is a maximal uncountable branch of  $T$ . Similar arguments show that if  $s \neq t$  have limit heights in  $T$  then they have different sets of predecessors. In particular  $T$  is normal.

Moreover, it is easy to see that for all  $t \in T$ , the set of all  $q \in Q$  such that for some  $\alpha \in \text{dom}(q)$ ,  $t \in q(\alpha)$  forms a dense subset of  $Q$ . This means that for each  $t \in T$  there is  $\alpha \in \omega_2$  such that  $t \in b_\alpha$ . Therefore, for each  $\nu \in \{0\} \cup \text{lim}(\omega_1)$ , the set  $[\nu, \nu + \omega)$  is a level of the tree  $T$ . In particular  $T$  is an  $\omega_1$ -tree whose levels are countable sets that are in the ground model.

### 3. Complete suborders of $Q$

When we analyze subtrees of the generic tree  $T$ , which is added by  $Q$ , it will be useful to know if there is a complete suborder of  $Q$  which adds the tree  $T$  but does not add certain branches. In this section we will find some subsets of  $Q$  which are complete suborders of it.

**Lemma 3.1.** *The poset  $Q_c$  is a complete suborder of  $Q$ . Moreover, if  $X \subset \omega_2$  is a set of ordinals of cofinality  $\omega_1$ , then  $Q_{\omega_2 \setminus X}$  is a complete suborder of  $Q$ .*

**Proof.** We only prove the first part of the lemma. The second part can be verified by a similar argument. Assume  $q \in Q$ . We will show that there is  $q' \in Q_c$  such that for all extensions  $p \leq q'$  in  $Q_c$ , the conditions  $p, q$  are compatible. Without loss of generality we can assume that  $q$  has the following extra property: For all  $\xi < \eta$  in  $\text{dom}(q)$  there are distinct  $m, n$  in  $\omega$  such that  $\max(q(\xi) \cap q(\eta)) + \omega + m \in q(\xi)$  and  $\max(q(\xi) \cap q(\eta)) + \omega + n \in q(\eta)$ . In particular,  $q$  decides  $\max(b_\xi \cap b_\eta)$  in the generic tree.

Assume  $\{\beta_i : i \in n\}$  is the increasing enumeration of all ordinals in  $\text{dom}(q)$  which have cofinality  $\omega_1$ . Also let  $C$  be the set of all ordinals in  $\text{dom}(q)$  which have countable cofinality. Define  $\beta'_i$ , for each  $i \in n$ , to be the least ordinal  $\xi$  such that:

- (1)  $\xi$  is a limit point of  $C_{\beta_i}$ ,
- (2)  $\xi$  is strictly above all elements of  $\text{dom}(q) \cap \beta_i$ ,
- (3) for all  $\alpha \in \text{dom}(q) \setminus \beta_i$ ,  $\rho(\xi, \alpha) \geq \rho(\beta_i, \alpha)$ ,
- (4)  $\text{otp}(C_\xi) > \max(q(\beta_i))$

Note that the third requirement can easily be arranged by Lemma 2.15. Let  $q'$  be the condition in  $Q_c$  such that  $\text{dom}(q') = C \cup \{\beta'_i : i \in n\}$ ,  $q'(\alpha) = q(\alpha)$  for all  $\alpha \in C$ , and  $q'(\beta'_i) = q(\beta_i)$  for each  $i \in n$ . It is easy to see that  $q' \in Q_c$ .

Now let  $p < q'$  be in  $Q_c$ . Let  $r$  be the condition in  $Q$  such that:

- (1)  $\text{dom}(r) = \text{dom}(p) \cup \{\beta_i : i \in n\}$ ,
- (2)  $r(\alpha) = p(\alpha)$  for each  $\alpha \in \text{dom}(p)$ , and
- (3)  $r(\beta_i) = p(\beta'_i) \cap (\max(q(\beta_i)) + 1)$ .

It is easy to see that  $r$  is a common extension of  $p$  and  $q$ , provided that it is in  $Q$ . We only show that Condition 4, of Definition 2.20 holds for  $r$ . Assume  $\alpha \in \text{dom}(p)$  and  $\beta$  is one of the  $\beta_i$ 's. If  $\alpha < \beta'$  then  $\rho(\alpha, \beta') = \rho(\alpha, \beta)$ . Therefore,  $\max(r(\alpha) \cap r(\beta)) \leq \max(p(\alpha) \cap p(\beta')) < \rho(\alpha, \beta') = \rho(\alpha, \beta)$ , which was desired. If  $\beta' \leq \alpha < \beta$ , then

$$\rho(\alpha, \beta) \geq \text{otp}(C_{\beta'}) > \max(q(\beta)) = \max(r(\beta)) \geq \max(r(\beta) \cap r(\alpha)).$$

If  $\beta < \alpha$  note that by Lemma 2.17 either  $\rho(\beta, \alpha) \geq \rho(\beta', \alpha)$  or  $\rho(\beta', \beta) = \rho(\beta', \alpha)$ . In the first case there is nothing to show, and for the second case we have  $\rho(\beta', \alpha) = \rho(\beta', \beta) = \text{otp}(C_{\beta'}) > \max(q(\beta))$ .

Now assume that  $\alpha < \beta$  are both in  $\{\beta_i : i \in n\}$ . Then

$$\max(r(\alpha) \cap r(\beta)) \leq \max(p(\alpha') \cap p(\beta')) = \max(q'(\alpha') \cap q'(\beta')).$$

Here the inequality is obvious. The equality follows from the facts that  $q'(\beta'_i) = q(\beta_i)$ , for each  $i \in n$ , and  $q$  satisfies the extra property in the beginning of the proof. Moreover,

$$\max(q'(\alpha') \cap q'(\beta')) = \max(q(\alpha) \cap q(\beta)) < \rho(\alpha, \beta).$$

This assures us that  $r$  satisfies condition 4 of Definition 2.20.  $\square$

It is well known that if there is a ccc poset  $P$  which adds a branch  $b$  to an  $\omega_1$ -tree  $U$ , then  $\{u \in U : \exists p \in P, p \Vdash u \in \dot{b}\}$  is a Souslin subtree of  $U$ . Here,  $Q$  is a ccc poset and  $Q_c$  is a complete suborder of  $Q$ . Moreover, if  $G \subset Q$  is generic then  $G \cap Q_c$  knows the generic tree  $T$ . Since there is a ccc poset  $R$  such that  $Q$  is equivalent to  $Q_c * \dot{R}$ ,  $T$  has lots of Souslin subtrees in any extension by  $Q_c$ . This leads to the following corollary. In the next section we prove a stronger statement which we will use to prove a fact about  $\rho$ . For now, this corollary helps us to have a better picture of the forcing  $Q$ .

**Corollary 3.2.** *The generic tree for  $Q_c$  has Souslin subtrees.*

**Lemma 3.3.** *Assume CH. Let  $\langle N_\xi : \xi \in \omega_1 \rangle$  be a continuous  $\in$ -chain of countable elementary submodels of  $H_\theta$  where  $\theta$  is a regular large enough cardinal,  $N_{\omega_1} = \bigcup_{\xi \in \omega_1} N_\xi$ , and  $\mu = \sup(N_{\omega_1} \cap \omega_2)$ . Then  $Q_\mu$  is a complete suborder of  $Q$ .*

**Proof.** We need to show that for all  $q \in Q$  there is  $p \in Q_\mu$  such that if  $r \leq p$  and  $r \in Q_\mu$  then  $r$  is compatible with  $q$ . Let  $R = \bigcup \text{range}(q)$ ,  $L = \text{dom}(q) \cap \mu$ , and  $H = \text{dom}(q) \setminus \mu = \{\beta_i : i \in k\}$  such that  $\beta_i$  is increasing. Fix  $\bar{\nu} \in \omega_1$  which is above all elements of  $R$  and all  $\rho(\alpha, \beta)$  where  $\alpha, \beta$  are in  $\text{dom}(q)$ . Using Lemma 2.15, fix  $\mu_0 \in \mu$  above  $\max(L)$  such that for all  $\beta \in H$  and for all  $\gamma \in \mu \setminus \mu_0$ ,  $\rho(\gamma, \beta) > \bar{\nu}$ . For each  $\beta \in H$  and  $\nu \in \bar{\nu}$  let  $A_{\nu, \beta} = \{\alpha \in \mu_0 : \rho(\alpha, \beta) = \nu\}$ .

Again by Lemma 2.15, for all  $\nu \in \bar{\nu}$  and  $\beta \in H$ ,  $A_{\nu, \beta}$  is a countable subset of  $\mu_0$ . Since CH holds, we can fix  $N = N_\xi$  such that  $\mu_0, \bar{\nu}, L, R, \langle A_{\nu, \beta} : \beta \in H, \nu \in \bar{\nu} \rangle$  are in  $N$ . By elementarity, there is  $H' = \{\beta'_i : i \in k\}$  which is in  $N$  and

- (1)  $\beta'_i$  is increasing,
- (2)  $\min(H') > \mu_0$
- (3) for all  $i \in k$  and for all  $\nu \in \bar{\nu}$ ,  $A_{\nu, \beta_i} = \{\alpha \in \mu_0 : \rho(\alpha, \beta'_i) = \nu\}$ , and
- (4) for all  $i < j$  in  $k$ ,  $\rho(\beta_i, \beta_j) = \rho(\beta'_i, \beta'_j)$ .

Let  $p$  be the condition such that  $\text{dom}(p) = L \cup H'$ , for all  $\xi \in L$ ,  $p(\xi) = q(\xi)$  and for all  $i \in k$ ,  $p(\beta'_i) = q(\beta_i)$ . Suppose  $r \leq p$  is in  $Q_\mu$ . We will find  $s \in Q$  which is a common extension of  $r, q$ . Pick  $s$  such that  $\text{dom}(s) = \text{dom}(r) \cup H$ ,  $s \upharpoonright \text{dom}(r) = r$ , and for all  $i \in k$   $s(\beta_i) = r(\beta'_i) \cap (\max(q(\beta_i)) + 1)$ .

We need to show that  $s$  is a condition in  $Q$ . All of the conditions in Definition 2.20 obviously hold, except for condition 4. If  $\alpha < \beta$  are in  $H$ , by the last requirement for  $H'$  and the fact that  $r$  is a condition,  $\max(s(\alpha) \cap s(\beta)) < \rho(\alpha, \beta)$ .

Now assume that  $\alpha \in \text{dom}(r)$  and  $\beta = \beta_i \in H$ . If  $\rho(\alpha, \beta) \geq \bar{\nu}$ , everything is obvious because  $\max(s(\beta)) < \bar{\nu}$ . Assume  $\rho(\alpha, \beta) = \nu < \bar{\nu}$ . So  $\alpha \in A_{\nu, \beta}$ . Since  $r \in Q_\mu$ , we have  $\max(s(\alpha) \cap s(\beta_i)) \leq \max(r(\alpha) \cap r(\beta'_i)) < \nu$ .  $\square$

Lemma 3.3 shows that for many ordinals  $\mu$  with cofinality  $\omega_1$ ,  $Q_\mu$  is a complete suborder of  $Q$ . It is natural to ask the same question for ordinals of countable cofinality. The following fact shows that quite often  $Q_\mu$  is not a complete suborder of  $Q$ , when  $\mu$  varies over ordinals of countable cofinality.

**Fact 3.4.** *Assume  $\text{cf}(\mu) = \omega$ ,  $\mu \in \omega_2$ , for some  $\beta > \mu$ ,  $\mu$  is a limit point of  $C_\beta$  and the set of all limit points of  $C_\mu$  is cofinal in  $\mu$ . Then  $Q_\mu$  is not a complete suborder of  $Q$ .*

**Proof.** Assume  $\beta > \mu$  such that  $\mu$  is a limit point of  $C_\beta$  and  $\text{cf}(\beta) = \omega$ . Let  $\nu = \text{otp}(C_\beta)$  and  $q = \{(\beta, \{\nu\})\}$ . We claim that for all  $p \in Q_\mu$  there is an extension  $\bar{p} \leq p$  in  $Q_\mu$  such that  $\bar{p}$  is incompatible with  $q$ . Fix  $p \in Q_\mu$ . Without loss of generality  $\nu \in \bigcup \text{range}(p)$  and  $p$  is compatible with  $q$ . Let  $\xi \in \text{dom}(p)$  such that  $\nu \in p(\xi)$ . Then  $p \cup \{(\beta, p(\xi) \cap (\nu + 1))\} \in Q$ . Let  $\alpha$  be a limit point of  $C_\mu$  which is above all elements of  $\text{dom}(p)$ . Then  $\bar{p} = p \cup \{(\alpha, p(\xi) \cap (\nu + 1))\}$  is a condition in  $Q_\mu$ . But  $\rho(\alpha, \beta) = \text{otp}(C_\alpha) < \nu$  and  $\nu \in \bar{p}(\alpha)$ . Hence  $\bar{p}, q$  are incompatible.  $\square$

**Lemma 3.5.** *Assume  $\mu \in \omega_2$ ,  $x \subset \omega_2$  is finite and  $Q_\mu \triangleleft Q$ . Then  $Q_\mu \triangleleft Q_{\mu \cup x} \triangleleft Q$ .<sup>1</sup>*

**Proof.** Obviously  $Q_\mu \triangleleft Q$  implies  $Q_\mu \triangleleft Q_{\mu \cup x}$ . It suffices to show  $Q_{\mu \cup x} \triangleleft Q$  for all  $\mu \in \omega_2$  with  $Q_\mu \triangleleft Q$  and finite  $x \subset [\mu, \omega_2)$ . Let  $q \in Q$ . We need to show there is  $p \in Q_{\mu \cup x}$  such that every extension  $p'$  of  $p$  in  $Q_{\mu \cup x}$  is compatible with  $q$ . Without loss of generality, by extending  $q$  if necessary, we can assume that

- $x \subset \text{dom}(q)$ ,
- $q$  forces that  $\dot{b}_\alpha \wedge \dot{b}_\beta = \max(q(\alpha) \cap q(\beta))$  for all  $\alpha, \beta$  in  $\text{dom}(q)$ ,

<sup>1</sup>  $P_0 \triangleleft P_1$  denotes that  $P_0$  is a complete suborder of  $P_1$ .

- if  $\alpha, \beta$  are in  $\text{dom}(q)$  and  $\nu \in \{0\} \cup \text{lim}(\omega_1)$ , then  $q(\alpha)$  intersects  $[\nu, \nu + \omega)$  if and only if  $q(\beta)$  does.

It is easy to see that if  $r \leq_{Q_x} q \upharpoonright x$  then  $r$  is compatible with  $q$ . This already shows that for all finite  $y \subset \omega_2$ ,  $Q_y \triangleleft Q$ . Aside from the extra assumptions on  $q$ , we can also assume that  $\mu$  is infinite.<sup>2</sup>

Let  $p_0 \in Q_\mu$  such that for all extensions  $p_1$  of  $p_0$  in  $Q_\mu$  the conditions  $p_1, q$  are compatible. Since  $\mu$  is infinite, by extending  $p_0$  if necessary, we can assume  $\bigcup \text{range}(q) \subset \bigcup \text{range}(p_0)$ . Consequently, for every  $\alpha \in \text{dom}(q)$  there is  $\alpha' \in \text{dom}(p_0)$  such that  $q(\alpha) \subset p_0(\alpha')$ . Define  $p$  on  $\text{dom}(p_0) \cup x$  as follows:

- If  $\alpha \in x$  let  $p(\alpha) = p_0(\alpha') \cap (\max(q(\alpha)) + 1)$ , where  $\alpha' \in \text{dom}(p_0)$  such that  $q(\alpha) \subset p_0(\alpha')$ .
- If  $\alpha \in \text{dom}(p_0)$  let  $p(\alpha) = p_0(\alpha)$ .

By Condition 3 of Definition 2.20, for all  $\alpha \in x$ ,  $p(\alpha)$  is independent of the choice of  $\alpha'$ . Compatibility of  $p_0$  and  $q$  implies that  $p$  is a condition in  $Q_{\mu \cup x}$ . Let  $p'$  be an extension of  $p$  in  $Q_{\mu \cup x}$ . Then  $p' \upharpoonright \mu$  is an extension of  $p_0$  in  $Q_\mu$ , hence it is compatible with  $q$ . Also the conditions  $p' \upharpoonright x$  and  $q$  are compatible in  $Q$ , since  $p' \upharpoonright x \leq_{Q_x} q \upharpoonright x$ . This means that  $p' = (p' \upharpoonright \mu) \cup (p' \upharpoonright x)$  is compatible with  $q$  and we are done.  $\square$

#### 4. Climbing Souslin trees to see $\rho$

In this section we analyze the external elements of the generic Kurepa tree that is added by the poset  $Q_c$ . The aim is to prove Lemma 4.3, which is a general fact about the function  $\rho$ . We use Lemma 4.3 to find more weakly external elements in the tree which is generic for  $Q$ .

**Proposition 4.1.** *Fix  $\kappa$  a regular cardinal greater than  $(2^{\omega_1})^+$ . Assume  $S$  is the set of all  $X \in [\omega_2]^\omega$  such that  $C_{\alpha_X} \subset X$  and  $\text{lim}(C_{\alpha_X})$  is cofinal in  $X$ . Define  $\Sigma = \{M \prec H_\kappa : M \cap \omega_2 \in S \wedge |L_{\alpha_M}| = \aleph_2\}$ . Then  $\Sigma$  is stationary in  $[H_\kappa]^\omega$ .*

**Proof.** Let  $E \subset [H_\kappa]^\omega$  be a club. Fix  $\theta$  a regular cardinal above  $(2^\kappa)^+$ . Let  $\langle M_\xi : \xi \in \omega_1 \rangle$  be a continuous  $\in$ -chain of countable elementary submodels of  $H_\theta$  such that for all  $\xi \in \omega_1$ ,  $M_\xi \cap \omega_2 \in S$  and  $M_\xi \cap H_\kappa \in E$ . Let  $\alpha_\xi = \sup(M_\xi \cap \omega_2)$  and  $\alpha = \sup\{\alpha_\xi : \xi \in \omega_1\}$ . By thinning out if necessary, without loss of generality we can assume that for all  $\xi \in \omega_1$ ,  $\alpha_\xi$  is a limit point of  $C_\alpha$ .

Let  $f : \{\eta \in \omega_2 : |L_\eta| \leq \aleph_1\} \rightarrow \omega_2$  by  $f(\eta) = \sup(L_\eta)$ , and  $C_f$  be the set of all ordinals that are  $f$ -closed. Obviously  $f \in M_0$  and for all  $\xi$ ,  $\alpha_\xi \in C_f$ . But for any  $\xi \in \omega_1$ ,  $\sup L_{\alpha_\xi} \notin M_{\xi+1}$ . So, for all  $\xi \in \omega_1$ ,  $M_\xi \cap H_\kappa \in E \cap \Sigma$ .  $\square$

**Lemma 4.2.** *Assume  $G \subset Q_c$  is generic and  $T$  is the Kurepa tree that is added by  $G$ . Assume  $Q/G$  is the quotient poset such that  $Q$  is equivalent to  $Q_c * (Q/G)$ . For each  $\alpha$  of cofinality  $\omega_1$ , let  $A_\alpha = \{x \in T : \exists q \in Q/G \ q \Vdash \text{“}x \in \dot{b}_\alpha\text{”}\}$ . Then each  $A_\alpha$  is a Souslin subtree of  $T$ . Moreover, there is  $\alpha \in \omega_2$  of cofinality  $\omega_1$  such that for all  $x \in A_\alpha$ ,  $T_x$  contains  $\aleph_2$  many  $b_\xi$  with  $\text{cf}(\xi) = \omega$ .*

**Proof.** It is trivial that  $A_\alpha$  is a Souslin subtree of  $T$ . For the rest of the lemma, let  $\theta > (2^{\omega_1})^+$  be a regular cardinal, and assume  $S$  is the set of all  $X \in [\omega_2]^\omega$  such that  $C_{\alpha_X} \subset X$  and  $\text{lim}(C_{\alpha_X})$  is cofinal in  $X$ . Let  $\langle M_\xi : \xi \in \omega_1 \rangle$  be a continuous  $\in$ -chain of countable elementary submodels of  $H_\theta$  such that for all  $\xi \in \omega_1$ ,  $M_\xi \cap \omega_2 \in S$ . Let  $\alpha_\xi = \sup(M_\xi \cap \omega_2)$  and  $\alpha = \sup\{\alpha_\xi : \xi \in \omega_1\}$ . Also fix  $q \in Q$  with  $\alpha \in \text{dom}(q)$ ,  $t \in q(\alpha)$ , and  $\gamma \in \omega_2$ . We find  $\eta > \gamma$  and  $p \leq q$  such that  $\text{cf}(\eta) = \omega$ ,  $\eta \in \text{dom}(p)$ , and  $t \in p(\eta)$ . Find  $\alpha' \in \text{lim}(C_\alpha)$  such that:

<sup>2</sup> In fact we can assume that  $\mu$  is uncountable. This is because there is no countably infinite  $A \subset \omega_2$  with  $Q_A \triangleleft Q$ . This is trivial from the ideas in the proof of Lemma 3.5 and we are not going to use it.

- (1)  $\text{dom}(q) \cap [\alpha', \alpha] = \emptyset$ ,
- (2)  $\text{otp}(C_{\alpha'})$  is above all elements of  $\bigcup \text{range}(q)$  and all  $\rho(\{\alpha, \beta\})$  for  $\beta \in \text{dom}(q)$ ,
- (3) for all  $\beta \in \text{dom}(q) \setminus \alpha$ ,  $\rho(\alpha, \beta) \leq \rho(\alpha', \beta)$ ,
- (4)  $|L_{\alpha'}| = \aleph_2$ .

Now pick  $\eta \in L_{\alpha'}$  which is above  $\gamma$  and all elements of  $\text{dom}(q)$  with  $\text{cf}(\eta) = \omega$ . Define  $p$  by  $\text{dom}(p) = \text{dom}(q) \cup \{\eta\}$ ,  $q(\zeta) = p(\zeta)$ , for all  $\zeta \in \text{dom}(q)$  and  $p(\eta) = q(\alpha) \cap (t + 1)$ . We show that for all  $\zeta \in \text{dom}(p)$ ,  $\max(p(\zeta) \cap p(\eta)) \leq \rho(\zeta, \eta)$ . If  $\zeta < \alpha'$ ,

$$\max(p(\zeta) \cap p(\eta)) \leq \max(q(\zeta) \cap q(\alpha)) < \rho(\zeta, \alpha) = \rho(\zeta, \alpha') = \rho(\zeta, \eta).$$

Also  $\max(p(\alpha) \cap p(\eta)) = \max(q(\alpha) \cap q(\eta)) = t < \text{otp}(C_{\alpha'}) \leq \rho(\alpha, \eta)$ . When  $\zeta$  is above  $\alpha$ ,

$$\max(p(\zeta) \cap p(\eta)) \leq \max(q(\zeta) \cap q(\alpha)) < \rho(\alpha, \zeta) \leq \text{otp}(C_{\alpha'}) \leq \rho(\zeta, \eta). \quad \square$$

Now we are ready to prove the main lemma of this section.

**Lemma 4.3.** *Let  $(2^{\omega_1})^+ < \kappa_0 < \kappa < \theta$  be regular cardinals such that  $(2^{\kappa_0})^+ < \kappa$ , and  $(2^\kappa)^+ < \theta$ . Let  $S$  be the set of all  $X \in [\omega_2]^\omega$  such that  $C_{\alpha_X} \subset X$  and  $\text{lim}(C_{\alpha_X})$  is cofinal in  $X$ . Assume  $\mathcal{A}$  is the set of all countable  $N \prec H_\theta$  with the property that if  $N \cap \omega_2 \in S$  then there is a club of countable elementary submodels  $E \subset [H_{\kappa_0}]^\omega$  in  $N$  such that for all  $M \in E \cap N$ ,*

$$\rho(\alpha_M, \alpha_N) \leq M \cap \omega_1.$$

*Then  $\mathcal{A}$  contains a club.*

**Proof.** Assume  $G$  is the  $V$ -generic filter over  $Q_c$  and  $T$  be the tree that is introduced by  $G$ . Assume  $\dot{A}$  is a  $Q_c$ -name for an Aronszajn subtree of  $T$  with the property that for all  $t \in \dot{A}$ , the set  $\{\xi \in \omega_2 : \text{cf}(\xi) < \omega_1 \text{ and } t \in b_\xi\}$  has size  $\aleph_2$ . Fix  $N \prec H_\theta$ , in  $V$ , with  $\dot{A} \in N$  and  $N \cap \omega_2 \in S$ . Suppose for a contradiction that

$$(*): \text{ for all clubs } E \subset [H_{\kappa_0}]^\omega \text{ in } N \text{ there is } M \in E \cap N \text{ such that } \rho(\alpha_M, \alpha_N) > M \cap \omega_1.$$

Let  $\delta_M, \delta_N$  be  $M \cap \omega_1$  and  $N \cap \omega_1$  respectively. Fix  $t \in [\delta_N, \delta_N + \omega)$ ,  $q \in Q_c$  such that  $q$  forces that  $t \in \dot{A}$ . Obviously,  $q$  forces that  $t$  is external to  $N[\dot{G}]$ . In other words,  $q$  forces that there is a club  $E \subset [H_{\kappa_0}[\dot{G}]]^\omega$  in  $N[\dot{G}]$  such that for all  $Z \in E \cap N[\dot{G}]$ ,  $Z$  does not capture  $t$ . Let  $\dot{E}$  be a name for the witness  $E$  above. So  $q$  forces that for all  $Z \in \dot{E} \cap N[\dot{G}]$ ,  $Z$  does not capture  $t$ . In order to reach a contradiction, it suffices to show  $(*)$  implies that there are  $M \prec H_\kappa$  in  $N$  and  $p \leq q$  in  $Q_c$  such that:

- (1)  $\dot{E} \in M$  and
- (2)  $p$  forces that  $M[\dot{G}]$  captures  $t$ .

We consider three cases. First, consider the case where  $t \notin \bigcup \text{range}(q)$ . Let  $\gamma \in (N \cap \omega_2) \setminus \text{dom}(q)$ , with  $\text{cf}(\gamma) = \omega$ . Let  $M \prec H_\kappa$  be in  $N$  such that  $\gamma, \dot{E}$  are in  $M$ . Let  $p$  be the condition such that  $\text{dom}(p) = \text{dom}(q) \cup \{\gamma\}$ ,  $\forall \xi \in \text{dom}(q) p(\xi) = q(\xi)$ , and  $p(\gamma) = \{t\}$ . It is obvious that  $p$  is an extension of  $q$  and it forces that  $M[\dot{G}]$  captures  $t$  via  $\dot{b}_\gamma$ .

Now suppose for some  $\xi \in \text{dom}(q) \cap N$ ,  $t \in q(\xi)$ . In this case assume  $M \prec H_\kappa$  is in  $N$  such that  $\dot{E}, \xi$  are in  $M$ . Then  $q$  forces that  $M[\dot{G}]$  captures  $t$  via  $\dot{b}_\xi$ .

For the last case, suppose  $t \in \bigcup \text{range}(q)$  but  $\forall \xi \in \text{dom}(q) \cap N t \notin q(\xi)$ . Since any element of  $\dot{A}$  is an element of  $\aleph_2$  many branches  $b_\xi \subset T$  with  $\text{cf}(\xi) < \omega_1$ , by extending  $q$  if necessary, we can assume that there

is  $\tau \in \text{dom}(q) \setminus \alpha_N$  such that  $t \in q(\tau)$ . We consider the partition  $\text{dom}(q) = H \cup L \cup R$  where  $R = \text{dom}(q) \cap N$  (rudimentary ordinals w.r.t.  $N$ ),  $L = (\text{dom}(q) \cap \alpha_N) \setminus R$  (low ordinals), and  $H = \text{dom}(q) \setminus \alpha_N$  (high ordinals). Let  $B_t$  be the set of all  $\xi \in \text{dom}(q)$  such that  $t \in q(\xi)$ . So  $B_t \cap R = \emptyset$  and  $\tau \in B_t$ . By Lemma 2.18 we have the following about the ordinals in  $H$ :

$$\exists \gamma_0 \in N \cap \omega_2 \forall \gamma \in N \setminus \gamma_0 \forall \xi \in H \rho(\gamma, \xi) \geq \rho(\gamma, \alpha_N) \tag{1}$$

For ordinals in  $L$ , let  $\gamma_1 = \max\{\min((N \cap \omega_2) \setminus \xi) : \xi \in L\}$ . Then

$$\forall \gamma \in N \setminus \gamma_1 \forall \xi \in L \rho(\xi, \gamma) \geq \delta_N. \tag{2}$$

In order to see (2), fix  $\xi \in L$  and let  $\xi' = \min(N \cap \omega_2) \setminus \xi$ . Observe that  $\text{cf}(\xi') = \omega_1$ . Let  $\gamma \in N$  be above  $\gamma_1$ . We show that  $\rho(\xi, \gamma) \geq N \cap \omega_1$ . Note that there is  $\alpha \in \xi'$  such that for all  $\eta \in (\alpha, \xi')$  the ordinal  $\text{otp}(C_{\xi'} \cap \eta)$  appears in the definition of  $\rho(\eta, \gamma)$ . Since  $\gamma, \xi'$  are in  $N$ , by elementarity, the witness  $\alpha$  exists in  $N$ . Since  $\xi \in (\alpha, \xi')$  the ordinal  $\text{otp}(C_{\xi'} \cap \xi)$  appears in the definition of  $\rho(\xi, \gamma)$ . But  $\text{otp}(C_{\xi'} \cap \xi) \geq \delta_N$ , which shows (2).

Now using (\*) choose  $M \prec H_\kappa$  in  $N$  such that  $\rho(\alpha_M, \alpha_N) > \delta_M$  and such that  $M$  has  $\gamma_0, \gamma_1, R, \bigcup \text{range}(q) \cap N, \dot{E}$  as elements. Let  $\gamma_3 > \max\{\gamma_0, \gamma_1\}$  be in  $M$  such that for all  $\gamma \in M$  that are above  $\gamma_3$ ,  $\rho(\gamma, \alpha_N) > \delta_M$ . The ordinal  $\gamma_3$  is guaranteed to exist by Lemma 2.14.

For every  $\xi \in R$  and  $\eta \in B_t$  by the initial segment requirement on the conditions in  $Q$ ,  $\max(q(\xi) \cap q(\eta)) = \max(q(\tau) \cap q(\xi))$ . If  $\max(q(\xi) \cap q(\tau)) \notin M$  for some  $\xi \in R$ , we are done. Assume  $\max(q(\xi) \cap q(\tau)) \in M$ , for all  $\xi \in R$ . By elementarity, fix  $\gamma > \gamma_3$  in  $M$  such that  $\text{cf}(\gamma) = \omega$  and

$$\forall \xi \in R \rho(\xi, \gamma) > \max(q(\tau) \cap q(\xi)). \tag{3}$$

Now define  $p \leq q$  as follows:

- $\text{dom}(p) = \text{dom}(q) \cup \{\gamma\}$ ,
- $\forall \xi \in \text{dom}(q) \setminus B_t \ p(\xi) = q(\xi)$ ,
- $\forall \xi \in B_t \ p(\xi) = q(\xi) \cup \{\delta_M\}$ ,
- $p(\gamma) = p(\tau) \cap (\delta_M + 1)$ .

Obviously,  $p$  forces that  $M[\dot{G}]$  captures  $t$  via  $\dot{b}_\gamma$ , provided that  $p \in Q_c$ . It is obvious that  $p$  fulfills the initial segment requirement. Moreover,  $\bigcup \text{range}(p) \setminus \bigcup \text{range}(q) = \{\delta_M\}$  because  $\bigcup \text{range}(q) \cap N \in M$ . We show for all  $\xi, \eta$  in  $\text{dom}(p)$ ,  $\rho\{\xi, \eta\} > \max(p(\xi) \cap p(\eta))$ . This can be done by managing the following six cases.

First assume that  $\xi, \eta$  are in  $\text{dom}(q)$  and at least one of them is not in  $B_t$ . Equivalently,  $\eta \in \text{dom}(q)$  and  $\xi \in \text{dom}(q) \setminus B_t$ . Then  $\delta_M \notin p(\xi)$  and  $p(\xi) = q(\xi)$ . Hence  $\max(p(\xi) \cap p(\eta)) = \max(q(\xi) \cap q(\eta)) < \rho\{\xi, \eta\}$  because  $q$  is a condition in  $Q$ .

For the second case assume  $\xi, \eta$  are both in  $B_t$ . Recall  $\delta_M < t$  and  $t \in q(\xi) \cap q(\eta)$ . Then  $\max(p(\xi) \cap p(\eta)) = \max(q(\xi) \cap q(\eta)) < \rho\{\xi, \eta\}$ . So far we have shown that condition 4 of Definition 2.20 holds for pairs of ordinals in  $\text{dom}(q)$ .

For the fourth case assume  $\xi \in H$  and  $\eta = \gamma$ . The way we chose  $\gamma_3$ , and (1) guarantees that  $\rho(\gamma, \xi) \geq \rho(\gamma, \alpha_N) > \delta_M = \max(p(\gamma))$ .

For the fifth case assume  $\xi \in L$  and  $\eta = \gamma$ . Then (2) implies that  $\rho(\xi, \gamma) \geq \delta_N > \max(p(\gamma))$ .

For the sixth case assume  $\xi \in R$  and  $\eta = \gamma$ . Then (3) implies that  $\rho(\xi, \gamma) > \max(q(\tau) \cap q(\xi)) = \max(p(\gamma) \cap p(\xi))$ . Therefore,  $p \in Q_c$ .  $\square$

## 5. $\rho$ introduces Aronszajn subtrees everywhere in $T$

In this section we will use Lemma 4.3 to show that every Kurepa subset of the generic Kurepa tree has an Aronszajn subtree. Here a subset  $Y$  of  $T$  is said to be a *Kurepa subset* if it is a Kurepa tree when it is considered with the order inherited from  $T$ . Note that  $Y$  is not necessarily downward closed. The theorems in this section are not using any large cardinal assumption.

**Lemma 5.1.** *Assume  $X \subset \omega_2$  is uncountable,  $Q_X \triangleleft Q$  and  $T$  is the generic tree for  $Q_X$ . Then  $\{b_\xi : \xi \in X\}$  is the set of all cofinal branches of  $T$  in the forcing extension by  $Q_X$ .*

**Proof.** Assume  $P = Q_X$  and  $\pi$  is a  $P$ -name for a branch that is different from all  $b_\xi$ ,  $\xi \in X$ . Inductively construct a sequence  $\langle p_\eta : \eta \in \omega_1 \rangle$  as follows. The condition  $p_0 \in P$  is arbitrary. If  $\langle p_\eta : \eta < \alpha \rangle$  is given, find  $p_\alpha \in P$  such that:

- $p_\alpha$  decides  $\min(\pi \setminus \bigcup\{b_\xi : \xi \in \bigcup\{\text{dom}(p_\eta) : \eta \in \alpha\}\})$  to be  $t_{p_\alpha}$ ,
- $t_{p_\alpha} \in \bigcup \text{range}(p_\alpha)$ ,
- for every  $\beta \in \text{dom}(p_\alpha)$ ,  $\text{ht}(\max(p_\alpha(\beta))) > \text{ht}(t_{p_\alpha})$ .<sup>3</sup>

Let  $A = \{p_\alpha : \alpha \in \omega_1\}$ . By going to a subset of  $A$  if necessary, we may assume that  $\{\text{dom}(p) : p \in A\}$  forms a  $\Delta$ -system with root  $d$ . Also  $\{\bigcup \text{range}(p) : p \in A\}$  forms a  $\Delta$ -system with root  $c$ . Moreover, we may assume that elements of  $A$  are pairwise isomorphic structures and the isomorphism between them fixes the root. By Lemma 2.19 there is an uncountable set  $B \subset A$  such that for every  $p, q$  in  $B$  if  $\alpha \in \text{dom}(p) \setminus \text{dom}(q)$ ,  $\beta \in \text{dom}(q) \setminus \text{dom}(p)$ , and  $\gamma \in d$ , then

- $\rho\{\alpha, \beta\} > \max(c)$  and
- $\rho\{\alpha, \beta\} \geq \min\{\rho\{\gamma, \alpha\}, \rho\{\gamma, \beta\}\}$ .

Note that for all  $p \in B$ ,  $\bigcup \text{range}(p) \subset \omega_1$ . So without loss of generality, by replacing  $B$  with an uncountable subset if necessary, we can assume the following: Whenever  $p, q$  are in  $B$  either

- $c < a = \bigcup \text{range}(p) \setminus c < b = \bigcup \text{range}(q) \setminus c$  or
- $c < b = \bigcup \text{range}(q) \setminus c < a = \bigcup \text{range}(p) \setminus c$ .

We claim that the elements of  $B$  are pairwise compatible. In order to see this, fix  $p, q$  in  $B$ . By symmetry, we can assume that

$$c < a = \bigcup \text{range}(p) \setminus c < b = \bigcup \text{range}(q) \setminus c.$$

We define the common extension  $r$  of  $p, q$  on  $\text{dom}(p) \cup \text{dom}(q)$  as follows: For  $\gamma \in d$  let  $r(\gamma) = p(\gamma) \cup q(\gamma)$ , and for  $\alpha \in \text{dom}(p) \setminus \text{dom}(q)$  let  $r(\alpha) = p(\alpha)$ . For  $\beta \in \text{dom}(q) \setminus \text{dom}(p)$  we have two cases. Either for all  $\gamma \in d$ ,  $\max(q(\gamma) \cap q(\beta)) \in c$  or there is a unique  $\gamma \in d$  such that  $\max(q(\gamma) \cap q(\beta)) \in b$ . In the first case let  $r(\beta) = q(\beta)$  and in the second case let  $r(\beta) = p(\gamma) \cup q(\beta)$ . In order to see that there is no possibility outside of these two cases, assume for a contradiction that  $\gamma_0, \gamma_1$  are in  $d$  and for  $i \in 2$ ,  $\max(q(\gamma_i) \cap q(\beta)) \in b \setminus c$ . In other words, both  $q(\gamma_0), q(\gamma_1)$  intersect  $q(\beta)$  above  $\max(c)$ . So there is  $\nu \in b \setminus c$  such that  $\nu \in q(\gamma_0) \cap q(\gamma_1)$ . Recall that the elements of  $B$  are isomorphic structures via the isomorphisms which fix the roots. Therefore,

<sup>3</sup> Note that the levels of the generic tree are in the ground model.

for each  $s \in B$  there is  $\nu_s \in \bigcup \text{range}(s) \setminus c$  such that  $\nu_s \in s(\gamma_0) \cap s(\gamma_1)$ . But this contradicts the fact that  $\rho(\gamma_0, \gamma_1)$  is countable, since  $\{\bigcup \text{range}(s) : s \in B\}$  is an uncountable  $\Delta$ -system with root  $c$ .

First we will show that  $r$  satisfies Condition 3. Note that if  $\gamma_1, \gamma_2$  are both in  $d$  then  $p(\gamma_1) \cap p(\gamma_2) \subset c$  and  $q(\gamma_1) \cap q(\gamma_2) \subset c$ . In order to see this, assume this is not the case. Then by the fact that the conditions in  $B$  are pairwise isomorphic,  $\sup\{\max(s(\gamma_1) \cap s(\gamma_2)) : s \in B\} = \omega_1$  which implies that  $\rho(\gamma_1, \gamma_2) \geq \omega_1$ . But this is absurd. Now assume  $i \in (p(\gamma_1) \cup q(\gamma_1)) \cap (p(\gamma_2) \cup q(\gamma_2))$ ,  $j < i$  and  $j \in (p(\gamma_1) \cup q(\gamma_1))$ . We will show that  $j \in p(\gamma_2) \cup q(\gamma_2)$ . Note that  $j \in c$ . Then  $j \in p(\gamma_1) \cap c = q(\gamma_1) \cap c$ . Since  $p, q$  both satisfy Condition 3 and  $i \in p(\gamma_2) \cup q(\gamma_2)$ , we have  $j \in p(\gamma_2) \cup q(\gamma_2)$ . If  $\alpha \in \text{dom}(p) \setminus \text{dom}(q)$  and  $\gamma \in d$  note that

$$r(\alpha) \cap r(\gamma) = p(\alpha) \cap (p(\gamma) \cup q(\gamma)) = p(\alpha) \cap p(\gamma).$$

But  $p(\alpha) \cap p(\gamma)$  is an initial segment of both  $p(\alpha)$  and  $r(\gamma)$  because  $a < b$ . If  $\beta \in \text{dom}(q) \setminus \text{dom}(p)$  and for all  $\gamma \in d$ ,  $\max(q(\gamma) \cap q(\beta)) \in c$  the argument is the same. So assume that for a unique  $\gamma_\beta \in d$ ,  $\max(q(\beta) \cap q(\gamma_\beta)) \in b$ . Then it is easy to see that  $r(\beta) \cap r(\gamma_\beta) = p(\gamma_\beta) \cup (q(\beta) \cap q(\gamma_\beta))$  is an initial segment of both  $r(\beta), r(\gamma_\beta)$ . If  $\beta \in \text{dom}(q) \setminus \text{dom}(p)$  and  $\gamma \in d \setminus \{\gamma_\beta\}$ , in order to see  $r(\beta) \cap r(\gamma)$  is an initial segment of both  $r(\beta), r(\gamma)$ , note that

$$r(\beta) \cap r(\gamma) = (p(\gamma_\beta) \cup q(\beta)) \cap (p(\gamma) \cup q(\gamma)) \subset c.$$

Then  $r(\beta) \cap r(\gamma) = p(\gamma_\beta) \cap p(\gamma)$  which makes Condition 3 trivial. We leave the rest of the cases to the reader.

For Condition 4, we only verify the case  $\alpha \in \text{dom}(p) \setminus \text{dom}(q)$  and  $\beta \in \text{dom}(q) \setminus \text{dom}(p)$ . If for all  $\gamma \in d$ ,  $\max(q(\gamma) \cap q(\beta)) \in c$ , there is nothing to prove. Assume for some unique  $\gamma \in d$ ,  $\max(q(\gamma) \cap q(\beta)) \in b$ . Obviously,  $r(\alpha) \cap r(\beta) = (p(\alpha) \cap p(\gamma)) \cup (p(\alpha) \cap q(\beta))$ . But  $\max(p(\alpha) \cap q(\beta)) \leq \max(c) < \rho\{\alpha, \beta\}$ . Moreover,

$$\max(p(\alpha) \cap p(\gamma)) \leq \max(a) < \min(b) \leq \max(q(\beta) \cap q(\gamma)) \leq \rho\{\gamma, \beta\}.$$

This means that  $\max(p(\alpha) \cap p(\gamma)) < \min\{\rho\{\alpha, \gamma\}, \rho\{\beta, \gamma\}\} \leq \rho\{\alpha, \beta\}$ . Therefore,  $\max(r(\alpha) \cap r(\beta)) < \rho\{\alpha, \beta\}$ .

We have two possible cases: either there is an uncountable  $C \subset B$  such that for all  $p \in C$ , there is  $\gamma \in d$  with  $t_p \in p(\gamma)$ , or there are only countably many  $p \in B$  such that for some  $\gamma \in d$ ,  $t_p \in p(\gamma)$ . If such an uncountable  $C$  exists, let  $s \in P$  such that  $s$  forces that the generic filter intersects  $C$  on an uncountable set. Then for some  $\gamma \in d$ ,  $s \Vdash |\pi \cap b_\gamma| = \aleph_1$ . But this contradicts the fact that  $\pi$  was a name for a branch that is different from all  $b_\xi$ 's.

Now assume that there is a countable set  $D \subset B$  such that if  $p \in B$  and for some  $\gamma \in d$ ,  $t_p \in p(\gamma)$  then  $p \in D$ . We can choose  $p, q$  in  $B \setminus D$  such that:

- (1) for some  $\alpha \in \text{dom}(p) \setminus \text{dom}(q)$ ,  $t_p \in p(\alpha)$ ,
- (2) for some  $\beta \in \text{dom}(q) \setminus \text{dom}(p)$ ,  $t_q \in q(\beta)$ ,
- (3)  $p$  forces that  $t_p$  is not in the branches that are indexed by the ordinals in  $d$ , and
- (4)  $\max(c) + \omega < t_p$  and  $t_p + \omega < t_q$ .

Obviously, (1), (2) are automatically true for any  $p, q$  in  $B \setminus D$ . We claim that there is at most one  $p_\eta \in B$  which does not force that  $t_{p_\eta}$  is not in the branches that are indexed by the ordinals in  $d$ . In order to see this, assume for a contradiction that  $\zeta < \eta < \omega_1$  and  $p_\eta, p_\zeta$  are counterexamples to our claim. Then  $p_\eta$  decides  $\min(\pi \setminus \bigcup\{b_i : i \in \bigcup\{\text{dom}(p_j) : j \in \eta\}\})$  to be  $t_{p_\eta}$ . In particular,  $p_\eta$  forces that  $t_{p_\eta} \notin \bigcup\{b_i : i \in \text{dom}(p_\zeta)\} \supset \bigcup\{b_i : i \in d\}$ . Therefore,  $p_\eta$  satisfies Condition (3), which is a contradiction. By the same argument, if  $p \neq q$  are in  $B$  then  $t_p \neq t_q$ . Therefore, it is easy to choose  $p, q$  in  $B$  such that the four conditions above hold.



Let  $a, b, c, d$  be as above. We will find a common extension of  $p, q$  which forces that  $t_p$  is not below  $t_q$ . This contradicts the fact that  $\pi$  was a name for a branch.

First consider the case in which for all  $\gamma \in d$ ,  $\max(q(\beta) \cap q(\gamma)) \in c$ . Let  $r$  be the common extension of  $p, q$  described as above. Recall that  $r(\beta) = q(\beta)$  in this case. Let  $\xi \in (t_p, t_p + \omega) \setminus (a \cup b)$ . Note that  $\xi > \max(c)$ . Let  $X = \{\eta \in \text{dom}(r) : \max(r(\beta) \cap r(\eta)) > \xi\}$ . Obviously,  $X \cap \text{dom}(p) = \emptyset$  and  $\beta \in X$ . Extend  $r$  to  $r'$  such that  $\text{dom}(r') = \text{dom}(r)$ ,  $r$  and  $r'$  agree on any element of their domain which is not in  $X$ , and  $r'(\eta) = r(\eta) \cup \{\xi\}$  for all  $\eta \in X$ . Checking  $r'$  is a condition is routine. The condition  $r'$  forces that in the generic tree  $\text{ht}(\xi) = \text{ht}(t_p)$  and they are distinct. Therefore, it forces that  $\xi < t_q$  and that  $t_p$  is not below  $t_q$ .

Now assume for some  $\gamma \in d$ ,  $\max(q(\beta) \cap q(\gamma)) \in b$ . Again assume that  $r$  is the common extension described above. So  $r(\beta) = p(\gamma) \cup q(\beta)$ , and  $r$  forces that  $\max(p(\gamma))$  is below  $t_q$  in the generic tree. Recall that  $\text{ht}(\max(p(\gamma))) > \text{ht}(t_p)$  and  $p$  forces that  $t_p$  is not in the branches indexed by the ordinals in the root  $d$ . Hence  $p$  forces that  $t_p$  is not below  $\max(p(\gamma))$ . Since  $r \leq p$ , it forces that  $t_p$  is not below  $t_q$  in the generic tree.  $\square$

Now we are ready to prove the main theorem of this section.

**Theorem 5.2.** *It is consistent that there is a Kurepa tree  $T$  such that every Kurepa subset of  $T$  has an Aronszajn subtree.*

**Proof.** Assume  $G$  is a generic filter for the forcing  $Q$ , and  $T$  is the tree introduced by  $G$ . Since  $Q$  is ccc, it preserves all cardinals and  $T$  is a Kurepa tree.

Assume  $U$  is a Kurepa subset of  $T$ , and  $X$  is the set of all  $\xi \in \omega_2$  such that  $b_\xi \cap U$  is uncountable. Let  $\langle N_\xi : \xi \in \omega_1 \rangle$  be a continuous  $\in$ -chain of countable elementary submodels of  $H_\theta$  such that  $U \in N_0$  and for all  $\xi \in \omega_1$ ,  $N_\xi \in \mathcal{A}$ , where  $\mathcal{A}$  is the same club as in Lemma 4.3. Let  $N_{\omega_1} = \bigcup_{\xi \in \omega_1} N_\xi$ ,  $\mu = N_{\omega_1} \cap \omega_2$ . Fix  $\eta \in X$  above  $\mu$ . By Proposition 2.3, it suffices to show that for some  $\xi \in \omega_1$ , the first element of  $b_\eta \cap U$  whose height is more than  $N_\xi \cap \omega_1$  is weakly external to  $N_\xi$  witnessed by some stationary set  $\Sigma$ .

Without loss of generality we can assume that for all  $\xi \in \omega_1$ :

- $\alpha_{N_\xi} = \sup(N_\xi \cap \omega_2)$  is a limit point of  $C_\mu$ ,
- $N_\xi \cap \omega_2 \supset C_{\alpha_{N_\xi}}$  and
- $\lim(C_{\alpha_{N_\xi}})$  is a cofinal in  $\alpha_{N_\xi}$ .

In order to see this, let  $f$  from  $\omega_1$  to  $\mu$  be the function which is defined as follows: For each  $\xi \in \omega_1$ ,  $f(\xi)$  is the least  $\zeta \in \omega_1$  with  $N_\zeta \supset C_\mu \cap \alpha_{N_\xi}$ . Now observe that if  $\xi$  is  $f$ -closed then it satisfies the second condition. For the other two conditions, note that the sets  $\{\alpha_{N_\xi} : \xi \in \omega_1\}$  and the set of all  $\gamma \in C_\mu$  which are limit of limit points in  $C_\mu$  are clubs in  $\mu$ .

Let  $\xi \in \omega_1$  be such that  $\text{otp}(C_{\alpha_{N_\xi}}) > \rho(\mu, \eta)$  and for all  $\zeta > \xi$ ,  $\rho(\alpha_{N_\zeta}, \eta) > \rho(\mu, \eta)$ . Then note that  $\rho(\mu, \eta) \in N_\xi$ . Use Lemma 4.3 to find  $E \in N_\xi$  which is a club of countable elementary submodels of  $H_{\omega_3}$  such that for all  $M \in E \cap N_\xi$ ,  $\rho(\mu, \eta) \in M$  and  $\rho(\alpha_M, \alpha_{N_\xi}) \leq M \cap \omega_1$ . Now let  $\Sigma$  be the set of all  $M \in E$  such that  $M \cap \omega_2 \supset C_{\alpha_M}$  and  $\lim(C_{\alpha_M})$  is a cofinal subset of  $\alpha_M$ . Obviously,  $\Sigma$  is stationary and in  $N_\xi$ . Let  $M \in \Sigma \cap N_\xi$ . We want to show that  $M$  does not capture  $b_\eta$ , as a branch of  $T$ . Equivalently, for all  $b \in M$  which is a cofinal branch of  $T$ ,  $\Delta(b, b_\eta) \in M$ . By the lemma above, it suffices to show that for all  $\gamma \in M$ ,  $\rho(\gamma, \eta) \leq M \cap \omega_1$ . Recall that:

$$\rho(\gamma, \eta) \leq \max\{\rho(\gamma, \alpha_M), \rho(\alpha_M, \mu), \rho(\mu, \eta)\}.$$

Fix  $\beta$  which is a limit point of  $C_{\alpha_M}$  and which is above  $\gamma$ . Since  $\beta \in M$  and  $\rho(\gamma, \beta) = \rho(\gamma, \alpha_M)$ , we have that

$$\rho(\gamma, \alpha_M) \in M.$$

Since  $M \in E$  and  $\alpha_{N_\xi} \in \text{lim}(C_\mu)$ , we obtain

$$\rho(\alpha_M, \mu) = \rho(\alpha_M, \alpha_{N_\xi}) \leq M \cap \omega_1.$$

Recall that  $\rho(\mu, \eta) \in M$ . Therefore,  $\rho(\gamma, \eta) \leq M \cap \omega_1$ .

Now assume  $M \in \Sigma \cap N_\xi$ ,  $t$  is the first element of  $b_\eta$  whose height is more than  $N_\xi \cap \omega_1$ . It suffices to show that  $M$  does not capture  $t$  as an element in  $U$ . Assume  $b \subset U$  is a cofinal branch of  $U$  which is in  $M$  and  $b$  contains  $\{s \in U \cap M : s < t\}$ . Since  $t \notin M$ , the set  $\{s \in U \cap M : s < t\}$  has order type  $M \cap \omega_1$ . Let  $b_\gamma$  be the downward closure of  $b$  in  $T$ . Then obviously  $\gamma \in M$ . But then the order type of  $b_\gamma \cap b_\eta$  is at least  $M \cap \omega_1$ , which is a contradiction.  $\square$

We finish this section by a corollary which relates the theorem above to Martin’s Axiom.

**Corollary 5.3.** *Assume  $\text{MA}_{\omega_2}$  holds and  $\omega_2$  is not a Mahlo cardinal in  $L$ . Then there is a Kurepa tree with the property that every Kurepa subset has an Aronszajn subtree.*

### 6. Taking Komjath’s inaccessible away

In this section we will show that if there is an inaccessible cardinal in  $L$  then there is a model of ZFC in which every Kurepa tree has an Aronszajn subtree. We will be using the following notation. If  $G \subset Q$  is a generic filter and  $X \subset \omega_2$  with  $Q_X \triangleleft Q$ , we use  $G_X$  in order to refer to  $G \cap Q_X$ . If  $X \subset A$  and  $Q_X \triangleleft Q_A \triangleleft Q$ ,  $R_{X,A}$  refers to the ccc poset such that  $Q_A = Q_X * \dot{R}_{X,A}$ . Note that the generic tree  $T$  is in  $V[G_X]$  if  $X$  is uncountable and  $Q_X \triangleleft Q$ . Then  $R_{X,A}$  can be described more explicitly in the forcing extension by  $G_X$  as follows. Let  $T$  be the generic tree for  $Q_X$  and  $b_\xi$  be the set of all  $t \in T$  such that  $t \in q(\xi)$  for some  $q \in G_X$ . Recall that  $b_\xi$  is an uncountable downward closed branch of  $T$ . Moreover, every branch of  $T$  in the forcing extension by  $G_X$  has to be  $b_\xi$  for some  $\xi \in X$ . The poset  $R_{X,A}$  consists of finite partial functions  $p$  from  $A \setminus X$  to  $T$  such that:

- (1) for every  $\alpha \in \text{dom}(p)$  and  $\xi \in X$ ,  $(p(\alpha) \wedge b_\xi) < \rho\{\xi, \alpha\}$  and
- (2) for all  $\alpha < \beta$  in  $\text{dom}(p)$ ,  $(p(\alpha) \wedge p(\beta)) < \rho(\alpha, \beta)$ .

In  $R_{X,A}$ ,  $q \leq p$  if  $\text{dom}(q) \supset \text{dom}(p)$  and  $p(\alpha) \leq_T q(\alpha)$  for all  $\alpha \in \text{dom}(p)$ . We sometimes use the notation  $R_A(B)$  in order to refer to  $R_{A,A \cup B}$  if  $A, B$  are disjoint.

Also, for finite  $x \subset [\mu, \omega_2)$ , let  $S^\mu[x]$  be the set of all  $\langle v_i : i \in |x| \rangle \in T^{\{|x|\}}$  such that for some  $q \in R_{\mu, \omega_2}$ :

- $\text{dom}(q) \supset x$  and
- for all  $i \in |x|$ ,  $q(x(i)) = v_i$ .

So in particular every condition in  $R_{\mu, \omega_2}$  force that  $\bigotimes_{\alpha \in x} \dot{b}_\alpha \subset S^\mu[x]$ . For  $\alpha \in \omega_2 \setminus \mu$ , we use  $S^\mu[\alpha]$  instead of  $S^\mu[\{\alpha\}]$ .

**Lemma 6.1.** *Assume  $\omega_1 < \mu < \omega_2$ ,  $Q_\mu \triangleleft Q$  in  $V$  and  $G \subset Q$  is  $V$ -generic. Let  $K$  be an  $\omega_1$ -tree in  $V[G_\mu]$  and  $b \subset K$  be a cofinal branch in  $V[G]$ . Then there is a finite  $x \subset [\mu, \omega_2)$  such that  $b \in V[G_{\mu \cup x}]$ .*

**Proof.** Work in  $V[G_\mu]$ . Let  $T$  be the generic tree that is introduced by  $G_\mu$ ,  $r \in R_{\mu, \omega_2} \cap G$ ,  $\tau \subset K \times \{q \in R_{\mu, \omega_2} : q \leq r\}$  be an  $R_{\mu, \omega_2}$ -name. Assume for all finite  $x \subset [\mu, \omega_2)$ ,

$$r \Vdash_{R_{\mu, \omega_2}} \text{“}\tau \text{ is a cofinal branch of } K \text{ which is not in } V[G_\mu * \dot{H}_x]\text{”} \tag{4}$$

where  $\dot{H}_x$  is the canonical name for the  $V[G_\mu]$ -generic filter of  $R_\mu(x)$ . For every  $u \in K$ , let  $C_u = \{q \leq r : q \Vdash_{R_{\mu, \omega_2}} u \in \tau\}$  and let  $E_u \subset C_u$  such that:

- (e<sub>1</sub>)  $E_u$  is an antichain that is maximal in  $C_u$ .
- (e<sub>2</sub>) If  $q \in E_u$  and  $\alpha \in \text{dom}(q)$  then  $\text{ht}_T(q(\alpha)) \geq \text{ht}_K(u)$ .
- (e<sub>3</sub>) If  $q \in E_u$  then  $q$  is a one-to-one function whose range consists of the elements of the same height in  $T$ .

The condition (e<sub>1</sub>) implies that  $E_u$  is countable because  $R_{\mu, \omega_2}$  is ccc. Let  $\tau' = \bigcup\{\{u\} \times C_u : u \in K\}$  and  $\tau'' = \bigcup\{\{u\} \times E_u : u \in K\}$ . Observe that  $r \Vdash \tau = \tau' = \tau''$ . Without loss of generality, we assume  $\tau = \tau''$ , or in other words  $\tau[\{u\}] = E_u$  for all  $u \in K$ . Let  $U$  be the set of all  $u \in K$  such that for some  $q \leq r$  in  $R_{\mu, \omega_2}$  the condition  $q$  forces that  $u \in \tau$ .  $R_{\mu, \omega_2}$  is ccc, so  $U$  is a Souslin tree in  $V[G_\mu]$ .

Let  $\Gamma \subset \text{range}(\tau)$  be uncountable such that  $\{\text{dom}(p) : p \in \Gamma\}$  forms a  $\Delta$ -system with root  $w$ . By thinning  $\Gamma$  out if necessary, we can assume the conditions in  $\Gamma$  have the same cardinality  $k + |w| \in \omega$ . Note that  $r \in R_\mu(w)$  and  $U = \text{dom}(\tau'') = \text{dom}(\tau)$ . Also note that an important consequence of (e<sub>3</sub>) is that if  $p \in \Gamma$  and  $p \upharpoonright w \in G_{\mu \cup w}$  then  $p$  is compatible with every condition in  $G_{\mu \cup w}$ .

By (e<sub>2</sub>), the set  $\Gamma_w = \{p \upharpoonright w : p \in \Gamma\}$  is an uncountable subset of  $R_\mu(w)$  consisting of conditions extending  $r$ . Therefore  $R_\mu(w)$  has an extension  $r'$  of  $r$  which forces that  $\Gamma_w \cap \dot{H}_w$  is uncountable. In order to contradict (4), we need to work in  $V[G_\mu * H_w]$  and some specific forcing extensions of this model. Here  $H_w \subset R_\mu(w)$  is a  $V[G_\mu]$ -generic filter which contains  $r'$  and consequently intersects  $\Gamma_w$  on an uncountable set. Due to similarity of arguments and for easier notation let's assume

$$|\Gamma_w \cap G_{\mu \cup w}| = \aleph_1$$

and work with  $V[G_{\mu \cup w}]$  instead of  $V[G_\mu * H_w]$ .

Let  $\mathcal{A} \in V[G_{\mu \cup w}]$  be the set of all  $p \upharpoonright (\text{dom}(p) \setminus w)$  such that  $p \in \Gamma$  and  $p \upharpoonright w \in G_{\mu \cup w}$ . Note that  $\mathcal{A}$  is uncountable. Here we use (e<sub>2</sub>) and the fact that  $\Gamma_w \cap G_{\mu \cup w}$  is uncountable.

For each  $p \in \mathcal{A}$ , let  $d_p : k \rightarrow \text{dom}(p)$  be the unique strictly increasing bijection. Let  $\langle I_l : 0 < l \leq \frac{k(k+1)}{2} + 1 \rangle$  be a sequence listing all  $I \subset k$  with  $0 < |I| \leq 2$  such that all singletons are listed before pairs. We are going to find  $\langle V_l, \mathcal{A}_l : l \leq \frac{k(k+1)}{2} + 1 \rangle$ , by induction on  $l$ , such that:

- $V_0 = V[G_{\mu \cup w}]$ .
- $V_{l+1}$  is an Aronszajn tree preserving and  $\omega_2$ -preserving forcing extension of  $V_l$ .
- $\mathcal{A}_l \in V_l$  is uncountable for all  $l$ .
- $\mathcal{A}_{l+1} \subset \mathcal{A}_l \subset \mathcal{A}_0 = \mathcal{A}$ .
- If  $\{p, q\} \subset \mathcal{A}_l$  then  $p \upharpoonright \{d_p(n) : n \in I_l\}$  and  $q \upharpoonright \{d_q(n) : n \in I_l\}$  are compatible in  $R_{\mu \cup w, \omega_2}$ .

We proceed by finding  $V_l, \mathcal{A}_l$  when  $V_{l-1}, \mathcal{A}_{l-1}, I_l$  are given. Work in  $V_{l-1}$ . First assume  $0 < l \leq k$ , which means  $I_l = \{n\}$  for some  $n \in k$ . This task can be done by managing the following cases:

- (1) The map  $p \mapsto p(d_p(n))$  is constant on some uncountable subset of  $\mathcal{A}_{l-1}$ .
- (2) The map  $p \mapsto p(d_p(n))$  is countable-to-one and the downward closure of  $\{p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$  has an uncountable branch.
- (3) The map  $p \mapsto p(d_p(n))$  is countable-to-one and the downward closure of  $\{p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$  is Aronszajn.

For the first case, fix uncountable  $\mathcal{B} \subset \mathcal{A}_{l-1}$  such that  $p \mapsto p(d_p(n))$  is constant on  $\mathcal{B}$ . Let  $\nu = p(d_p(n))$  for some (any)  $p \in \mathcal{B}$ . Let  $\mathcal{A}_l \subset \mathcal{B}$  be uncountable such that if  $p \neq q$  are in  $\mathcal{A}_l$  then  $\rho\{d_p(n), d_q(n)\} > \nu$ . It is easy to see that  $\mathcal{A}_l$  together with  $V_l = V_{l-1}$  works.

For the second case let  $W$  be the downward closure of  $\{p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$  in  $T$ . By Lemmas 2.8 and 5.1, let  $\xi \in \mu \cup w$  such that  $b_\xi \subset W$ . Let  $\langle p_i : i \in \omega_1 \rangle$  be a sequence in  $\mathcal{A}_{l-1}$  such that  $\langle p_i(d_{p_i}(n)) \wedge b_\xi : i \in \omega_1 \rangle$  is strictly increasing. Let  $\Gamma_0 \subset \omega_1$  be uncountable such that  $\langle \alpha_i = d_{p_i}(n) : i \in \Gamma_0 \rangle$  and  $\langle \rho\{\alpha_i, \xi\} : i \in \Gamma_0 \rangle$  are both strictly increasing. Recall that  $\rho\{\alpha_i, \xi\} \geq b_\xi \wedge p_i(\alpha_i)$ , so this is possible. Find uncountable  $\Gamma_1 \subset \Gamma_0$  such that  $\rho(\alpha_i, \alpha_j) \geq \min\{\rho\{\alpha_i, \xi\}, \rho\{\alpha_j, \xi\}\}$  for  $i < j$  in  $\Gamma_1$ . In order to see  $\mathcal{A}_l = \{p_i : i \in \Gamma_1\}$  and  $V_l = V_{l-1}$  work, assume for a contradiction that  $p_i(\alpha_i) \wedge p_j(\alpha_j) \geq \rho(\alpha_i, \alpha_j)$  for some  $i < j$  in  $\Gamma_1$ . Then

$$\rho\{\xi, \alpha_i\} > p_i(\alpha_i) \wedge b_\xi = p_i(\alpha_i) \wedge p_j(\alpha_j) \geq \rho(\alpha_i, \alpha_j) \geq \rho\{\alpha_i, \xi\},$$

which obviously is a contradiction.

For the third case, let  $W$  be a pruned downward closed uncountable subtree of the downward closure of  $\{p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$  in  $T$ . Let  $V_l$  be a forcing extension of  $V_{l-1}$  which adds an antichain  $A \subset W$  as in Lemma 2.8. From now on we work in  $V_l$ . Fix  $\gamma > \sup\{d_p(n) : p \in \mathcal{A}_{l-1}\}$  in  $\omega_2$  and  $\langle t_i : i \in \omega_1 \rangle$  in  $A$  such that if  $i < j$  then  $\text{ht}(t_i) < \text{ht}(t_j)$ . Since  $W$  is pruned, for every  $t \in W$  there are uncountably many  $p$  in  $\mathcal{A}_{l-1}$  with  $t \leq_T p(d_p(n))$ . Since  $\omega_2$  is preserved, the square sequence of  $V_{l-1}$  is a square sequence in  $V_l$ . Therefore, for each  $i \in \omega_1$  there is  $p_i \in \mathcal{A}_{l-1}$  such that  $t_i \in \rho(d_{p_i}(n), \gamma)$  and  $t_i <_T p_i(d_{p_i}(n))$ . Let  $\alpha_i = d_{p_i}(n)$ . Find uncountable  $\Gamma_0 \subset \omega_1$  such that  $\langle \alpha_i : i \in \Gamma_0 \rangle$  and  $\langle \rho(\alpha_i, \gamma) : i \in \Gamma_0 \rangle$  are both strictly increasing. Find uncountable  $\Gamma_1 \subset \Gamma_0$  such that

$$\rho(\alpha_i, \alpha_j) \geq \min\{\rho(\alpha_i, \gamma), \rho(\alpha_j, \gamma)\}$$

whenever  $i < j$  in  $\Gamma_1$ . In order to see  $\mathcal{A}_l = \{p_i : i \in \Gamma_1\}$  works, assume  $i < j$  are in  $\Gamma_1$ . Then

$$p_i(\alpha_i) \wedge p_j(\alpha_j) < t_i < \rho(\alpha_i, \gamma) = \min\{\rho(\alpha_i, \gamma), \rho(\alpha_j, \gamma)\} \leq \rho(\alpha_i, \alpha_j),$$

as desired. This finishes our induction for the singleton sets  $I_l$ .

Before we deal with the induction steps in which  $I_l$  is a pair, let's make an observation.

**Observation 6.2.** Assume  $\mathbb{V}$  is a forcing extension of  $V[G_{\mu \cup w}]$  by a forcing described in Lemma 2.8. Let  $m < n < k$  and assume in  $\mathbb{V}$ ,  $\mathcal{B} \subset \mathcal{A}$  is uncountable such that the maps  $p \mapsto p(d_p(n))$  and  $p \mapsto p(d_p(m))$  are countable-to-one on  $\mathcal{B}$ . Then either

- (a) there are incomparable  $s, t$  in  $T$  and uncountable  $\mathcal{B}_0 \subset \mathcal{B}$  such that for all  $p \in \mathcal{B}_0$ ,  $s <_T p(d_p(m))$  and  $t <_T p(d_p(n))$ , or
- (b)  $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{B}\}$  is uncountable.

**Proof of Observation 6.2.** Assume  $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{B}\}$  is countable. Assume  $u \in T$  such that for uncountably many  $p \in \mathcal{B}$ ,  $p(d_p(m)) \wedge p(d_p(n)) = u$  and let  $\delta = \text{ht}(u) + 1$ . Then there are incomparable  $s, t$  above  $u$  in  $T_\delta$  such that

$$\mathcal{B}' = \{p \in \mathcal{B} : (p(d_p(m)) \upharpoonright (\delta + 1), p(d_p(n)) \upharpoonright (\delta + 1)) = (s, t)\}$$

is uncountable. Since both maps  $p \mapsto p(d_p(m))$  and  $p \mapsto p(d_p(n))$  are countable-to-one, there is an uncountable  $\mathcal{B}_0 \subset \mathcal{B}'$  as desired in (a). Therefore, the dichotomy in Observation 6.2 holds.  $\square$

Assume  $V_{l-1}, \mathcal{A}_{l-1}, I_l$  are given and  $I_l = \{m, n\}$  is a pair. Based on Observation 6.2, we can assume at least one of the following cases holds:

- (0) At least one of the maps  $p \mapsto p(d_p(n))$  or  $p \mapsto p(d_p(m))$  is not countable-to-one on  $\mathcal{A}_{l-1}$ .
- (a) There are incomparable  $s, t$  in  $T$  and uncountable  $\mathcal{B}_0 \subset \mathcal{A}_{l-1}$  such that for all  $p \in \mathcal{B}_0$ ,  $s <_T p(d_p(m))$  and  $t <_T p(d_p(n))$ . Moreover, the maps  $p \mapsto p(d_p(n))$  and  $p \mapsto p(d_p(m))$  are countable-to-one on  $\mathcal{A}_{l-1}$ .
- (b.1) The downward closure of  $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$  in  $T$  has an uncountable branch and the maps  $p \mapsto p(d_p(n))$  and  $p \mapsto p(d_p(m))$  are countable-to-one on  $\mathcal{A}_{l-1}$ .
- (b.2) The downward closure of  $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$  in  $T$  is an Aronszajn tree and the maps  $p \mapsto p(d_p(n))$  and  $p \mapsto p(d_p(m))$  are countable-to-one on  $\mathcal{A}_{l-1}$ .

For case (0), the forcing extension is the trivial forcing extension. Find uncountable  $\mathcal{B} \subset \mathcal{A}_{l-1}$  and  $t \in T$  such that one of the maps  $p \mapsto p(d_p(n))$  or  $p \mapsto p(d_p(m))$  is constantly  $t$  on  $\mathcal{B}$ . Let  $\nu = t + 1$  and let  $\mathcal{A}_l \subset \mathcal{B}$  be uncountable such that for  $p \neq q$  in  $\mathcal{A}_l$ ,  $\rho\{d_p(n), d_q(m)\} > \nu$ . So for all distinct  $p, q$  in  $\mathcal{A}_l$ ,  $p(d_p(n)) \wedge q(d_q(m)) < \nu < \rho\{d_p(n), d_q(m)\}$ . By the symmetry and since we have already dealt with the one element subsets of  $k$ , this finishes case (0).

For case (a), the forcing extension is the trivial forcing extension. Fix  $s, t, \mathcal{B}_0$  as in (a) of Observation 6.2. Let  $\mathcal{A}_l \subset \mathcal{B}_0$  be uncountable such that for  $p \neq q$  in  $\mathcal{A}_l$ ,  $t < \rho\{d_p(n), d_q(m)\}$ . Then for all  $p \neq q$  in  $\mathcal{A}_l$ ,  $p(d_p(n)) \wedge p(d_q(m)) = s \wedge t < t < \rho\{d_p(n), d_q(m)\}$ . Because of symmetry and the fact that we dealt with the one element sets in the previous steps, this finishes case (a).

For case (b.1), the forcing extension is the trivial forcing extension. Assume  $W$  is the downward closure of the uncountable set  $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$  in  $T$ . Using Lemmas 2.8 and 5.1, let  $\xi \in \mu \cup w$  such that  $b_\xi \subset W$ . We can find  $\{p_i : i \in \omega_1\} \subset \mathcal{A}_{l-1}$  such that  $\langle p_i(d_{p_i}(m)) \wedge p_i(d_{p_i}(n)) \wedge b_\xi : i \in \omega_1 \rangle$  is strictly increasing. Find uncountable  $\Gamma_0 \subset \omega_1$  such that the sequences

- $\langle \alpha_i = d_{p_i}(n) : i \in \Gamma_0 \rangle$ ,
- $\langle \beta_i = d_{p_i}(m) : i \in \Gamma_0 \rangle$ ,
- $\langle \{(p_i(\alpha_i) \wedge b_\xi), (p_i(\beta_i) \wedge b_\xi)\} : i \in \Gamma_0 \rangle$ ,
- $\langle \{\rho\{\alpha_i, \xi\}, \rho\{\beta_i, \xi\}\} : i \in \Gamma_0 \rangle$

are all strictly increasing. Find uncountable  $\Gamma_1 \subset \Gamma_0$  such that

$$\rho\{\alpha_i, \beta_j\} \geq \min\{\rho\{\alpha_i, \xi\}, \rho\{\beta_j, \xi\}\}, \tag{5}$$

for  $i \neq j$  in  $\Gamma_1$ .

Assume  $i < j$  are in  $\Gamma_1$ . Then

$$p_i(\alpha_i) \wedge p_j(\beta_j) = p_i(\alpha_i) \wedge b_\xi < \rho\{\alpha_i, \xi\} = \min\{\rho\{\alpha_i, \xi\}, \rho\{\beta_j, \xi\}\}.$$

From (5) it follows that  $p_i(\alpha_i) \wedge p_j(\beta_j) < \rho\{\alpha_i, \beta_j\}$ . Again, by symmetry and the fact that we have already dealt with the one element sets before,  $\mathcal{A}_l = \{p_i : i \in \Gamma_1\}$  and  $V_l = V_{l-1}$  works. This finishes case (b.1).

In case (b.2), let  $W$  be the downward closure of the uncountable set  $\{p(d_p(m)) \wedge p(d_p(n)) : p \in \mathcal{A}_{l-1}\}$  in  $T$ . Let  $W'$  be an uncountable downward closed pruned subtree of  $W$ . Let  $V_l$  be a forcing extension of  $V_{l-1}$  in which  $W'$  has an uncountable antichain  $A$  as in Lemma 2.8. Work in  $V_l$  and let  $\{t_i : i \in \omega_1\} \subset A$  such that  $\langle \text{ht}_T(t_i) : i \in \omega_1 \rangle$  is strictly increasing. Let  $\gamma \in \omega_2$  be above all ordinals in  $\{p(d_p(n)) + p(d_p(m)) : p \in \mathcal{A}_{l-1}\}$ . For each  $i \in \omega_1$ , find  $p_i \in \mathcal{A}_{l-1}$  such that

- $t_i <_T (p_i(\alpha_i) \wedge p_i(\beta_i))$  where  $\alpha_i = d_{p_i}(n)$  and  $\beta_i = d_{p_i}(m)$ ,
- $t_i \in \rho(\alpha_i, \gamma)$ , and
- $t_i \in \rho(\beta_i, \gamma)$ .

This is possible because the maps  $p \mapsto p(d_p(n))$  and  $p \mapsto p(d_p(m))$  are countable-to-one and  $W'$  is pruned. Let  $\Gamma_0 \subset \omega_1$  be uncountable such that:

- $\rho\{\alpha_i, \beta_j\} \geq \min\{\rho(\alpha_i, \gamma), \rho(\beta_j, \gamma)\}$  for all distinct  $i, j$  in  $\Gamma_0$ , and
- $\langle \{\rho(\alpha_i, \gamma), \rho(\beta_i, \gamma)\} : i \in \Gamma_0 \rangle$  is strictly increasing.

Now we show that  $\mathcal{A}_l = \{p_i : i \in \Gamma_0\}$  works. Assume  $i < j$  in  $\Gamma_0$ . Then

$$p_i(\alpha_i) \wedge p_j(\beta_j) < t_i \in \rho(\alpha_i, \gamma) = \min\{\rho(\alpha_i, \gamma), \rho(\beta_j, \gamma)\} \leq \rho(\alpha_i, \beta_j).$$

As in the previous case, by symmetry and the fact that we have already dealt with the one element  $I_l$ 's, this finishes the work for case (b.2).

Let  $z = \frac{k(k+1)}{2} + 1$ ,  $\mathcal{A}' = A_z$ . In  $V_z$  define

$$\Gamma^* = \{p \in \Gamma : p \upharpoonright w \in G_{\mu \cup w} \text{ and } p \upharpoonright (\text{dom}(p) \setminus w) \in \mathcal{A}'\}.$$

By the refinement process above,  $\mathcal{A}' \in V_z$  is uncountable and consists of pairwise compatible elements. So by  $(e_3)$ ,  $\Gamma^*$  is uncountable and consists of pairwise compatible elements. Again by  $(e_3)$ , every condition in  $\Gamma^*$  is compatible with every condition in  $G_{\mu \cup w}$ .

Define  $b' = \{u \in U : \exists p \in \Gamma^* (u, p) \in \tau\}$ . Since elements of  $\Gamma^*$  are pairwise compatible, the elements of  $b'$  are pairwise comparable. By  $(e_1)$  and the fact that  $\Gamma^*$  is uncountable,  $b'$  is uncountable. Therefore, the downward closure of  $b'$ , which we call  $b$ , is an uncountable branch of  $U$  in  $V[G_{\mu \cup w}]$ . Here we use the fact that  $V_z$  is a forcing extension of  $V[G_{\mu \cup w}]$  using forcings which satisfy the properties listed in Lemma 2.8. Let  $\pi = \{(u, p) \in \tau : \forall q \in G_{\mu \cup w} p \text{ is compatible with } q\}$ . Note that  $\pi$  is the natural transition of  $\tau$  from  $V[G_\mu]$  to  $V[G_{\mu \cup w}]$ . Recall that  $r \in G_{\mu \cup w}$  and it forces that  $\tau$  is an uncountable branch. So in  $V[G_{\mu \cup w}]$ , the trivial element of  $R_{\mu \cup w, \omega_2}$  forces that  $\pi$  is an uncountable branch. By  $(e_1)$ ,  $\{p \in \text{range}(\pi) : \exists u \in b (u, p) \in \pi\}$  is uncountable. Let  $s \in R_{\mu \cup w, \omega_2}$  which forces that the generic filter of  $R_{\mu \cup w, \omega_2}$  intersects  $\{p \in \text{range}(\pi) : \exists u \in b (u, p) \in \pi\}$  on an uncountable set. In particular in  $V[G_{\mu \cup w}]$ , the condition  $s$  forces that the downward closure of  $\pi$ ,  $\tau$  is the same as  $b$ . In other words, the condition  $s$  forces that  $\tau$  is a branch in  $V[G_{\mu \cup w}]$ . But this contradicts (4) because  $s$  is compatible with  $r$ .  $\square$

Now we are ready to prove Theorem 1.2. Assume  $\lambda$  is the first inaccessible cardinal in  $L$  and  $V$  is the generic extension of  $L$  by the Levy collapse forcing with countable conditions which makes  $\lambda$  the second uncountable cardinal. Note that  $V$  is a model of  $\square_{\omega_1}$ . Assume  $G \subset Q$  is  $V$ -generic and  $T, \langle b_\xi : \xi \in \lambda \rangle$  are the generic tree and branches that are defined from  $G$  as usual. We show for every Kurepa tree  $K$  in  $V[G]$  there is a Kurepa subtree of  $T$  which club embeds into  $K$ . By Theorem 5.2, this finishes the proof of Theorem 1.2.

Assume for a contradiction that  $K \in V[G]$  is a Kurepa tree,  $\check{K}$  is a  $Q$ -name for  $K$ , and  $p_0 \in G$  forces that  $\check{K}$  is a Kurepa tree such that no Kurepa subtree of  $\check{T}$  club embeds into  $\check{K}$ . Let  $\mu_0 \in \omega_2$  such that  $Q_{\mu_0} \triangleleft Q$ ,  $K$  and  $T$  are in  $V[G_{\mu_0}]$  and  $p_0 \in G_{\mu_0}$ . Note that in  $V[G_{\mu_0}]$ ,

$$R_{\mu_0, \omega_2} \Vdash \text{“no Kurepa subtree of } \check{T} \text{ club embeds into } \check{K} \text{.”} \tag{6}$$

Let  $Y \in V[G_{\mu_0}]$  be the set of all  $(\tau, p, x, A)$  such that:

- $(a_0)$   $x$  is a finite subset of  $[\mu_0, \omega_2)$ ,
- $(a_1)$   $\tau$  is an  $R_{\mu_0}(x)$ -name,
- $(a_2)$   $p \Vdash_{R_{\mu_0}(x)}$  “ $\tau$  is a cofinal branch of  $\check{K}$  which is not in  $V[G_{\mu_0} * \dot{H}_{x'}]$ , for any finite  $x'$  which is a proper subset of  $x$ ”, where  $\dot{H}_{x'}$  is the canonical name for the  $V[G_{\mu_0}]$ -generic filter of  $R_{\mu_0}(x')$ ,
- $(a_3)$   $p$  is a one-to-one function,  $\text{dom}(p) = x$  and  $\text{range}(p)$  consists of the elements of the same height in  $T$ ,

$$(a_4) \ A = \{u \in K : \exists q \in R_{\mu_0}(x) \ q \leq p \wedge q \Vdash \check{u} \in \tau\}.$$

For  $i \in \{1, 2, 3, 4\}$  let  $Y_i$  be the projection of  $Y$  on the  $i$ 'th component. By Lemmas 2.6 and 6.1,  $|Y_3| = \aleph_2$ . Let  $\langle x_\xi : \xi \in \omega_2 \rangle$  be an enumeration of  $Y_3$ .

Let  $n \in \omega$  and  $\Gamma_0 \subset \omega_2$  be of size  $\aleph_2$  such that  $\{x_\xi : \xi \in \Gamma_0\}$  is a  $\Delta$ -system with root  $w$  and  $|x_\xi| = n + |w|$  for  $\xi \in \Gamma_0$ . By thinning  $\Gamma_0$  out if necessary we can assume that  $\langle y_\xi = x_\xi \setminus w : \xi \in \Gamma_0 \rangle$  is strictly increasing. For every  $\xi \in \Gamma_0$  let  $\tau'_\xi, p'_\xi, A'_\xi$  be such that  $(\tau'_\xi, p'_\xi, x_\xi, A'_\xi) \in Y$ . By thinning  $\Gamma_0$  out again we assume that for all  $i \in n + |w|$ ,  $p'_\xi(x_\xi(i))$  does not depend on  $\xi$ . There is a condition  $r \in R_{\mu_0, \omega_2}$  which forces that for  $\aleph_2$  many  $\xi \in \Gamma_0$ ,  $p'_\xi$  is in the generic filter  $\dot{H}_{[\mu_0, \omega_2]}$ , where  $\dot{H}_{[\mu_0, \omega_2]}$  is the canonical name for the  $V[G_{\mu_0}]$ -generic filter of  $R_{\mu_0, \omega_2}$ . In order to contradict (6), we need to work with a  $V[G_{\mu_0}]$ -generic filter of  $R_{\mu_0, \omega_2}$  which intersects  $\{p'_\xi : \xi \in \Gamma_0\}$  on a set of size  $\aleph_2$ . Due to similarity of arguments and for easier notation let's assume without loss of generality that

$$|G \cap \{p'_\xi : \xi \in \Gamma_0\}| = \aleph_2. \tag{7}$$

Fix  $\mu \in \omega_2 \setminus \mu_0$  above  $\max(w)$  such that  $Q_\mu \triangleleft Q$ . From now on, we work in  $V[G_\mu]$  unless otherwise stated. Define  $\Gamma_1 \in V[G_\mu]$  to be the set of all  $\xi \in \Gamma_0$  such that  $\min(y_\xi) > \mu$  and  $p'_\xi \upharpoonright w \in G_\mu$ . Obviously  $|\Gamma_1| = \aleph_2$  by (7). For each  $\xi \in \Gamma_1$  let  $p_\xi = p'_\xi \upharpoonright y_\xi$ . Note that by (a<sub>3</sub>) and the definition of  $\Gamma_1$ ,  $p_\xi$  is compatible with every condition in  $G_\mu$ . Via the natural transition of objects  $\tau'_\xi, A'_\xi$  from  $V[G_{\mu_0}]$  to  $V[G_\mu]$ , we can find  $\tau_\xi, A_\xi$  in  $V[G_\mu]$  such that for all  $\xi \in \Gamma_1$  the statement (a<sub>i</sub>) above implies (b<sub>i</sub>) below:

- (b<sub>1</sub>)  $\tau_\xi$  is an  $R_\mu(y_\xi)$ -name,
- (b<sub>2</sub>)  $p_\xi \in R_\mu(y_\xi)$  forces that  $\tau_\xi$  is a cofinal branch of  $\check{K}$  which is not in  $V[G_\mu]$ ,
- (b<sub>3</sub>)  $p_\xi$  is a one-to-one function and the elements in  $\text{range}(p_\xi)$  have the same height in  $T$ ,
- (b<sub>4</sub>)  $A_\xi = \{u \in K : \exists q \in R_\mu(y_\xi) \ q \leq p_\xi \wedge q \Vdash \check{u} \in \tau_\xi\}$ .

We only show how we obtain (b<sub>2</sub>). Assume for a contradiction that  $\xi \in \Gamma_1$ ,  $r \in G_\mu \cap R_{\mu_0, \mu}$  is an extension of  $p'_\xi \upharpoonright w$  and  $\bar{p}_\xi \in R_{\mu_0, \omega_2}$  is an extension of  $p_\xi$  such that:

$$r * \bar{p}_\xi \Vdash_{R_{\mu_0, \omega_2}} \tau'_\xi \text{ is a cofinal branch in } V[G_{\mu_0} * \dot{H}_{\mu_0, \mu}].$$

Since  $r * \bar{p}_\xi$  extends  $p'_\xi$ , by (a<sub>2</sub>),

$$r * \bar{p}_\xi \Vdash_{R_{\mu_0, \omega_2}} \tau'_\xi \text{ is a cofinal branch in } V[G_{\mu_0} * \dot{H}_{\mu_0, \mu}] \cap V[G_{\mu_0} * \dot{H}_{x_\xi}].$$

This contradicts (a<sub>2</sub>) because by Lemma 5.1, for every  $V[G_{\mu_0}]$ -generic filter  $H \subset R_{\mu, \omega_2}$ ,  $V[G_{\mu_0} * H_x] \cap V[G_{\mu_0} * H_{\mu_0, \mu}] = V[G_{\mu_0} * H_{x \cap \mu}]$  and  $x_\xi \cap \mu$  is a proper subset of  $x_\xi$ . Hence (b<sub>2</sub>) holds.

Note that by Lemma 2.6, all finite powers of  $T$  and  $K$  have at most  $\aleph_1$  many cofinal branches and Souslin subtrees in  $V[G_\mu]$ . Let  $\Gamma_2 \subset \Gamma_1$  be of size  $\aleph_2$  such that for all  $\xi$  and  $\eta$  in  $\Gamma_2$  the following hold:

- $S^\mu[y_\xi(i)] = S^\mu[y_\eta(i)]$  for all  $i \in n$ ,
- $S^\mu[y_\xi] = S^\mu[y_\eta]$ ,
- $A_\xi = A_\eta$ .

Observe that if  $y \in \{y_\xi : \xi \in \Gamma_2\}$  and  $\bar{v} = \langle v_i : i \in n \rangle$  is an element of  $S^\mu[y]$ , and  $v_i$ 's are pairwise distinct then  $\bigotimes_{i \in n} (S^\mu[y(i)])_{v_i} = (S^\mu[y])_{\bar{v}}$ . Moreover, this tree does not depend on the choice of  $y$ . For  $i \in n$ , let  $t_i = p_\xi(y_\xi(i))$  for some  $\xi \in \Gamma_2$ . The properties of  $\Gamma_0$  guarantee that  $t_i$  does not depend on the choice of  $\xi$ . Let  $\Gamma_3 \subset \Gamma_2$  with  $|\Gamma_3| = \aleph_2$  such that if  $\xi < \eta$  are in  $\Gamma_3$ ,  $\alpha \in y_\xi$ ,  $\beta \in y_\eta$ , then  $\rho(\alpha, \beta) > \max\{t_i : i \in n\}$ .

For every  $\zeta \in \Gamma_3$  define  $\varphi_\zeta$  from  $\bigotimes_{i \in n} (S^\mu[y(i)])_{t_i}$  to the poset consisting of all extensions of  $p_\zeta = \{(y_\zeta(i), t_i) : i \in n\}$  in  $R_\mu(y_\zeta)$  as follows. For every  $\bar{s} = \langle s_i : i \in n \rangle$  in  $\bigotimes_{i \in n} (S^\mu[y(i)])_{t_i}$ , let  $\varphi_\zeta(\bar{s})$  be the function defined on  $y_\zeta$  which sends  $y_\zeta(i)$  to  $s_i$ . It is easy to see that  $\varphi_\zeta$  is an isomorphism from its domain to a dense subset of the set of all extensions of  $p_\zeta$  in  $R_\mu(y_\zeta)$ . Let  $S = \bigcup_{i \in n} (S^\mu[y(i)])_{t_i}$  and  $U = A_\zeta$ . Obviously,  $U$  is Souslin in  $V[G_\mu]$ . Also  $V[G_\mu]$  thinks that there is a derived tree of  $S$ , namely  $\bigotimes_{i \in n} (S^\mu[y(i)])_{t_i}$ , which adds a branch to  $U$ .

**Claim 6.3.** *All derived trees of  $S$  are Souslin in  $V[G_\mu]$ .*

**Proof.** Assume  $\langle s_j^i : i \in n \wedge j \in m \rangle$  are pairwise distinct elements of  $S$  with the same height  $\delta$  such that  $t_i \leq s_j^i$  for all  $i, j$ . We will show that  $\prod\{S_{s_j^i} : i \in n \wedge j \in m\}$  is the set of all possible points of a branch of  $T^{[mn]}$  which is added by a ccc poset in  $V[G_\mu]$ . Let  $\langle \xi_j : j \in m \rangle$  be a strictly increasing sequence in  $\Gamma_3$  such that for all  $j < k < m$  if  $\alpha \in y_{\xi_j}$  and  $\beta \in y_{\xi_k}$  then  $\rho(\alpha, \beta) > \delta + \omega$ . Let  $z_j = y_{\xi_j}$ . Define  $p : \bigcup_{j \in m} z_j \rightarrow T$  by  $p(z_j(i)) = s_j^i$ . By the requirement on  $\Gamma_3$  and the fact that  $\varphi_{\xi_j}$  is an isomorphism,  $p \upharpoonright z_j \in R_\mu(z_j)$  for all  $j \in m$ . The way we chose the  $z_j$ 's implies that  $p \in R_\mu(\bigcup_{j \in m} z_j)$ .

Obviously, the set of all extensions of  $p$  in  $R_\mu(\bigcup_{j \in m} z_j)$  is a ccc poset in  $V[G_\mu]$  and it adds a new branch to  $T^{[mn]}$ . We show that the set  $\prod\{S_{s_j^i} : i \in n \wedge j \in m\}$  is the set of all possible points of this branch. In order to see this, assume  $a_j^i \geq s_j^i$  is in  $S^\mu[y(i)]$ . Then the function  $r$  on  $\bigcup_{j \in m} z_j$  defined by  $r(z_j(i)) = a_j^i$  is a condition in  $R_\mu(\bigcup_{j \in m} z_j)$ . This can be seen in the same way as we showed  $p \in R_\mu(\bigcup_{j \in m} z_j)$ . Moreover,  $r$  forces that  $\langle a_j^i : i \in n \wedge j \in m \rangle$  is in the new branch that is added by  $R_\mu(\bigcup_{j \in m} z_j)$ . Therefore,  $\prod\{S_{s_j^i} : i \in n \wedge j \in m\}$  is the set of all possible points of the new branch that is added by  $R_\mu(\bigcup_{j \in m} z_j)$ , which is a ccc poset in  $V[G_\mu]$ . This shows the derived tree of  $S$  generated by  $\langle s_j^i : i \in n \wedge j \in m \rangle$  is a Souslin tree.  $\square$

**Claim 6.4.** *Assume  $\langle v_j : j \in k \rangle$  is a sequence of pairwise distinct elements of the same height in  $S$ . Then in  $V[G_\mu]$ , there is a condition  $q$  in  $R_{\mu, \omega_2}$  which forces that each  $S_{v_j}$  is Kurepa.*

**Proof.** Fix  $\Gamma_4 \subset \Gamma_3$  such that  $|\Gamma_4| = \aleph_2$  and for all  $\xi < \eta$  in  $\Gamma_4$ , for all  $\alpha \in y_\xi$ , for all  $\beta \in y_\eta$ ,

$$\rho(\alpha, \beta) > \max\{v_i : i \in k\}.$$

For every increasing  $\sigma = \langle \xi_l : l \in k \rangle$  in  $\Gamma_4$ , let  $q_\sigma : \bigcup_{l \in k} y_{\xi_l} \rightarrow S$  be a function such that  $q_\sigma(y_{\xi_l}(i)) = v_j$  if  $v_j$  is the  $l$ 'th ordinal in  $\langle v_j : j \in k \rangle$  that is above  $t_i$  in  $T$ . If there is no  $l$ 'th ordinal in  $\langle v_j : j \in k \rangle$  that is above  $t_i$  in  $T$ , let  $q_\sigma(y_{\xi_l}(i)) = t_i$ . The same argument as in Claim 6.3 shows that  $q_\sigma \in R_{\mu, \omega_2}$ .

Let  $\Gamma_5 \subset [\Gamma_4]^k$  be a collection of pairwise disjoint sets with  $|\Gamma_5| = \aleph_2$ . Since  $R_{\mu, \omega_2}$  is ccc, there is a condition  $q \in R_{\mu, \omega_2}$  which forces that for  $\aleph_2$  many  $\sigma \in \Gamma_5$ ,  $q_\sigma$  is in the generic filter. But then  $q$  forces that  $S_{v_j}$  is Kurepa for all  $j \in k$ .  $\square$

In  $V[G_\mu]$ , let  $\bigotimes_{i \in k} S_{v_i}$  be a derived tree of  $S$  which adds a branch to  $U$  and which has the minimum dimension with this property. Such a derived tree exists because  $\bigotimes_{i \in n} S_{t_i}$  adds a branch to  $U$ . By Lemma 2.7 and Claim 6.3, there is a club embedding  $f$  from  $\bigotimes_{i \in k} S_{v_i}$  to  $U$  in  $V[G_\mu]$ . By Claim 6.4, there is a condition  $q \in R_{\mu, \omega_2}$  which forces that all  $S_{v_i}$ 's are Kurepa subtrees of  $T$  in  $V[G]$ . Let  $j \in k$  and  $c_i \in V[G]$  be a cofinal



branch of  $S_{v_i}$ , for  $i \in k \setminus \{j\}$ . In  $V[G]$  let  $g$  be the restriction of  $f$  to the tree  $(\bigotimes_{i \in k \setminus \{j\}} c_i) \otimes S_{v_j}$ , which obviously is isomorphic to  $S_{v_j}$ . Then  $q$  forces that  $g$  is a club embedding from an isomorphic copy of  $S_{v_j}$  into  $U$ , and  $S_{v_j}$  is a Kurepa subtree of  $T$ . This contradicts (6).

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Data availability

No data was used for the research described in the article.

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